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A full ranking method using integrated DEA models and its application to modify GA for finding Pareto optimal solution of MOP problem

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Abstract: This paper uses integrated Data Envelopment Analysis (DEA) models to rank all extreme and non-extreme efficient Decision Making Units (DMUs) and then applies integrated DEA ranking method as a criterion to modify Genetic Algorithm (GA) for finding Pareto optimal solutions of a Multi-Objective Programming (MOP) problem. The researchers have used ranking method as a shortcut way to modify GA to decrease the iterations of GA. The modified algorithm reduces the computational efforts to find Pareto optimal solutions of MOP problem and can be used to find Pareto optimal solutions of MOP with convex and non-convex efficient frontiers. An example is given to illustrate the modified algorithm.

Keywords: *Data envelopment analysis (DEA); Ranking; Integrated DEA models; Multi-objective programming (MOP); Genetic algorithm (GA); Efficiency*

1. Introduction

Data Envelopment Analysis (DEA) and Multiple Objective Programming (MOP) are tools that can be used in management control and planning. DEA introduced by Charnes *et al.* in 1978, is a mathematical programming based approach that evaluates the efficiency of an organization or, in general, a Decision Making Unit (DMU) relative to a set of comparable organizations. DEA considers multiple inputs and outputs simultaneously, requiring neither a priori weights nor a functional form for input/output relationships. DEA utilizes mathematical programming to construct an empirical production possibility set, and provides a single efficiency score for that DMU by comparing to a ‘‘virtual producer’’ on the efficient frontier. After introducing the first model in DEA, the CCR model by Charnes *et al.* (1978), Banker *et al.* (1984) developed the DEA technique by providing the BCC and FDH models (Tulkens, 1993). An additional characteristic of DEA that has challenged users to improve on insights and values derived from the methodology involves the efficient set of DMUs, i.e., those with a score of 1.0. Andersen and Petersen (1993) proposed a super-efficiency procedure for ranking efficient DMUs. This method ranks only non-extreme efficient DMUs. In some cases, the AP model is infeasible. In addition to this difficulty, the AP model may be unstable because of extreme sensitivity to small variations in the data when some DMUs have relatively small values for some of their inputs.

In many practical problems such as engineering design problems, criteria functions cannot be given explicitly in terms of design variables. Under this circumstance, values of criteria functions for given values of design variables are usually obtained by some analyses such as structural analysis, thermodynamically analysis or fluid mechanical analysis. These analyses require considerably much computation time. Therefore, it is unrealistic to apply existing interactive optimization methods to those problems.

MOP is the simultaneous consideration of two or more objective functions that are completely or partially in conflict with each other. The optimality of such optimizations is largely defined through the Pareto optimality. Recently, MOP methods using genetic algorithms (GA) have been studied actively by many authors (Arakawa *et al*., 1998), (Fonseca and Fleming, 1993), (Schafer, 1985) and (Tamaki *et al*., 1996). GAs are useful for generating efficient frontiers with two or three objective functions. Decision making can be easily performed on the basis of visualized efficient frontiers. However, these methods have some shortcomings; there is a tendency for Vector Evaluated Genetic Algorithms (VEGA) (Schafer, 1985) to generate such solutions that one of the objective functions is extremely good.

Arakawa *et al.* (Arakawa *et al*., 1998) used the

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synthetic GA and DEA to generate Pareto optimal solution of MOP problem. Yun *et al.* (2001) proposed a method by combining generalized data envelopment analysis (GDEA) and GA to generate efficient frontiers in MOP problems. GDEA removes dominated design alternatives faster than methods based on only GA. The proposed method can yield desirable efficient frontiers. This method, however, has its own deficiencies. The most important deficiency is when a MOP problem has more than two or three objective functions.

In some situations many DMUs are efficient and the decision maker wants to select only one DMU among them as the most efficient DMU. For instance, to identify the most efficient, Facility Layout Design (FLD) Ertay *et al.* (2006) proposed a DEA/AHP methodology. Amin and Toloo (2007) extended their work and proposed an integrated DEA model based on a common set of weights to select the most DMU under- assumption of constant returns to scale (CRS). Amin (2008) showed in some situations the most efficient DMU among DMUs with CRS is not single and proposed a nonlinear programming problem to obtain a single most CCR-efficient DMU.

This paper uses Amin's model (2008) as an integrated ranking DEA model to rank all extreme and non-extreme efficient DMUs. Then, the researchers have used integrated ranking DEA model as a criterion and modify GA to find Pareto optimal solution of MOP problem. This method can be used to find Pareto optimal solutions of a MOP problem with more than tree objective functions or problems with non-convex efficient frontiers. It reduces the computational efforts to solve MOP problems.

The rest of this paper is organized as follows: Section 2 provides a background of MOP and DEA. In Section 3 the researchers have used integrated DEA ranking model to rank all extreme and non-extreme efficient DMUs. In Section 4, the researchers have used integrated DEA ranking method to modify GA. Section 5 discusses an example that applies the modified GA. Finally, Section 6 presents the concluding remarks.

2. Background

2.1. The MOP problem

An MOP problem is defined as follows:

$$min\, f(x) = \left(f_1(x),\dots,f_l(x)\right)^r$$

$$s.t.\quad x \in S = \left\{x \in R^n \,\middle|\, g_j(x) \le 0, j = 1,\dots,l\right\} \quad (1)$$

where, $f_1(x),\dots,f_l(x)$ are objective functions, $x=(x_1,\dots,x_n)^r$ is decision variables vector and S represents the feasible region of Problem (1).

Definition 1. $\hat{x} \in S$ is a Pareto optimal solution of problem (1) if there exist no $\hat{x} \in S$ such that $f(x) \le f(\hat{x})$ and $f(x) \ne f(\hat{x})$.

The Pareto optimal solutions of Problem (1) are found by the approaches such as the aspiration level technique, lexicography (Sawaragi *et al.*, 1985). But, these approaches require a long time particularly when the problems have many objective functions.

2.2. The DEA models

Suppose we have a set of n peer DMUs, DMU$_j$ $(j=1,\dots,n)$, with multiple inputs $x_{ij}=(1,\dots,m)$, and multiple outputs $y_{rj}(r=1,\dots,s)$. The DEA model for measuring the relative efficiency of DMU$_o$ under an assumption of constant returns to scale is the CCR model, Charnes *et al.* (1978). This model is a fractional linear program which can be transformed into the following linear program:

$$\begin{aligned}
&\theta_o^* = max \sum_{r=1}^{s} u_r y_{ro} \\
&s.t. \sum_{i=1}^{m} w_i x_{io} = 1, \\
&\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} w_i x_{io} \le 0, j = 1,\dots,n \\
&w_i \ge \varepsilon,\ i = 1,\dots,m \\
&u_r \ge \varepsilon, r = 1,\dots,s.
\end{aligned} \quad (2)$$

In some situations the efficient DMU is not unique and the decision maker wants to select the most efficient DMU among efficient DMUs. Amin (2008) proposed the following programming problem to obtain a single most CCR-efficient DMU:

$$\begin{aligned}
&min\, M \\
&s.t.\ M - d_j \ge 0, j = 1,\dots,n \\
&\sum_{i=1}^{m} w_i x_{ij} \le 1, j = 1,\dots,n \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0, j = 1,\dots,n
\end{aligned}$$

$$\sum_{j=1}^{n}\theta_j = n-1,$$
$$\theta_j - d_j\beta_j = 0, j=1,...,n$$
$$\beta_j \geq 1, d_j \geq 0, \theta_j \in \{0,1\}, j=1,...,n$$
$$w_i \geq \varepsilon^*, i=1,...,m \quad (3)$$
$$u_r \geq \varepsilon^*, r=1,...,s$$

where, ε^* is computed by the following model (2008):

$$\varepsilon^* = max\,\varepsilon$$
$$s.t. \sum_{i=1}^{m} w_i x_{ij} \leq 1, j=1,...,n \quad (4)$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} \leq 0, j=1,...,n$$
$$w_i \geq \varepsilon, i=1,...,m$$
$$u_r \geq \varepsilon, r=1,...,s.$$

Arakawa *et al.* (1998) used the combination of the DEA and the GA method to find efficient solutions of an MOP problem. In this method individual's values in constraints considered as the DEA model inputs and criterion function values for each individual considered as DEA model outputs. See Figure 1, where each DMU has two inputs and one same output. DEA efficient frontier can be used as an approximation to the MOP efficient frontier (see Figure 1, (Arakawa *et al.*, 1998; Yun *et al.*, 2001)).

Yun *et al.* (2001) proposed GDEA method which, also, includes basic DEA models the efficiency of DMUs in GDEA model is obtained by the following model.

$$max\,\Delta$$
$$s.t.\ \Delta \leq \tilde{d}_j - \alpha\sum_{i=1}^{m} v_i(-F_i(x_k) + F_i(x_j)),$$
$$j=1,...,n \quad (5)$$
$$\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1,$$
$$u_r, v_i \geq \varepsilon,\ i=1,...,m,\ r=1,...,s.$$

where α is defined in terms of the problem variables and for *j=1,…,n,* we have;

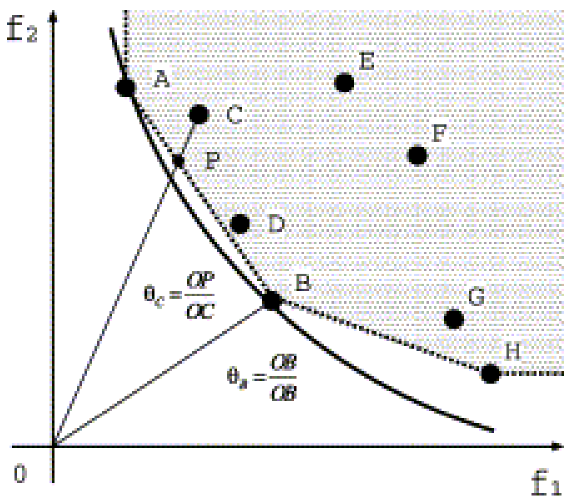

Figure 1: GA with DEA method and DEA efficiency.

$$\tilde{d}_j = \max\{u_r(y_{rk} - y_{rj}),\ v_i(-x_{ik} + x_{ij})\},$$
$$i=1,...,m,\ r=1,...,s.$$

Efficient frontier in GDEA model based on different α is showed in Figure 2, (Yun *et al.*, 2001).

Yun *et al.* (2001) proposed a method to eliminate the defects which were in the Arakawa *et al.* (1998) model. They defined inputs for GDEA model as follows;

$$F_i(x) = f_i(x) + \sum_{j=1}^{l} p_j \times \left[\{g_j(x)\}_+\right]^{\alpha}, i=1,...,s \quad (6)$$

where p_j represents a penalty and α represents penalty component and

$$\{g_j(x)\}_+ = max\{0, g_j(x)\}.$$

Yun *et al.* (2001) used (6) and converted model (1) to the following model:

$$min\,F(x) = (F_1(x),...,F_s(x))^T$$
$$s.t.\ x \in E^n \quad (7)$$

and, therefore, the GDEA model is converted as follows (Arakawa *et al.*, 1998):

$$\Delta_o^* = max\,\Delta_o$$
$$s.t.\ \Delta_o \leq \tilde{d}_j - \alpha\sum_{i=1}^{m} v_i(-F_i(x_o) + F_i(x_j)), \quad (8)$$
$$j=1,...,n,$$
$$\sum_{i=1}^{m} v_i = 1, \quad v_i \geq \varepsilon, i=1,...,m.$$

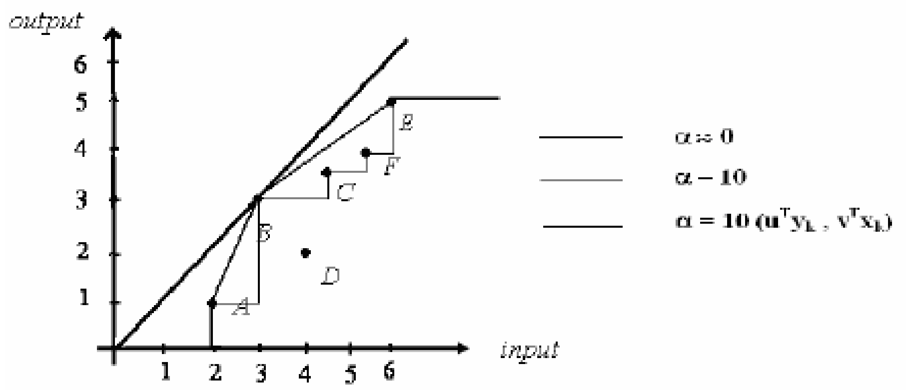

Figure 2: Efficient frontiers with different α .

We will use the properties of GDEA model and integrated DEA ranking method (see Section 3) to generate convex and non-convex efficient frontiers of the MOP problems.

3. Ranking all of efficient DMUs using the integrated DEA models

This section uses the proposed model by Amin (2008), Model (3), to rank all extreme and non-extreme efficient DMUs. To this end, let $E_1 = \{j_1,\ldots,j_q\}(\subseteq \{1,\ldots,n\})$ be the indices set of CCR extreme and non-extreme efficient DMUs. Because, by CCR model at least a DMU is efficient, therefore E_1 is nonempty.

To determine most efficient DMU among DMU $j_1,\ldots,$ DMU j_q , using common set of weights, we consider Model (3) with CCR efficient DMUs. Therefore, we have the following model:

$$
\begin{aligned}
& min\, M \\
& s.t.\ M - d_j \geq 0, \\
& \sum_{i=1}^{m} w_i x_{ij} \leq 1, j \in E_1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0, j \in E_1 \\
& \sum_{j \in E_1}^{n} \theta_j = |E_1| - 1, \\
& \theta_j - d_j \beta_j = 0, j \in E_1 \\
& \beta_j \geq 1, d_j \geq 0, \theta_j \in \{0,1\}, j \in E_1 \\
& w_i \geq \varepsilon^*, i = 1,\ldots,m \\
& u_r \geq \varepsilon^*, r = 1,\ldots,s
\end{aligned} \qquad (9)
$$

where $|E_1|$ is the cardinality of E_1 and ε^* is computed by the following model, which is a modification of Model (4):

$$
\begin{aligned}
& \varepsilon^* = max\, \varepsilon \\
& s.t. \sum_{i=1}^{m} w_i x_{ij} \leq 1, j \in E_1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} \leq 0, j \in E_1 \\
& w_i \geq \varepsilon, i = 1,\ldots,m \\
& u_r \geq \varepsilon, r = 1,\ldots,s.
\end{aligned} \qquad (10)
$$

Let $d_i^* = 0$ using Model (11), where $i_1 \in E_1$. As noted in Amin (2008), DMU i_1 is the most efficient DMU among DMU $j_1,\ldots,$.DMU j_q. Therefore, it has the highest rank, i.e. its rank is 1. Now, we remove DMU i_1 among CCR efficient DMUs and define E_2 as:

$$E_2 = \{j_1,\ldots,j_q\} - \{i_1\} = \{r_1,\ldots,r_{q-1}\}.$$

If we set E_2 instead of E_1 in Models (9) and (10), then the second most efficient DMU is founded. Now, suppose that:

$$E_p = \{j_1,\ldots,j_q\} - \{i_1,\ldots,i_{p-1}\} = \{k_1,\ldots,k_{q-p+1}\}.$$

To determine p^{th} most efficient DMU among DMU $k_1,\ldots,$ DMU k_{q-p+1}, we use Model (3) and consider the following model:

$$
\begin{aligned}
& min\, M \\
& s.t.\ M - d_j \geq 0, \quad j \in E_p \\
& \sum_{i=1}^{m} w_i x_{ij} \leq 1, j \in E_p
\end{aligned}
$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0, j \in E_p$$

$$\sum_{j \in E_p}^{n} \theta_j = |E_p| - 1,$$

$$\theta_j - d_j \beta_j = 0, j \in E_p$$

$$\beta_j \geq 1, d_j \geq 0,\ \theta_j \in \{0,1\},\ j \in E_{p1}$$

$$w_i \geq \varepsilon^*, i = 1,...,m$$

$$u_r \geq \varepsilon^*, r = 1,...,s \quad (11)$$

where DMUi_1,…, DMUi_{p-1} are 1^{st},…,$(p-1)^{th}$ most efficient DMUs using E_1, E_2, …, E_{p-1}, respectively. And, ε^* is computed using the following model:

$$\varepsilon^* = max\, \varepsilon$$

$$s.t. \sum_{i=1}^{m} w_i x_{ij} \leq 1, j \in E_p$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} \leq 0, j \in E_p$$

$$w_i \geq \varepsilon, i = 1,...,m$$

$$u_r \geq \varepsilon, r = 1,...,s. \quad (12)$$

If $d_i^* = 0$ using Model (12), where $i_p \in E_p$, then the rank of DMUi_p is p. Therefore, to rank all of extreme and non-extreme efficient DMUs, we have the following algorithm.

Stage 1: set $p=1$ and solve Model (9), and let DMU$_{i_p}$ be the most efficient DMU,

Stage 2: set $p=p+1$ and $E_{p+1}=E_p-\{i_p\}$, where $p \geq 1$,

Stage 3: if $|E_p| = 1$, then stop, where $|E_p|$ is the cardinality of E_p.

Stage 4: solve Model (11) and go to Stage 2.

Theorem: By Models (9) and (11) rank score of every efficient DMU is unique.

Proof: According to Theorem in Amin (2008) each iteration, say p^{th} iteration, Model (11) finds a single DMU, say DMUi_p, as most efficient DMU among DMUs, DMUk_1,…, DMUk_{q-p+1}. So, the rank of DMUi_p as a single DMU is p.

Therefore, using the above algorithm we can rank all extreme and non-extreme efficient DMUs and every efficient DMU has unique rank score.

4. Application the integrated DEA ranking model to modify GA

For each $x \in S$, using (6) we can construct a DMU with s inputs and one output, where output quantity is 1. Therefore, when p members are selected in the primary stage of the proposed algorithm, we can construct p DUMs with s inputs and one output. Then, we evaluate these DMUs by Model (2) (or Model (8)) to obtain θ_o^* (or Δ_o^*).

If we multiple the inputs of DMU$_o$ by θ_o^*, then it lies on the DEA efficient frontier. In other words we approach to the MOP efficient frontier (see Figures 4.a and 4.b which is corresponding to DEA efficient frontier), while with Yun *et al.* (2001) or Arakawa *et al.* (1998) algorithms we should repeat them about 20 to 30 times, so that the member or the originated DMUs can be placed on the frontier. Therefore, we may use Model (11) as a criterion to select the next population iteration. That is by ranking all DMUs using Model (11), extreme and non-extreme efficient DMUs, which lie on efficient frontier DUMs with better ranks are selected to generate the next population. This method is a shortcut to decrease the iterations of GA to find the Pareto efficient solution of MOP problem. Therefore, in brief we have the following algorithm which is a modification of proposed algorithm by Yun *et al.* (2001) and is called the modified genetic algorithm.

Stage 1: (Initialization). Generate p-individuals randomly. Here, the number of p is given before hand.

Stage 2: (Crossover-Mutation). Make $p/2$- pairs randomly among the population. Making crossover each pair generates a new population. Mutate them according to the given probability of mutation.

Stage 3: (Evaluation of fitness by GDEA). Evaluate the GDEA-efficiency by solving the problem (8) and project all DMUs on DEA efficient frontier.

Stage 4: (Selection). Select p- $p-$ individuals from current population in the third stage, in terms of the obtained ranking scores from Model (11).

The process Step 2-Step 4 is continued until the number of generations attains a given number.

5. Example

Consider the following nonlinear bi-objective problem.

$$min\ (\ f_1(\ x\ ), f_2(\ x\ )) = (\ x_1, x_2\ )$$

$$s.t.\ \ (\ x_1 - 2\ )^2 + (\ x_2 - 2\ )^2 \leq 4$$

$$x_1, x_2 \geq 0.$$

Now we compare the obtained results of the proposed algorithm with Arakawa *et al.* (1998) and Yun *et al.* (2001) algorithms. Figures 5.a and 5.b show the results with 30 repetitions using Arakawa *et al.* (1998) and Yun *et al.* (2001) algorithms, respectively. There exist some deficient solutions, which they lead us to an inaccurate frontier. But in the modified method (see Figure 5.c) the efficient frontier which is appropriate, will exist after a several repetition, adding that 90% of the solutions are efficient. Moreover, if we have more repetition, up to 30 times, the accuracy enhances up to %99. It is remarkably attractive to mention that the members dominated by other members in the repetition are closed to the optimized Pareto's solution.

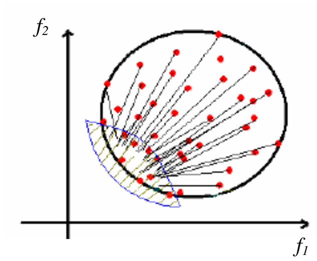

Figure 3.a: Objective functions values space.

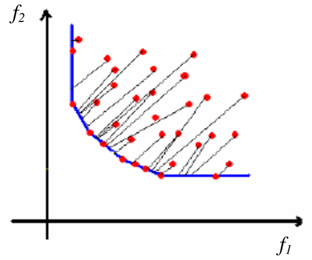

Figure 3.b: Corresponding DEA efficient frontier and projection.

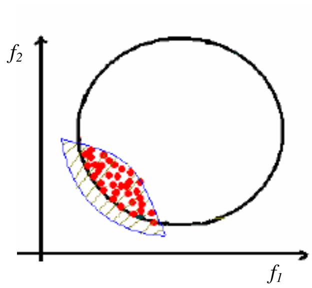

Figure 4.a: Some Pareto optimal solutions.

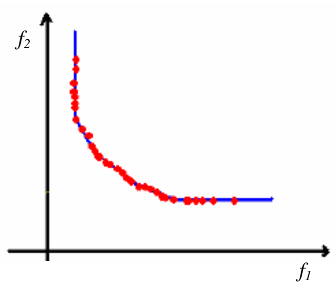

Figure 4.b: All DMUs on the corresponding DEA efficient frontier.

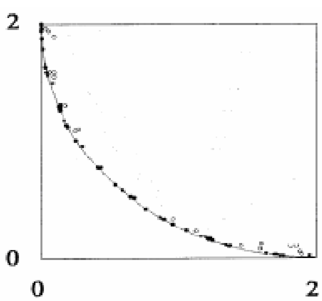

Figure 5.a: Arakawa *et al.*'s (1998) method.

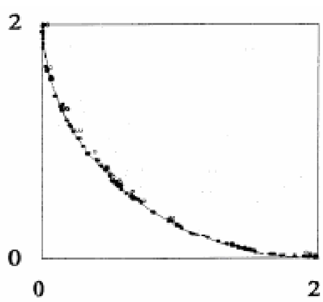

Figure 5.b: Yun *et al.*'s (2001) method.

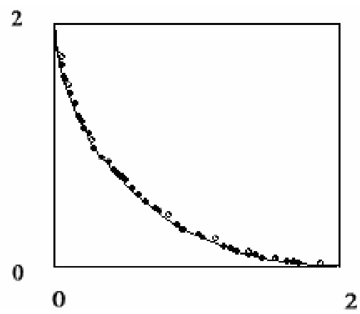

Figure 5.c: Using the proposed method.

6. Conclusion

This paper used the integrated DEA models to rank all of efficient DMUs. Then, the researchers used the proposed integrated DEA ranking method to modify GA for finding the Pareto efficient solutions of MOP problem. The modified GA reduces the computational efforts and can be used to generate convex and non-convex Pareto efficient frontiers of MOP problems, and can also be applied to solve the MOP problems with two or several objective functions. In other words the researchers have used a shortcut in order to decrease the iterations of GA for finding the Pareto efficient solutions of MOP problem.

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