Original Research

Multivariate incapability index for high technology manufacturing processes in presence of the measurement errors: A case study in electronic industry

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Abstract

Process capability indices play a vital role in evaluating the conformity of the process properties to the required specifications. Process incapability indices are created by transformation in the process capability indices, leading to the separation of information related to the process accuracy and precision. This separation of information can be very beneficial to specify whether the process is capable or not and to detect deviations in the production processes that produce high-tech products, such as the electronics industry. The main goal of this study is to propose a process incapability index by considering the measurement error for processes with multivariate quality characteristics. The efficiency of this index is then examined by a numerical example using Monte Carlo simulation method. Moreover, the performance of proposed approach is compared with the case where there is no measurement error. In addition, as a practical example, this index is compared with a number of recently proposed indices in the literature, and sensitivity analysis is conducted, as well. The simulation results showed that the measurement error has a significant effect on process capability and incapability indices. Therefore, we strongly suggest that the measurement error has to be considered in the process analysis.

Keywords: Multivariate process incapability index; Measurement errors; Multivariate normal distribution; High technology manufacturing processes.

1. INTRODUCTION

One of the most important factors to achieve customer's satisfaction is producing high quality product. One aspect of the process quality analysis is the process capability analysis. Process capability indices are applied to analyze the capability of the process to achieve the required specifications. By using these indices, the process capability is reported as a number indicating the degree of conformity of the manufactured products to the determined specifications. Process capability indices are divided into two categories: univariate and multivariate indices. If the quality of the products can be described by one characteristic, univariate process indices, and if the quality of the products can be described by multiple characteristics, multivariate indices are used.

To monitor of processes in which the sensitivity of production is high and also the deviations are very low, the process capability indices are not suitable to use any more as they always show a great degree of capability; therefore, in such cases, applying process incapability indices can be helpful because these low deviations are well recognized by these indices. For example, the electronic products have more strict tolerance and specifications in comparison to other types of products. Accordingly, the process performance will have a greater impact on the product quality management. So, it is necessary to use a high accuracy index.

The structure of this study is as follows: In the next section, the related literature will be reviewed. In Section 3, the multivariate process incapability index is developed by considering the measurement error and its formulations are presented. Section 4 provides a numerical example using Monte Carlo simulation approach. Moreover, a practical example is also given and compared with the process incapability index without considering the measurement error. Finally, in Section 5, the findings are presented and some suggestions for future research are made.

2. Review of the Literature

In the following, we examine the studies that have been done so far on the univariate and multivariate process capability indices and univariate and multivariate incapability indices, as well as the measurement error.

2.1. Univariate process capability indices

Suppose that X is a quality characteristic and follows normal distribution with the mean μ and variance σ^2. Also, LSL and USL are the upper and lower specification limits and T is the target value of the quality characteristic. M is the midpoint of the specification limits. If $T = M = \frac{(LSL+USL)}{2}$, then it is called symmetric tolerance, and if $T \neq M$, it is called asymmetric tolerance.

Vannman [1] introduced a new superstructure of process capability indices according to the Equation (1) for processes with the symmetric tolerance, which includes C_p, C_{pk}, C_{pm}, and C_{pmk} indices.

$$C_p(u,v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad u,v \geq 0 \tag{1}$$

where $d = \frac{(USL-LSL)}{2}$ is half of the specification limit tolerance length and u and v are the weighting factors for the mean deviation from the target value and the process variation. By setting the values of 0 and 1 for the parameters u and v, the indices $C_p(0,0) = C_p$, $C_p(0,1) = C_{pm}$, $C_p(1,0) = C_{pk}$, $C_p(1,1) = C_{pmk}$ are made.

The above indices are not, however, suitable for processes where the specification limits tolerance is asymmetric. Therefore, for such processes, the true specification limits $(T - D_l, T - D_u)$ are placed with symmetric tolerance $(T - d^*, T - d^*)$, where $d^* = \min\{D_l, D_u\}$, $D_l = T - LSL$ and $D_u = USL - T$. A superstructure of capability indices is then introduced according to the Equation (2).

$$C_p^*(u,v) = \frac{d^* - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad u,v \geq 0 \tag{2}$$

By setting the values 0 and 1 for the parameters u and v, the indices $C_p^*(0,0) = C_p^*$, $C_p^*(0,1) = C_{pm}^*$, $C_p^*(1,0) = C_{pk}^*$, and $C_p^*(1,1) = C_{pmk}^*$ are created [2-5]. In this method, in some processes, it leads to underestimation of capability because actual specification limits are replaced by smaller tolerance.

Another approach in this case is to replace actual specification limits with $(T - d', T - d')$, where $d' = \frac{(D_l + D_u)}{2}$ is midpoint of D_l and D_u; so, a new superstructure of capability indices is developed according to the Equation (3).

$$C_p'(u,v) = \frac{d' - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad u,v \geq 0 \tag{3}$$

By setting the values of 0 and 1 for the parameters u and v, the indices $C_p'(0,0) = C_p'$, $C_p'(0,1) = C_{pm}'$, $C_p'(1,0) = C_{pk}'$, $C_p'(1,1) = C_{pmk}'$ are obtained [3, 5]. The class of indices in equation (3) leads to over estimation of capability in some processes and for some other processes, leads to underestimation.

Chen and Pearn [2] stated that when the process standard deviation is fixed, the maximum value of the above indices doesn't occur at $\mu = T$ but between T and M. So presented a series of new indices according to the Equation (4).

$$C_p''(u,v) = \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}} \quad u,v \geq 0 \tag{4}$$

where,

$$F^* = \max\left\{\frac{d^*(T-\mu)}{D_l}, \frac{d^*(\mu - T)}{D_u}\right\} \tag{5}$$

and

$$F = \max\left\{\frac{d(T-\mu)}{D_l}, \frac{d(\mu - T)}{D_u}\right\} \tag{6}$$

From the Equation (4), the indices $C_p''(0,0) = C_p''$ ، $C_p''(0,1) = C_{pm}''$ ،$C_p''(1,0) = C_{pk}''$ and $C_p''(1,1) = C_{pmk}''$ are obtained. Abbasi Ganji and Sadeghpour Gildeh [6] declared that a major problem in the class of $C_p''(u,v)$

indices is that the process deviation from the target value is evaluated without considering the direction of the deviation. To solve this problem, they proposed a class of process capability indices $C_p'''(u,v)$ as follows:

$$C_p'''(u,v) = \frac{d^* - uA^*}{3\sqrt{\sigma^2 + vA^2}} \qquad u,v \geq 0 \tag{7}$$

where,

$$A^* = \frac{(\mu - T)^2}{D_u} I\{\mu > T\} + \frac{(T-\mu)^2}{D_l} I\{T \leq \mu\} \tag{8}$$

and

$$A^2 = \frac{d^2(\mu - T)^2}{D_u^2} I\{\mu > T\} + \frac{d^2(T-\mu)^2}{D_l^2} I\{T \leq \mu\} \tag{9}$$

In the Equations (8) and (9), $I\{.\}$ is an indicator function which is defined as follows:

$$I\{x\} = \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases} \tag{10}$$

Based on the above indicator, process capability decreases faster as the mean shifts from the target value to the closer specification limit than the case that the mean deviates from the target value to the farther specification limit.

2.2. Multivariate process capability indices

Let us assume that $\boldsymbol{X}$ is defined as a $(p \times 1)$ vector of p quality characteristics which follows multivariate normal distribution with $(p \times 1)$ mean vector $\boldsymbol{\mu}$ and $(p \times p)$ variance-covariance matrix $\boldsymbol{\Sigma}$. Since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown in most processes, a sample of size n is represented as a $p \times n$ matrix for estimation of process parameters. $\bar{\boldsymbol{X}}$ is a $(p \times 1)$ vector that contains the sample mean of each quality characteristic, and $\mathbf{S}$ is a $p \times p$ variance-covariance matrix of the observed sample. For a better view please see Equations (11) to (16).

$$\boldsymbol{\mu} = \left(\mu_1, \mu_2, \ldots, \mu_p\right)' \tag{11}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{pmatrix} \tag{12}$$

$$\bar{\boldsymbol{X}} = \left(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_p\right)' \tag{13}$$

$$\mathbf{S} = \begin{pmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & \ddots & \vdots \\ s_{p1} & \cdots & s_{pp} \end{pmatrix} \tag{14}$$

where,

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{15}$$

and

$$S_{ij} = \frac{\sum_{k=1}^{n}\left((x_{ik} - \bar{x}_i)(x_{jk} - \bar{x}_j)\right)}{n-1} \qquad i,j = 1,2,\ldots,p \tag{16}$$

In the Equation (14), s_{ij} is the sample covariance between the quality characteristic i and j (for $i,j = 1,2,\ldots,p$).

Taam et al. [7] presented a multivariate capability index as a ratio of two volumes as follows:

$$MC_p = \frac{vol.(R_1)}{vol.(R_2)} \tag{17}$$

where R_1 is modified tolerance region and the largest ellipsoid centering at the target vector that is completely within the actual tolerance region, and R_2 is the elliptical process region, covering 99.73% of the multivariate normal process. They considered the possible shift of process mean from the target vector and taking into account an adjustment factor that measures the closeness between the target vector and the process mean. Then, they rewrote the MC_p index as:

$$MC_{pm} = \frac{MC_p}{\boldsymbol{D}} \tag{18}$$

Based on a random sample collected, the MC_{pm} index estimator is obtained by Equation (19).

$$\widehat{MC}_{pm} = \frac{\widehat{MC}_p}{\widehat{\boldsymbol{D}}} \tag{19}$$

where,

$$\widehat{MC}_p = \frac{vol.(modified\ tolerance\ region)}{vol.(estimated\ 99.73\%\ process\ region)} = \frac{vol.(modified\ tolerance\ region)}{|\mathbf{S}|^{\frac{1}{2}}\left(\pi\chi^2_{p,0.9973}\right)^{\frac{p}{2}}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}} \tag{20}$$

and

$$\widehat{\boldsymbol{D}} = \left[1 + \frac{n}{n+1}(\bar{\boldsymbol{X}} - \boldsymbol{T})'\mathbf{S}^{-1}(\bar{\boldsymbol{X}} - \boldsymbol{T})\right]^{\frac{1}{2}} \tag{21}$$

In the Equation (20), $\chi^2_{p,0.9973}$ is 99.73 percentile of the chi-square distribution with p degrees of freedom, $|\mathbf{S}|$ is the determinant of sample variance-covariance matrix and $r_i(u)$ is the radius of ellipsoid and $\Gamma\left(\frac{p}{2}+1\right)$ is gamma function.

Shahriari et al. [8] presented a multivariate process capability vector that had been developed based on Hubble et al. [9] study. Their capability vector consists of three components, $MPCV = [C_{pM}, PV, LI]$. The first component of this vector is defined as follows:

$$C_{pm} = \left[\frac{\prod_{i=1}^{p}(USL_i - LSL_i)}{\prod_{i=1}^{p}(UPL_i - LPL_i)}\right]^{\frac{1}{2}} \tag{22}$$

where,

$$UPL_i = \mu_i + \sqrt{\frac{\chi^2_{p,\alpha}|\mathbf{\Sigma}_i^{-1}|}{|\mathbf{\Sigma}^{-1}|}} \tag{23}$$

and

$$LPL_i = \mu_i - \sqrt{\frac{\chi^2_{p,\alpha}|\mathbf{\Sigma}_i^{-1}|}{|\mathbf{\Sigma}^{-1}|}} \tag{24}$$

In the Equations (23) and (24), $\chi^2_{p,\alpha}$ is the upper α^{th} percentile of the chi-square distribution with p degrees of freedom and $|\mathbf{\Sigma}_i^{-1}|$ is the determinant of the matrix $\mathbf{\Sigma}_i^{-1}$ and $\mathbf{\Sigma}_i$ is the matrix obtained by deleting the i^{th} row and column of the matrix $\mathbf{\Sigma}$.

The second component of the above vector is the substantial level of the observed value with the Hotelling's T^2 statistic and is defined as:

$$PV = P\left(F_{p,n-p} > \frac{n-p}{p(n-1)}t^2\right) \tag{25}$$

where,

$$t^2 = n(\overline{\boldsymbol{X}} - \boldsymbol{T})'\mathbf{S}^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{T}) \tag{26}$$

where $F_{p,n-p}$ is a random variable that follows Fisher distribution with p and n-p degrees of freedom. The PV value never exceeds 1, and when the PV value is close to zero, it indicates that the process mean is away from the target vector [10]. Finally, the third component of the above vector is defined as Equation 27:

$$LI = \begin{cases} 1 & \text{entire modified process region is contained within the tolerance region} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

Due to the $MPCV$, the process is capable when the value of the first component is greater than 1, the value of the second component is greater than or equal to the substantial level, which is usually considered as 0.05, and the third component is equal to 1.

Shahriari and Abdollahzadeh [11] modified the first component of the above vector and developed a multivariate vector as $NMPCV = [NMC_{pM}, PV, LI]$. They claimed that the best way to introduce the modified tolerance region is to apply the largest ellipsoid, the center of which is the target vector and its axes are parallel to the axes of the process ellipsoid and should be completely within the actual tolerance region. Also, the direction of the process ellipsoid axes depends completely on the variance-covariance structure.

Abbasi Ganji and Sadeghpour Gildeh [12] expressed that when the tolerance region is asymmetric, the main problem with the introduced multivariate indices is that the mean vector deviation from the target vector is evaluated without considering the direction of the deviation. To solve this problem they defined a multivariate capability vector with two components as $MPCVG = [MC_{PG}, PV]$. The first component of this vector is defined as follows:

$$MC_{PG} = \frac{C}{\sqrt{\chi^2_{p,0.9973}}} \tag{28}$$

where,

$$C = \min\left\{\frac{(USL_i - \mu_i)}{\sigma_i}, \frac{(\mu_i - LSL_i)}{\sigma_i}\right\}; i = 1, 2, \dots, p \tag{29}$$

The second component of the above vector is similar to the Equations (25) and (26). According to this vector, a process is capable when the MC_{PG} index is at least equal to 1 and PV is at least equal to α (which is usually considered as 0.05).

2.3. Univariate process incapability indices

Greenwich and Jahr-Schaffrath [13] created a new index by transformation on C^*_{pm} index provided by Chan et al [14]. They called it the process incapability index, C_{pp} which is introduced as follows:

$$C_{pp} = \left(\frac{1}{C^*_{pm}}\right)^2 = \left(\frac{\mu - T}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 \tag{30}$$

where $D = \frac{d^*}{3}$. The C_{pp} can be written as $C_{pp} = C_{ia} + C_{ip}$, which is the sum of the inaccuracy index $C_{ia} = \left(\frac{\mu - T}{D}\right)^2$ and the imprecision index $C_{ip} = \left(\frac{\sigma}{D}\right)^2$. In the index C_{ia}, subscript *ia* refers to inaccuracy and in the index C_{ip}, subscript *ip* refers to imprecision.

The inaccuracy index indicates the relative process departure (the process mean shift from the target value) and the imprecision index measures the process variation relative to the specification limits. For a process that has more capability to meeting the required specifications, C_{pp} has a smaller value, while for a process with less capability, it has a larger value. The process is in the most capable state when the variance is very small, next to zero (not zero), and the index $C_{pp}$is next to zero (not zero). For this purpose, the process mean should be equal to the target value ($\mu = T$) and the process variance should be next to zero (not zero).

We believe that C_{pp} is superior than C^*_{pm}. Since the above conversion is a bijective transformation, the C_{pp} index contains the same information as C^*_{pm} does, i.e., inaccuracy and imprecision. In addition, the C_{pp} index provides uncontaminated separation (without integration) of process accuracy and precision; while this type of information separation is not achieved with the C^*_{pm} Index. C^*_{pm} is often used to evaluate process capability. As the process accuracy is very important, this information separation is very useful as it emphasizes how much process inaccuracy is effective and the process being incapable of meeting the specifications. Unlike other process capability indices, process capability decreases with growing the C_{pp} value; for this reason, this index is named a process incapability index [13].

Chen [15] said that for processes with asymmetric tolerance, C_{pp} may not be able to accurately calculate process performance precisely because it measures the distance between the target value and the process mean without considering the mean position. To address this issue, he generalized the C_{pp} incapability index provided by Greenwich and Jahr-Schaffrath [13] and presented the C''_{pp} index as follows:

$$C''_{pp} = \left(\frac{A}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 \tag{31}$$

where,

$$A = max\left\{\frac{(\mu - T)d}{D_u}, \frac{(T-\mu)d}{D_l}\right\} \tag{32}$$

where $d = \frac{(USL-LSL)}{2}$. The C''_{pp} Index can be rewritten as $C''_{pp} = C''_{ia} + C_{ip}$. C''_{pp} includes asymmetric tolerance of the specification limits D_u and D_l. It makes the process capability index more accurate than C_{pp}.

Abbasi Ganji [16], using a transformation in the process capability index $C'''_{pp}(u,v)$ that had been provided by Abbasi Ganji and Sadeghpour Gildeh [6], proposed a new process incapability index, $IC'''_{pp}(u,v)$ as follows:

$$IC'''_{pp}(u,v) = \left(\frac{1}{C'''_{pp}(u,v)}\right)^2 = \left(\frac{6\sigma}{2(d^* - uA^*)}\right)^2 + \frac{9vA^2}{(d^*-uA^*)^2} \qquad u,v \geq 0 \tag{33}$$

where,

$$IC'''_{pp}(u,v) = C'''_{ia}(u,v) + C'''_{ip}(u) \qquad u,v \geq 0 \tag{34}$$

where inaccuracy index is defined as $C'''_{ia}(u,v) = \frac{9vA^2}{(d^*-uA^*)^2}$ and imprecision index is defined as $C'''_{ip}(u) = \left(\frac{6\sigma}{2(d^*-uA^*)}\right)^2$. If the value of $IC'''_{pp}(u,v)$ exceeds 1, it is assumed that the process is "*incapable*".

2.4. Multivariate process incapability indices

As mentioned earlier, many studies have been done on the multivariate process capability indices. However, a few studies have been conducted on multivariate process incapability indices. Abbasi Ganji [16] first introduced a multivariate process incapability vector including of two components for processes in which the quality characteristic follows the multivariate normal distribution. This index is calculated due to the ratio of the volume of a tolerance region to the volume of the process region. According to her approach, the multivariate process incapability vector is defined as follows:

$$MPICV = [MIC'''_{pp}(u,v), LI] \tag{35}$$

where, the first component of the above vector is formulated according to the Equation (36).

$$MIC'''_{pp}(u,v) = MC'''_{ia}(u,v) + MC'''_{ip}(u) \qquad u,v \geq 0 \tag{36}$$

where $MIC'''_{pp}(u,v)$, $MC'''_{ia}(u,v)$ and $MC'''_{ip}(u)$ are the multivariate modes of $IC'''_{pp}(u,v)$, $C'''_{ia}(u,v)$ and $C'''_{ip}(u)$, respectively. In the multivariate mode, instead of the expression $2(d^* - uA^*)$, as used in Equation (33), the ellipsoid and instead of 6σ, the region that covers 99.73% of the multivariate normal process is used. Finally, she presented the multivariate imprecision index as:

$$MC'''_{ip}(u) = \left(\frac{vol.(99.73\%\ of\ process\ region)}{vol.(ellipsoid\ with\ radius\ r_i)}\right)^2 = \left[\frac{|\boldsymbol{\Sigma}|^{\frac{1}{2}}(\pi\chi^2_{0.0027,v})^{\frac{p}{2}}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right]^2 \qquad u \geq 0 \tag{37}$$

where $v = p$ and p denotes the number of quality characteristics. The imprecision index for processes that have two quality characteristics is as follows:

$$MC''''_{ip}(u) = \frac{|\boldsymbol{\Sigma}| \times 11.829^2}{\left(r_1(u)r_2(u)\right)^2} \qquad u \geq 0 \tag{38}$$

In addition, the multivariate inaccuracy index is in accordance with the Equation (39).

$$MC''''_{ia}(u,v) = \frac{9v\boldsymbol{A}'\boldsymbol{A}}{\boldsymbol{r}'(u)\boldsymbol{r}(u)} \qquad u,v \geq 0 \tag{39}$$

In the above Equations, the following equations hold:

$$d_i^* = min\{D_{l_i}, D_{u_i}\}, D_{l_i} = T_i - LSL_i, D_{u_i} = USL_i - T_i \tag{40}$$

$$A_i = \frac{d_i(\mu_i - T_i)}{D_{u_i}} I\{\mu_i > T_i\} + \frac{d_i(T_i - \mu_i)}{D_{l_i}} I\{\mu_i \leq T_i\} \tag{41}$$

$$A_i^* = \frac{(\mu_i - T_i)^2}{D_{u_i}} I\{\mu_i > T_i\} + \frac{(T_i - \mu_i)^2}{D_{l_i}} I\{\mu_i \leq T_i\} \tag{42}$$

$$r_i(u) = |d_i^* - uA_i^*| \tag{43}$$

The second component of the vector presented in the Equation (35) is defined as Equation 44:

$$LI = \begin{cases} 1; & \textit{The process region is completely within the tolerance region} \\ 0; & \textit{Otherwise} \end{cases} \tag{44}$$

If $MIC''''_{pp}(u,v) \geq 1$ or LI = 0, the process is called "*incapable*". If $0.56 \leq MIC''''_{pp}(u,v) < 1$ and LI = 1, the process is called "*capable*".

If $0.44 \leq MIC''''_{pp}(u,v) < 0.56$ and LI = 1, the process is called "*satisfactory*". If $0.25 \leq MIC''''_{pp}(u,v) < 0.44$ and LI = 1, the process is called "*excellent*" and if $MIC''''_{pp}(u,v) < 0.25$ and LI = 1, the process is called "*super*" [16].

2.5. Measurement error

Measurement error is defined as the difference between the measured value and the original/actual value. It is almost impossible to accurately measure without error in any production and service environment. In other words, the error caused by the measuring instrument or the human is unavoidable. Due to the existence of errors, even if measured with the most advanced and accurate tools, the error value never reaches zero. Variations in the production process and measuring instruments can lead to incorrect measurements. In practical environments, the measured value of quality characteristic is often affected by measurement error. Accordingly, a proper process capability index is required to evaluate process performance under such conditions.

2.5.1. The process incapability indices considering the measurement error

There are many studies in the literature that have examined the impact of measurement errors on various aspects of statistical process monitoring. Maleki et al. [17] presented a review study regarding the impact of measurement errors on various aspects of statistical process monitoring, which includes many articles in this field. They reviewed 60 papers from 1954 to 2016, 19 of which were on process capability indices.

Sadeghpour Gildeh and Abbasi Ganji [18], for the first time, presented the process incapability index by taking into account measurement errors and examined its statistical properties and obtained its Maximum Likelihood Estimation (MLE). They assumed that X is a quality characteristic and follows normal distribution, $N(\mu, \sigma^2)$, and E is measurement error that is independently normally distributed with mean of zero and variance σ_e^2, i.e., $N(0, \sigma_e^2)$. For taking measurement errors into account, they consider the variable $G = X + E$, which is distributed as $N(\mu, \sigma_G^2)$, where $\sigma_G^2 = \sigma^2 + \sigma_e^2$. Eventually, they proposed the C''^{G}_{pp} index according to Equation (45) as a measure of the process incapability with measurement error.

$$C''^{G}_{pp} = \left(\frac{A}{D}\right)^2 + \frac{\sigma^2 + \sigma_e^2}{D^2} = \left(\frac{A}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2 + \left(\frac{\sigma_e}{D}\right)^2 = C''_{pp} + \left(\frac{\sigma_e}{D}\right)^2 \tag{45}$$

Table (1) summarizes the most important studies that have been done on the process capability index. According to the Table (1), the innovation of the present study is the combination of the following assumptions:

• The process quality characteristics follows multivariate normal distribution.

• **Multivariate process incapability Index:** In the practical environments, the quality characteristic often depends on more than one variable; so, considering this assumption leads to more practical results.

• **Measurement error:** As mentioned earlier, all processes have some errors and it is impossible to reduce the error to zero. Therefore, considering the measurement error leads to higher accuracy in calculating the performance indices and future decision making.

TABLE 1. A REVIEW OF STUDIES CONDUCTED ON THE INCAPABILITY INDEX

Author/s	Year	Assumptions				
		Symmetric tolerance	Asymmetric tolerance	Univariate	Multivariate	Measurement error
Greenwich and Jahr-Schaffrath [13]	1995	✓		✓		
Chen [15]	1998		✓	✓		
Sadeghpour Gildeh and Abbasi Ganji [18]	2019		✓	✓		✓
Abbasi Ganji [16]	2019		✓	✓	✓	
Current research		✓	✓		✓	✓

3. METHODOLOGY

Let us suppose that the quality characteristic $\boldsymbol{X}$ is defined as a $p \times 1$ vector which follows multivariate normal distribution, $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\boldsymbol{\mu}$ and the variance-covariance matrix $\boldsymbol{\Sigma}$. The upper and lower specification limits for each quality characteristic X_i, $i = 1, 2, \dots, p$ are denotes by USL_i and LSL_i, respectively. Target vector $\boldsymbol{T}$ consists of p target values where each of its components is between the specification limits. A sample with the size n of the process can be represented as a $p \times n$ matrix. $\bar{\boldsymbol{X}}$ is a $p \times 1$ vector that includes the sample means of p characteristics and $\mathbf{S}$ is $p \times p$ sample variance-covariance matrix. The measurement error ($\boldsymbol{E}$) which defines as a $p \times 1$ vector follows multivariate normal distribution, $N_p(\boldsymbol{\mu}_E, \boldsymbol{\Sigma}_E)$, with zero mean vector ($\boldsymbol{\mu}_E = \mathbf{0}$) and the variance-covariance matrix $\boldsymbol{\Sigma}_E$ and E is independent of X.By considering the measurement error, the quality characteristic in the multivariate mode is defined as $\boldsymbol{G} = \boldsymbol{X} + \boldsymbol{E}$, which has a distribution of $N_p(\boldsymbol{\mu}_G, \boldsymbol{\Sigma}_G)$, where $\boldsymbol{\mu}_G = \boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_G = \boldsymbol{\Sigma} + \boldsymbol{\Sigma}_E$.

The index $MIC'''_{pp}(u, v)$ provided by Abbasi Ganji [16] will be $MIC'''^{G}_{pp}(u, v)$ in the presence of the measurement error and we call it "*error-affected multivariate incapability index*" where u and v are the weighting factors for the mean vector deviation from the target vector and the process variation. To calculate this index, we use the division of the tolerance region into the process region. Because the index $MIC'''_{pp}(u, v)$ consists of two indices of inaccuracy and imprecision, the measurement error effect must be considered on both indices. Therefore, the error-affected multivariate imprecision index can be calculated as Equation 46:

$$\begin{aligned} MC'''^{G}_{ip}(u) &= \left(\frac{vol.(99.73\%\ of\ process\ and\ measurement\ error\ region)}{vol.(ellipsoid\ with\ radius\ r_i)}\right)^2 \\ &= \left(\frac{|\boldsymbol{\Sigma}_G|^{\frac{1}{2}} \left(\pi\chi^2_{0.0027,v}\right)^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2 \\ &= \left(\frac{|\boldsymbol{\Sigma} + \boldsymbol{\Sigma}_E|^{\frac{1}{2}} \left(\pi\chi^2_{0.0027,v}\right)^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2 ; \quad u \geq 0 \end{aligned} \tag{46}$$

where subscript ip means imprecision, and $\chi^2_{0.0027,v}$ is the upper 0.0027 percentile of the chi-square distribution with v degrees of freedom where $v = p$ and p denotes the number of quality characteristic. Also, $r_i(u)$ is radius of ellipsoid and $\Gamma\left(\frac{p}{2}+1\right)$ is gamma function. In the following, the gauge capability in the multivariate mode is calculated by

$$\lambda^M = \frac{vol.(99.73\%\ of\ process\ region)}{vol.(ellipsoid\ with\ radius\ r_i)} = \frac{|\boldsymbol{\Sigma}_E|^{\frac{1}{2}} (\pi\chi^2_{0.0027,v})^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)} \tag{47}$$

where superscript M in λ^M denotes the multivariate mode of gauge capability. The determinant of the variance-covariance matrix **E** is obtained according to the Equation (48).

$$|\boldsymbol{\Sigma}_E|^{\frac{1}{2}} = \frac{\lambda^M(\pi r_1(u) r_2(u) \dots r_p(u))}{(\pi\chi^2_{0.0027,v})^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}} \tag{48}$$

If **A** and **B** are two-dimensional matrices, the following Equation holds [49]:

$$\begin{aligned}|\mathbf{A}+\mathbf{B}| &= \left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right| = \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22}\end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{vmatrix}\end{aligned} \tag{49}$$

According to the above equation, the value of $|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E|$ can be obtained as follows:

$$\begin{aligned}|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E| &= \left|\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{bmatrix} + \begin{bmatrix} \sigma^E_{11} & \sigma^E_{12} \\ \sigma^E_{21} & \sigma^E_{22}\end{bmatrix}\right| = \begin{vmatrix} \sigma_{11}+\sigma^E_{11} & \sigma_{12}+\sigma^E_{12} \\ \sigma_{21}+\sigma^E_{21} & \sigma_{22}+\sigma^E_{22}\end{vmatrix} \\ &= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21}+\sigma^E_{21} & \sigma_{22}+\sigma^E_{22}\end{vmatrix} + \begin{vmatrix} \sigma^E_{11} & \sigma^E_{12} \\ \sigma_{21}+\sigma^E_{21} & \sigma_{22}+\sigma^E_{22}\end{vmatrix} \\ &= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma^E_{21} & \sigma^E_{22}\end{vmatrix} + \begin{vmatrix} \sigma^E_{11} & \sigma^E_{12} \\ \sigma_{21} & \sigma_{22}\end{vmatrix} + \begin{vmatrix} \sigma^E_{11} & \sigma^E_{12} \\ \sigma^E_{21} & \sigma^E_{22}\end{vmatrix}\end{aligned} \tag{50}$$

where $\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma^E_{21} & \sigma^E_{22}\end{vmatrix} + \begin{vmatrix} \sigma^E_{11} & \sigma^E_{12} \\ \sigma_{21} & \sigma_{22}\end{vmatrix}$ is assumed as the constant value of C; so, the value of $|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E|$ can be rewritten as Equation 51.

$$|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E| = |\boldsymbol{\Sigma}| + |\boldsymbol{\Sigma}_E| + C \tag{51}$$

Therefore the $MC'''^{G}_{ip}(u)$ can be rewritten as follows:

$$\begin{aligned}MC'''^{G}_{ip}(u) &= \left(\frac{|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E|^{\frac{1}{2}} (\pi\chi^2_{0.0027,v})^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2 \\ &= \left(\frac{(|\boldsymbol{\Sigma}| + |\boldsymbol{\Sigma}_E| + C)^{\frac{1}{2}} (\pi\chi^2_{0.0027,v})^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2 \\ &= \left(\frac{\left(\frac{|\boldsymbol{\Sigma}|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\right)(|\boldsymbol{\Sigma}| + |\boldsymbol{\Sigma}_E| + C)^{\frac{1}{2}} (\pi\chi^2_{0.0027,v})^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2 \\ &= MC'''_{ip}(u) \times \left(\left(\frac{|\boldsymbol{\Sigma}| + |\boldsymbol{\Sigma}_E| + C}{|\boldsymbol{\Sigma}|}\right)^{\frac{1}{2}}\right)^2 = MC'''_{ip}(u) \times \left(1 + \frac{(\lambda^M)^2}{MC'''_{ip}(u)} + \frac{C}{|\boldsymbol{\Sigma}_G - \boldsymbol{\Sigma}_E|}\right)\end{aligned} \tag{52}$$

Given that:

$$|\boldsymbol{\Sigma}+\boldsymbol{\Sigma}_E| = |\boldsymbol{\Sigma}_G| = |\boldsymbol{\Sigma}| + |\boldsymbol{\Sigma}_E| + C \tag{53}$$

the value of C is obtained by Equation (54):

$$C = |\mathbf{\Sigma} + \mathbf{\Sigma}_E| - (|\mathbf{\Sigma}| + |\mathbf{\Sigma}_E|) = |\mathbf{\Sigma}_G| - (|\mathbf{\Sigma}| + |\mathbf{\Sigma}_E|) \quad (54)$$

Finally, the $MC'''^{G}_{ip}(u)$ index can be obtained as follows:

$$\begin{aligned} MC'''^{G}_{ip}(u) &= MC'''_{ip}(u) \times \left(1 + \frac{(\lambda^M)^2}{MC'''_{ip}(u)} + \frac{C}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) = MC'''_{ip}(u) \times \left(\frac{(\lambda^M)^2}{MC'''_{ip}(u)} + \frac{C + (|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|)}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) \\ &= MC'''_{ip}(u) \times \left(\frac{(\lambda^M)^2}{MC'''_{ip}(u)} + \frac{|\mathbf{\Sigma}_G| - |\mathbf{\Sigma}| - |\mathbf{\Sigma}_E| + (|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|)}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) \\ &= MC'''_{ip}(u) \times \left(\frac{(\lambda^M)^2}{MC'''_{ip}(u)} + \frac{|\mathbf{\Sigma}_G| - |\mathbf{\Sigma}_E|}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) \end{aligned} \quad (55)$$

The $MC'''^{G}_{ip}(u)$ index can be rewritten such as:

$$MC'''^{G}_{ip}(u) = (\lambda^M)^2 + \left(MC'''_{ip}(u) \times \frac{|\mathbf{\Sigma}_G| - |\mathbf{\Sigma}_E|}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) \qquad u \geq 0 \quad (56)$$

Note that, as there is no standard deviation σ in the inaccuracy index formulation $C'''_{ia}(u,v) = \frac{9vA^2}{(d^* - uA^*)^2}$, the measurement error has no effect in this part of the incapability index formula. So, the multivariate inaccuracy index $MC'''_{ia}(u,v)$ is calculated according to Equation (57).

$$MC'''_{ia}(u,v) = \frac{9v\boldsymbol{A}'\boldsymbol{A}}{\boldsymbol{r}'(u)\boldsymbol{r}(u)}; \qquad u,v \geq 0 \quad (57)$$

where *ia* denotes inaccuracy and $\boldsymbol{A}'$ denotes the transpose of vector $\boldsymbol{A}$. Note that, in the above equations, the following equations hold.

$$d_i^* = \min\{D_{l_i}, D_{u_i}\}, D_{l_i} = T_i - LSL_i, D_{u_i} = USL_i - T_i \quad (58)$$

$$d_i = \frac{USL_i - LSL_i}{2} \quad (59)$$

$$A_i = \frac{d_i(\mu_i - T_i)}{D_{u_i}} I\{\mu_i > T_i\} + \frac{d_i(T_i - \mu_i)}{D_{l_i}} I\{\mu_i \leq T_i\} \quad (60)$$

$$A_i^* = \frac{(\mu_i - T_i)^2}{D_{u_i}} I\{\mu_i > T_i\} + \frac{(T_i - \mu_i)^2}{D_{l_i}} I\{\mu_i \leq T_i\} \quad (61)$$

$$r_i(u) = |d_i^* - uA_i^*| \quad (62)$$

$$\boldsymbol{d}^* = \begin{pmatrix} d_1^* \\ d_2^* \\ \vdots \\ d_p^* \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix}, \quad \boldsymbol{A}^* = \begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_p^* \end{pmatrix}, \quad \boldsymbol{r}(u) = \begin{pmatrix} r_1(u) \\ r_2(u) \\ \vdots \\ r_p(u) \end{pmatrix} \quad (63)$$

In the Equations (60) and (61), $I\{.\}$ is an indicator function which is defined as follows:

$$I\{x\} = \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases} \quad (64)$$

Finally, the error-affected multivariate incapability index $MIC'''^{G}_{pp}(u,v)$ can be summarized and rewritten as Equation 65.

$$MIC'''^{G}_{pp}(u,v) = MC'''_{ia}(u,v) + MC'''^{G}_{ip}(u) = \frac{9v\boldsymbol{A}'\boldsymbol{A}}{\boldsymbol{r}'(u)\boldsymbol{r}(u)} + (\lambda^M)^2 + \left(MC'''_{ip}(u) \times \frac{|\mathbf{\Sigma}_G| - |\mathbf{\Sigma}_E|}{|\mathbf{\Sigma}_G - \mathbf{\Sigma}_E|}\right) \qquad u,v \geq 0 \quad (65)$$

3.1. Estimation of the $MIC'''^{G}_{pp}(u,v)$

Assume that $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ are n random samples of a multivariate quality characteristic that follows multivariate normal distribution with the mean vector $\boldsymbol{\mu}$ and the variance-covariance matrix $\mathbf{\Sigma}$. $\overline{\boldsymbol{X}}$ is a $p \times 1$ vector that includes the sample means and $\mathbf{S}$ is a $p \times p$ matrix which includes sample variance and covariance. $\overline{\boldsymbol{X}}_E$ is a $p \times 1$ vector that contains the sample means of errors, and $\mathbf{S}_E$ is a $p \times p$ sample variance-covariance matrix of errors. To estimate the index $MIC'''^{G}_{pp}(u,v)$, we use the estimators of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$, according to the Equations (15) and (16). Therefore, we have

$$\widehat{MC}_{ip}^{\prime\prime\prime G}(u) = \left(\frac{vol.(estimated\ 99.73\%\ of\ affected - error\ process\ region\ )}{vol.(ellipsoid\ with\ radius\ r_i)}\right)^2$$
$$= \left(\frac{|\mathbf{S}_G|^{\frac{1}{2}}\left(\pi\chi^2_{0.0027,v}\right)^{\frac{p}{2}}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2$$
$$= \left(\frac{|\mathbf{S}+\mathbf{S}_E|^{\frac{1}{2}}\left(\pi\chi^2_{0.0027,v}\right)^{\frac{p}{2}}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi r_1(u) r_2(u) \dots r_p(u)}\right)^2; \quad u \ge 0 \tag{66}$$

$$\widehat{MC}_{ip}^{\prime\prime\prime G}(u) = (\lambda^M)^2 + \left(\widehat{MC}_{ip}^{\prime\prime\prime}(u) \times \frac{|\mathbf{S}_G| - |\mathbf{S}_E|}{|\mathbf{S}_G - \mathbf{S}_E|}\right); \quad u \ge 0 \tag{67}$$

$$\widehat{MC}_{ia}^{\prime\prime\prime}(u,v) = \frac{9v\widehat{\boldsymbol{A}}'\widehat{\boldsymbol{A}}}{\widehat{\boldsymbol{r}}'(u)\,\widehat{\boldsymbol{r}}(u)}; \quad u,v \ge 0 \tag{68}$$

Finally,

$$\widehat{MIC}_{pp}^{\prime\prime\prime G}(u) = \widehat{MC}_{ia}^{\prime\prime\prime}(u,v) + \widehat{MC}_{ip}^{\prime\prime\prime G}(u) = \frac{9v\widehat{\boldsymbol{A}}'\widehat{\boldsymbol{A}}}{\widehat{\boldsymbol{r}}'(u)\,\widehat{\boldsymbol{r}}(u)} + (\lambda^M)^2 + \left(\widehat{MC}_{ip}^{\prime\prime\prime}(u) \times \frac{|\mathbf{S}_G| - |\mathbf{S}_E|}{|\mathbf{S}_G - \mathbf{S}_E|}\right); \ u,v \ge 0 \tag{69}$$

In the above Equations, the following equations hold:

$$\hat{A}_i = \frac{d_i(\bar{X}_i - T_i)}{D_{u_i}} I\{\bar{X}_i > T_i\} + \frac{d_i(T_i - \bar{X}_i)}{D_{l_i}} I\{\bar{X}_i \le T_i\} \tag{70}$$

$$\hat{A}_i^* = \frac{(\bar{X}_i - T_i)^2}{D_{u_i}} I\{\bar{X}_i > T_i\} + \frac{(T_i - \bar{X}_i)^2}{D_{l_i}} I\{\bar{X}_i \le T_i\} \tag{71}$$

$$\hat{r}_i(u) = \left|d_i^* - u\hat{A}_i^*\right| \tag{72}$$

$$\widehat{MC}_{ip}^{\prime\prime\prime}(u) = \left[\frac{|\mathbf{S}|^{\frac{1}{2}}\left(\pi\chi^2_{0.0027,v}\right)^{\frac{p}{2}}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}}{\pi \hat{r}_1(u)\hat{r}_2(u) \dots \hat{r}_p(u)}\right]^2 \quad u \ge 0 \tag{73}$$

According to the relation $\boldsymbol{G} = \boldsymbol{X} + \boldsymbol{E}$, the smaller E, the closer $\boldsymbol{X}$ and $\boldsymbol{G}$ get to each other, and it becomes more difficult to distinguish capability in sensitive and high-tech industries where a small percentage of error is catastrophic. If there is a measurement error, $\boldsymbol{G}$ is inadvertently inserted into the mean vector in Equation (70) because the collected data is contaminated with error and this leads to the superiority of the $\widehat{MIC}_{pp}^{\prime\prime\prime G}(u)$index and the innovation of this article in Giving this incapability is transparent and separate information.

4. Analysis

Here, in order to examine the performance of the proposed multivariate incapability index, a numerical example (Case (I)) will be presented and compared with the case where the measurement error is not considered. We applied Monte Carlo simulation approach for computing the indices. A practical example (Case (II)) is also provided and compared with other classical indices.

4.1. Numerical instance (Case (I))

Here, a numerical instance which has been adopted from Ouyang [20] and is on the production process of H-type chip resistor is presented. The H-type chip resistor has five quality characteristics including length (a), upper width (b), width (c), height (d), and lower width (e) (see Figure (1)).

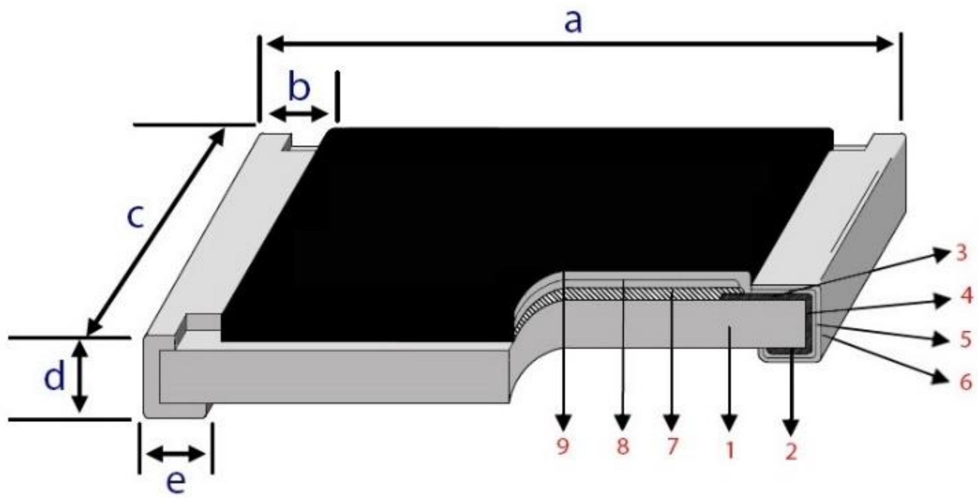

FIGURE 1. H-TYPE CHIP RESISTOR

The constituent parts of this resistor, according to the numbering shown in Figure (1), are given in Table (2).

TABLE 2. COMPONENTS OF THE H-TYPE CHIP RESISTOR

1	Alumina Substrate	4	Edge Electrode	7	Resistor Layer
2	Bottom Electrode	5	Barrier Layer	8	Primary Overcoat
3	Top Electrode	6	External Electrode	9	Secondary Overcoat

For this example, we used the Monte Carlo simulation method by coding in Matlab software. The steps of this approach are as follows:

***Step* 1**: Select a number randomly from the mean and variance intervals according to Table (3) for each quality characteristic and generate 100 data from normal distribution. According to Table (3), it is clear that the process tolerance is asymmetric

***Step* 2**: Calculate the relationships presented in the previous section and estimate the $MIC'''_{pp}(u,v)$ and $MIC'''^{G}_{pp}(u,v)$ indices.

***Step* 3**: Go back to the step 1 and repeat the above steps 10,000 times.

TABLE 3. SPECIFICATIONS OF THE PRODUCTION PROCESS OF H-TYPE CHIP RESISTORS

Quality characteristic	(USL, LSL)	T	The intervals for generating mean	The intervals for generating variances
a	(1.9, 2.1)	2.03	(1.6, 1.69)	(0.025, 0.03)
c	(1.15, 1.35)	1.21	(0.88, 0.95)	(0.025, 0.03)
d	(0.35, 0.65)	0.52	(0.5, 0.55)	(0.035, 0.04)
e	(0.15, 0.55)	0.34	(0.35, 0.4)	(0.45 0.05)
b	(0.2, 0.5)	0.36	(0.35,0.4)	(0.035, 0.04)

Shishebori and Hamadani [21] studied the multivariate gauge capability as $\lambda^M = 0.1$ to calculate the MC_p regarding to the gauge measurement error. In this case, we followed their approach and considered $\lambda^M = 0.1$. After 10,000 replications, the mean, variance and coefficient of variation (CV) of $MIC'''_{pp}(u,v)$ and $MIC'''^{G}_{pp}(u,v)$ are obtained and summarized in Table (4).

TABLE 4. SIMULATION RESULTS OF $MIC'''_{pp}(u,v)$ AND $MIC'''^{G}_{pp}(u,v)$

Index	Mean	Variance	CV
$MIC'''_{pp}(u,v)$	0.945465	0.010050	0.106034
$MIC'''^{G}_{pp}(u,v)$	1.068278	0.014624	0.113200

Note that, The Coefficient of Variation is calculated by dividing the standard deviation by the mean i.e. $\mathrm{CV} = \frac{\sigma}{\mu} \times 100$.

As shown in the Table (4), when the measurement error is not considered, the value of the $MIC'''_{pp}(u,v)$ is equal to 0.945465 that shows the process is "*capable*". However, by considering the measurement error, the value of the $MIC'''^{G}_{pp}(u,v)$ is 1.068278 that represents the process is "*incapable*". Therefore, we can conclude that taking measurement error into account in calculating incapability indices affects the final decision about the performance of the process under study and can lead to wrong decisions.

In other words, the process may actually be incapable, but indices that do not take into account the error measurement show that the process is capable. In this case, the manufactured products are considered without any problems and reach the customers, and it causes a lot of damage, including the loss of factory credit, the cost of rework and a lot of financial damage to the high technology industries.

4.2. Comparative example (Case (II))

Jackson [22] studied the Film-developing solution process in which two components including Elon (E) and Hydroguinone (H) are monitored. Process information includes the specification limits and target value for both factors are as follows:

$$E\begin{cases} LSL = 235 \\ USL = 295 \\ T = 250 \end{cases} \tag{74}$$

$$H\begin{cases} LSL = 440 \\ USL = 500 \\ T = 460 \end{cases} \tag{75}$$

According to above information, it is clear that the process tolerance is asymmetric. From a random sample of size 75 [22], the sample variance-covariance matrix is calculated as:

$$\mathbf{S} = \begin{bmatrix} 102.65 & 68.87 \\ 68.87 & 107.96 \end{bmatrix} \tag{76}$$

To compare the performance of our developed approach with that Abbasi Ganji [16] suggested, we used the mean vectors provided in Table (5) and assumed that $\lambda^M = 0.1$ to calculate the $MIC'''_{pp}(u,v)$ and $MIC'''^{G}_{pp}(u,v)$. The results are summarized in Table (5).

By taking into account the sample mean $\overline{\boldsymbol{X}} = \binom{251}{461}$ and the process information according to Equations (74), (75) and (76), the method of calculating the index $MC'''^{G}_{ip}(1)$ in the first row of Table (5) will be explained.

The Equation (76) indicates the sample variance-covariance matrix and its determinant value is equal to $\det(\mathbf{S}) = 6339.017$. By inserting the value $\lambda^M = 0.1$ [21] in the Equation (48), the value of $|\mathbf{S}_E|^{\frac{1}{2}} = \frac{\lambda^M(r_1(u)r_2(u))}{11.829} = \frac{0.1\times14.97778\times19.975}{11.829} = 2.529217$ is obtained.

Shishebori and Hamadani [21] created a diagonal matrix to find the variance-covariance error matrix; the main elements of its main diagonal was the second root of the variance-covariance error matrix determinant. We also created a diagonal matrix with the elements $\sqrt{|\mathbf{S}_E|} = 2.529217$ to calculate the error variance-covariance matrix as follows:

$$\mathbf{S}_E = \begin{pmatrix} 2.529217 & 0 \\ 0 & 2.529217 \end{pmatrix} \tag{77}$$

As mentioned before, the error affected variance-covariance matrix of data is created by the sum of Equations (76) and (77) as follows.

$$\mathbf{S}_G = \begin{pmatrix} 105.1792 & 68.87 \\ 68.87 & 110.4892 \end{pmatrix} \tag{78}$$

The determinant of $\mathbf{S}_G$ matrix is equal to $|\mathbf{S}_G| = 6878.092$; and by inserting this number , $|\mathbf{S}_E|$ and $|\mathbf{S}_G - \mathbf{S}_E| = |\mathbf{S}|$ values into the Equation (67), the index $\widehat{MC}'''^{G}_{ip}(1) = 10.75$ is obtained.

As mentioned earlier, each of Indices $MIC'''_{pp}(1,1)$ and $MIC'''^{G}_{pp}(1,1)$ consists of two parts, imprecision index and inaccuracy index. According to the Table (5), as you can see, by considering the measurement error the value of $MIC'''^{G}_{pp}(1,1)$ is larger than $MIC'''_{pp}(1,1)$. For example, for the case 2, the $MIC'''_{pp}(1,1)$ is equal to 10.09 while the $MIC'''^{G}_{pp}(1,1)$ is equal to 10.93.

The same trend is observed for $MC'''^{G}_{ip}(1)$ compared to the $MC'''_{ip}(1)$. For example in case 2, $MC'''_{ip}(1)$ is equal to 10, while the $MC'''^{G}_{ip}(1)$ is equal to 10.84. In addition, it is observed that the $MC'''_{ia}(1,1)$ is the same in both classical and proposed indices; as there is no standard deviation σ

in the inaccuracy index $MC''''_{ia}(u,v)$, the measurement error has no effect in this part.

According to the Table (5) the index $MIC''''_{pp}(1,1)$ calculates the inaccuracy and imprecision of the process separately. The main problem of the previous index is that it does not take into account the measurement error and this leads to an erroneous estimation of capability, but in the new proposed index, the indices of inaccuracy and imprecision are calculated along with the measurement error.

For sensitivity analysis, we set the process information as follows:

$$E\begin{cases} LSL = 235 \\ USL = 295 \\ T = 265 \end{cases} \quad (79)$$

$$H\begin{cases} LSL = 440 \\ USL = 500 \\ T = 470 \end{cases} \quad (80)$$

From a random sample of size 75 [22], the sample mean vector is shown in Equation (81).

$$\bar{\mathbf{X}} = \begin{bmatrix} 264.32 \\ 471.48 \end{bmatrix} \quad (81)$$

TABLE 5. COMPARISON OF THE DEVELOPED INDICES WITH THE CLASSICAL ONES

	Mean	Classical Indices		Proposed Indices	
Case	$\bar{\mathbf{X}}$	$\left(MC''''_{ia}(1,1), MC''''_{ip}(1)\right)$	$MIC'''_{pp}(1,1)$	$\left(MC''''_{ia}(1,1), MC''''^{G}_{ip}(1)\right)$	$MIC''''^{G}_{pp}(1,1)$
1	$\begin{pmatrix}251\\461\end{pmatrix}$	$(0.01, 9.91)$	9.92	$(0.01, 10.75)$	10.76
2	$\begin{pmatrix}249\\459\end{pmatrix}$	$(0.09, 10)$	10.09	$(0.09, 10.84)$	10.93
3	$\begin{pmatrix}252\\462\end{pmatrix}$	$(0.06, 10.07)$	10.13	$(0.06, 10.92)$	10.98
4	$\begin{pmatrix}248\\458\end{pmatrix}$	$(0.37, 10.42)$	10.79	$(0.37,11.28)$	11.65
5	$\begin{pmatrix}253\\463\end{pmatrix}$	$(0.13, 10.36)$	10.49	$(0.13,11.22)$	11.35
6	$\begin{pmatrix}247\\457\end{pmatrix}$	$(0.86, 11.19)$	12.05	$(0.86, 12.08)$	12.94
7	$\begin{pmatrix}254\\464\end{pmatrix}$	$(0.24, 10.77)$	11.01	$(0.24,11.64)$	11.88
8	$\begin{pmatrix}246\\456\end{pmatrix}$	$(1.60,12.39)$	13.99	$(1.60,13.33)$	14.93

Given the Equations (79) and (80), it is clear that the target vector is set to the midpoint of the specification limits and we are dealing with the symmetric tolerance. The variance-covariance matrix and the sample mean are obtained according to the Equations (76) and (81), respectively.

Tables (6) to (8) shows the values of the $MC'''_{ia}(u,v)$, ${MC'''_{ip}}^{G}(u)$ and ${MIC'''_{pp}}^{G}(u,v)$ for different values of u and v, which are the weighting factors for the mean vector deviation from the target vector and the process variation.

According to the Table (6) when u increases, the value of ${MC'''_{ip}}^{G}(u)$ increases and leads to an increased imprecision; for example for $u = 0$, the value of ${MC'''_{ip}}^{G}(u)$ is 1.381859 and for $u = 0.2$, the value of ${MC'''_{ip}}^{G}(u)$ is 1.383315.

According to the Table (7) and Figure (2) for $v = 0$, the value of $MC'''_{ia}(u,v)$ for different value of u is equal to zero; because v is in the numerator of $MC'''_{ia}(u,v)$. As u and v grow simultaneously, the value of the $MC'''_{ia}(u,v)$ increases. If u is fixed and v increases, the value of the $MC'''_{ia}(u,v)$ increases, and if v is fixed and u increases, the value of the index $MC'''_{ia}(u,v)$ also increases. The increasing rate of the case where u is constant and v is incremental is higher than the case where v is constant and u is incremental.

According to the Table (8) and Figure (3) when u and v grow simultaneously, the value of the ${MIC'''_{pp}}^{G}(u,v)$ increases. If u is considered to be fixed and v increases, the value of the ${MIC'''_{pp}}^{G}(u,v)$ increases, and if v is fixed and u increases, the value of the ${MIC'''_{pp}}^{G}(u,v)$ also increases. The increasing rate of the case where u is fixed and v is incremental is higher than the case where v is fixed and u is incremental.

According to the Table (9), as λ^M increases, the value of the ${MIC'''_{pp}}^{G}(1,1)$ increases and leading to an increase incapability; for example when $\lambda^M = 0.05$, the value of ${MIC'''_{pp}}^{G}(1,1)$ is equal to 1.256147 while for $\lambda^M = 0.1$, the value of ${MIC'''_{pp}}^{G}(1,1)$ is equal to 1.415765.

The gauge capability (λ^M) is directly related to the measurement error. Therefore, increasing the λ^M leads to increase the values of variance-covariance error matrix elements. It should be noted that this matrix is usually calculated by the calibration or quality control department. To manage and prevent the increase of λ^M, which leads to the rejection, it is recommended to use calibrated, new, quality measuring devices, as well as expert operators.

According to Figure (4), the simultaneous increase of the weighting factors u and v as well as the increase in the λ^M leads to an increase in the value of the ${MIC'''_{pp}}^{G}(u,v)$.

It is clear that, increasing the ${MIC'''_{pp}}^{G}(u,v)$ index leads to increase of process incapability. To manage this fact, the following headlines can be considered:

- ✓ apply accurate and advanced devices as well as calibrated and quality instruments for measurement to use by trained operators and experts,
- ✓ estimate the variance-covariance matrix accurately,
- ✓ the target of the process should be set accurately,
- ✓ the process design and specification limits should be determined carefully,
- ✓ all the factors that affect the process including intrinsic, human and environmental factors should be evaluated for future decisions.

We calculated the ratio of ${MIC'''_{pp}}^{G}(u,v)$ and $MIC'''_{pp}(u,v)$ denoted by R as follows:

$$R = \frac{{MIC'''_{pp}}^{G}(u,v)}{MIC'''_{pp}(u,v)} \qquad (82)$$

By increasing the values of weighting factors, both ${MIC'''_{pp}}^{G}(u,v)$ and $MIC'''_{pp}(u,v)$ increases, however the R values decrease according to the Figure (5). Based on Figure (5), it is clear that considering the measurement error has improved the $MIC'''_{pp}(u,v)$; because by increasing the value of weighting factors, the increase rate of the ${MIC'''_{pp}}^{G}(u,v)$ is less than the increase rate of the $MIC'''_{pp}(u,v)$.

TABLE 6. VALUES OF $MC_{ip}^{\prime\prime\prime G}(u)$ FOR DIFFERENT VALUES OF U

u	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$MC_{ip}^{\prime\prime\prime G}(u)$	1.381859	1.383315	1.384772	1.386232	1.387694	1.389159	1.390625	1.392094	1.393565	1.395039	1.396514

TABLE 7. THE VALUES OF $MC_{ia}^{\prime\prime\prime}(u,v)$ FOR DIFFERENT VALUES OF U AND V

	u										
v	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0	0	0	0	0	0	0	0	0	0	0	0
0.2	0.002653	0.002654	0.002656	0.002657	0.002659	0.002661	0.002662	0.002664	0.002665	0.002667	0.002668
0.4	0.005306	0.005309	0.005312	0.005315	0.005318	0.005321	0.005324	0.005328	0.005331	0.005334	0.005337
0.6	0.007958	0.007963	0.007968	0.007972	0.007977	0.007982	0.007987	0.007991	0.007996	0.008001	0.008005
0.8	0.010611	0.010617	0.010624	0.01063	0.010636	0.010643	0.010649	0.010655	0.010661	0.010668	0.010674
1	0.013264	0.013272	0.01328	0.013287	0.013295	0.013303	0.013311	0.013319	0.013327	0.013335	0.013342
1.2	0.015917	0.015926	0.015936	0.015945	0.015954	0.015964	0.015973	0.015983	0.015992	0.016002	0.016011
1.4	0.01857	0.018581	0.018592	0.018602	0.018613	0.018624	0.018635	0.018646	0.018657	0.018668	0.018679
1.6	0.021222	0.021235	0.021247	0.02126	0.021273	0.021285	0.021298	0.02131	0.021323	0.021335	0.021348
1.8	0.023875	0.023889	0.023903	0.023917	0.023932	0.023946	0.02396	0.023974	0.023988	0.024002	0.024016
2	0.026528	0.026544	0.026559	0.026575	0.026591	0.026606	0.026622	0.026638	0.026653	0.026669	0.026685

TABLE 8. THE VALUE OF $MIC_{pp}^{'''G}(u,v)$ FOR DIFFERENT VALUES OF U AND V

	u										
v	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0	1.381859	1.383315	1.384772	1.386232	1.387694	1.389159	1.390625	1.392094	1.393565	1.395039	1.396514
0.2	1.384512	1.385969	1.387428	1.38889	1.390353	1.391819	1.393287	1.394758	1.39623	1.397705	1.399183
0.4	1.387165	1.388623	1.390084	1.391547	1.393012	1.39448	1.39595	1.397422	1.398896	1.400372	1.401851
0.6	1.389818	1.391278	1.39274	1.394205	1.395671	1.39714	1.398612	1.400085	1.401561	1.403039	1.40452
0.8	1.39247	1.393932	1.395396	1.396862	1.39833	1.399801	1.401274	1.402749	1.404227	1.405706	1.407188
1	1.395123	1.396586	1.398052	1.39952	1.40099	1.402462	1.403936	1.405413	1.406892	1.408373	1.409857
1.2	1.397776	1.399241	1.400708	1.402177	1.403649	1.405122	1.406598	1.408077	1.409557	1.41104	1.412525
1.4	1.400429	1.401895	1.403364	1.404835	1.406308	1.407783	1.409261	1.41074	1.412223	1.413707	1.415194
1.6	1.403082	1.40455	1.40602	1.407492	1.408967	1.410444	1.411923	1.413404	1.414888	1.416374	1.417862
1.8	1.405734	1.407204	1.408676	1.41015	1.411626	1.413104	1.414585	1.416068	1.417553	1.419041	1.420531
2	1.408387	1.409858	1.411332	1.412807	1.414285	1.415765	1.417247	1.418732	1.420219	1.421708	1.423199

TABLE 9. THE VALUES OF $MIC_{pp}^{'''G}(1,1)$ FOR DIFFERENT VALUES OF λ^M

λ^M	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
$MIC_{pp}^{'''G}(u,v)$	1.256147	1.402462	1.553777	1.710091	1.871406	2.037721	2.209036	2.385351	2.566666	2.752981	2.944269

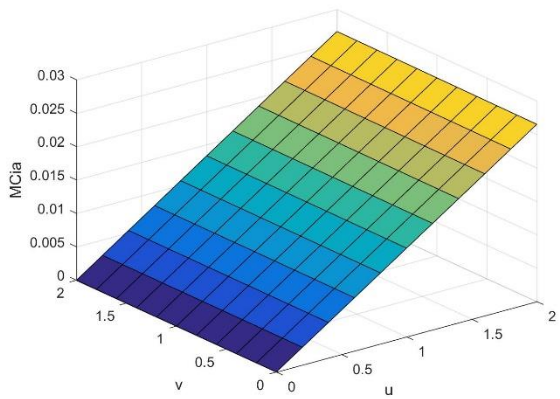

FIGURE 2. THE VALUES OF $MC'''_{ia}(u,v)$ FOR DIFFERENT VALUES OF U AND V

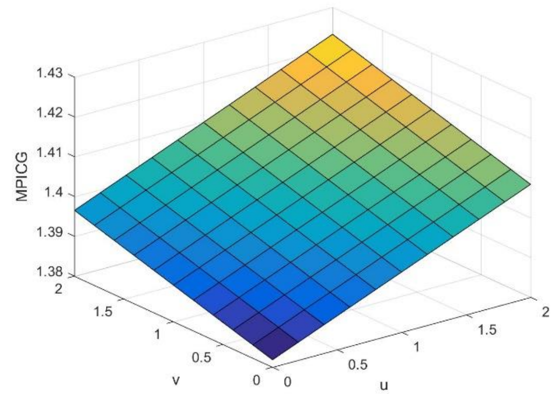

FIGURE 3. THE VALUE OF $MIC'''^{G}_{pp}(u,v)$ FOR DIFFERENT VALUES OF U AND V AND $\lambda^M = 0.1$

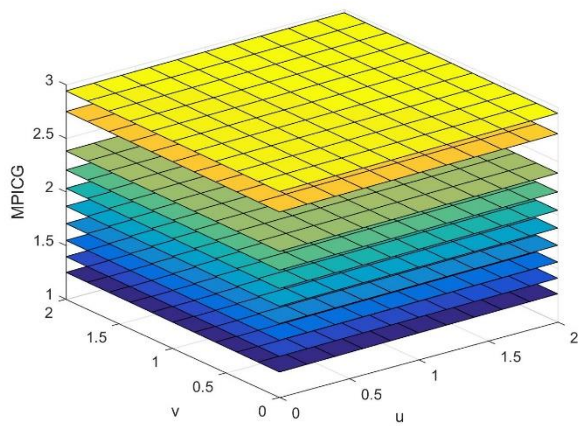

FIGURE 4. THE VALUE OF $MIC'''^{G}_{pp}(u,v)$ FOR DIFFERENT VALUES OF U AND V AND λ^M

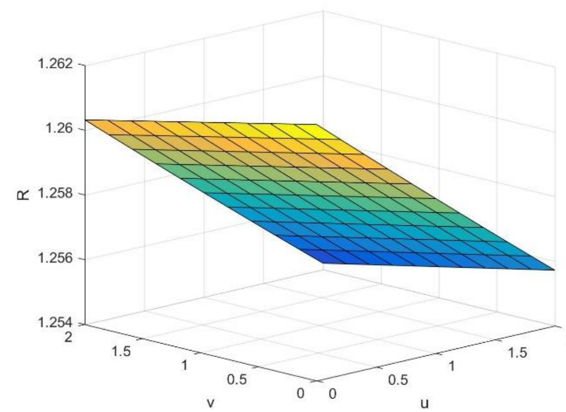

FIGURE 5. RATIO OF $MIC'''^{G}_{pp}(u,v)$ AND $MIC'''_{pp}(u,v)$ FOR DIFFERENT VALUES OF U AND V

5. Conclusion

For processes in which the production process is very sensitive and high-tech, it is necessary to use the process incapability indices. Process incapability indices can be defined as numerical values to monitor the process performance in conformance with the process specifications. In several processes in practical environment, the characteristic measurements are affected by errors, which over/underestimates the process incapability.

In this paper, a multivariate process incapability index is presented for the first time by taking into account the measurement error. Then, the performance of the developed index was compared with the previous classical index. The simulation results showed that the new developed index is superior to the previous classical ones.

In addition, a sensitivity analysis was carried out on the parameters u, v and λ^M. Due to the simulation findings, the following conclusions could be drawn.

Not considering the measurement error will lead to the miscalculation of process capability and to financial and credit losses. If both or one of the parameters u and v increases, the value of the proposed index also increases and as the value of the gauge capability (λ^M) grows, the proposed index increases as well. Increasing the value of the proposed index indicates more capability of the process. As suggestions for future research in this direction, calculation of the multivariate process incapability index considering the measurement error in the fuzzy environment could be helpful. In the practical environment, there are several situations that we cannot cluster the parameters exactly; so using fuzzy sets can solve this problem in statistical quality control methods and calculation of the process incapability index.

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