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# Evaluation of the behavior of robust process capability estimators based on their bootstrap confidence intervals for Gamma distribution

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**Abstract**
Today, producing a product with high quality, according to the customer needs requires a clear strategy of the manufacturers in the market. To produce a good product, measures are taken to measure and control products at all production levels among, which the analysis of process capability indices is of great importance in the industry. In this context, the usability indicators can be effective when the data follow a normal distribution. On the other hand, if the data aren't standard normal, evaluation of the process's capability based on these indices will typically be confronted with the problem. In this paper, after investigating the behavior and characteristics of the median absolute deviation *(MAD)* and interquartile range *(IQR)* and ($Q_n$), their analysis is conducted for the Gamma distribution. Then, the bias errors and standard errors are obtained using the jackknife method. Three estimators are evaluated in three different modes according to the bootstrap methods and based on their confidence intervals. Finally, by analyzing the results of this research, the reliability, and performance of the estimators are evaluated in different states.

**Keywords-** Robust estimators; Gamma distribution; Process capability indices; Bootstrap confidence intervals

## INTRODUCTION

In recent years, massive research on capability indices has been done. Today, several process capability indices have been introduced by researchers. Process capability indices can be helpful when products follow a normal distribution. Careful examination of process capabilities is important to identify strengths and weaknesses of the reliability and to produce the products with high quality. Estimators such as mean and variance are used to evaluate classical process capability indices. However, some of the essential features in the industry perhaps move away from the normal situation. This leads to the study of non-normal behavior, which even affects reliability, from another aspect. With the change of the problem data, the process characteristic dispersion may fluctuate and change the index. Most quality features ignore the assumption of normality. Therefore, standard-based process capability indices are not widely used and cannot express the process performance well. Besseris refers to determining the capability indicators based on the median absolute deviation for abnormal data and concludes that much progress is achieved if the issue data are highly asymmetric. Study the robust interquartile range ($IQR$) and ($Q_n$) for Weibull process capability indices using bootstrap techniques has been performed to obtain 95% confidence intervals[1]-[2].. By examining Gamma and normal distributions, we can see some features of Gamma distribution in many studies [3]. If the distribution deviates from the normal state, the index formula loses its process capability efficiency. Non-normal distribution has a significant effect on classical PCI. Because the standard deviation for non-normal distributions loses its effective performance. Therefore, the terms *IQR* and *MAD* are using for analysis [4]. In some studies, mentioned that *MAD* is a good approximation of the standard deviation for non-normal distribution. Confidence intervals in the correct interpretation of process capability indices can be effective [2]. Creating confidence intervals was first considered by researchers in [5]. The creating confidence intervals was first used in Chou's research [6]. Ouyang et al. (2024) studied the development of robust confidence intervals for the cost-based process capability index, and by using bootstrap methods, it evaluates process capability indices in normal mode [7]. Saha et al. (2022) also used six different methods of estimation to obtain the estimates of the PCI and also compare three bootstrap confidence intervals (BCIs)[8]. Various methods based on process capability indicators have been further studied by other researchers, including Afshari et al. (2022), Day et al. (2023)[9]-[10]. Afshari has studied the effects of measurement tool error on multivariable process capability estimation and assessment. Day et al. has pointed out the applications of process capability indicators based on cost and loss in electronics industries. And it studies 5 bootstrap methods in normal mode to compare process capability indicators. and Kashif et al. (2023) investigated PCIs under non-normal distributions, particularly for applications involving the Weibull process[11]. Many researchers claim that the median absolute deviation of the best alternative to the standard deviation for the data is non-normal. In Weibull distribution, *IQR* and *MAD* are used to measure *PCI* variability. Kashif points out that *MAD* performs better than *IQR* considering the Weibull distribution and the bootstrap percent confidence interval [12]-[13]. Statistical tools are necessary to improve the process performance. One of these methods is the applying process capability indices (*PCI*), Which the most important of them are $C_p$ and $C_{pm}$ and $C_k$ and $C_{mk}$. Other indices are derived from these indices [2].

Bootstrap is a nonparametric method based on the observed n-sample. Some studies suggest bootstrap resampling to estimate the accuracy of estimators. The bootstrap method, the size of the bias accuracy, variance, confidence interval, and hypothesis test can be estimated. First, the bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ is obtained by placing the observed sample $x_1$, $x_2, \ldots,\ x_n$ by random sampling method. Then, the bootstrap estimator is calculated [14]. The jackknife method works somewhat similar to the bootstrap method, and researchers have contributed a lot to the progress of this approach. It is noteworthy that the jackknife method is also a method for calculating bias error, and standard error, which according to Quenouille studies and the invention of this method in 1950, much progress was made in the industry. However, the jackknife method performs better than the bootstrap methods when the data are fewer, which follows more straightforward calculations. When the number of data is more than 200, it is best to use bootstrap methods because the jackknife method loses its usefulness [15]-[16]. Process capability indices have a wide range that directly or indirectly affects the evaluation, and production of products. For example, *MAD* is used in producing synthetic composite composites. If the confidence interval or accuracy of the process capability indices don't include in the design or production of the product, it may lead to erroneous decisions and causing losses on the line. Even getting waves of frequencies is accomplished by antennas in a given range that can be measured and analyzed through the capability indices of the process to achieve a specific frequency. Statistical distributions have much application in the issue of quality control and decisions. Many processes in production do not follow normal distributions, and Clements was the first person to broach the subject [17].

One of the reasons for the emergence of process capability indices is the need in factories to estimate and measure the efficiency of production processes in producing standard items. Our goal is to evaluate and identify the best estimator to examine the behavior of problem data in the production processes and control the deviation and variance of an abnormal process. In this way, the customer reaches the appropriate quality product. So far, the evaluation of capability indices has not been done considering the Gamma distribution. The Gamma distribution can show a more excellent attribute of the evaluation of the process capability indices. In this study, we intend to study three robust process capability indices with Gamma distribution to see how changes in the number of data studied, along with the shape and scale parameter, change the evaluation of the indices. In the present study, the formulas are expressed in the software format after reviewing the points and definitions. In the following, the discussion of results is discussed. Finally, the conclusion is derived from the data.

Gamma distribution, like Weibull distribution, is one of the non-normal distributions, and each of them has many applications in data evaluation in the industry. Gamma distribution can cover gaps in industry and other distributions. Gamma distribution is used in military industries such as telecommunication systems and similar in fiberglass industry and power generation. It is worth mentioning that so far, no research has been done using the gamma distribution. Wang et al. Studied process capability indices for abnormal distributions. To show the efficiency of robust process capability indices, he mentioned abnormal distributions such as Weibull distribution. He also mentioned the gamma distribution as a new candidate in future studies, which can also be used in future studies [18]. In their study, Lee et al. Evaluated LCD products using process capability and gamma distribution indices and came to a satisfactory conclusion [19]. Piao et al noted that if data distribution is not normal, gamma distribution can be used to study process capability indices [20]. Evaluation of classical process capability indices is not possible everywhere, and for more accurate evaluation, robust process capability indices are used, which can lead us to a better answer [21]. Even process capability indices can be studied in other studies for fuzzy sets [22].

## PROCESS CAPABILITY INDICES *(PCI)*

The classical process capability indices, $C_p$, was first introduced by Juran after World War II to provide consultative services to the Japanese industry and expressed as follows [3]:

$$c_p = \frac{USL - LSL}{6\hat{\sigma}} \tag{1}$$

*USL* and *LSL*, are the upper and lower specification limits of a process characteristic, respectively. The standard deviation, which is one characteristic of a process, is defined with the Sigma notation. As mentioned in the previous sections, if the data deviates from normal, then the above formula loses its effectiveness. The difference expression (*USL-LSL*) is the process characteristic specification range (*PCSS*), and the expression ($6\hat{\sigma}$) is the process characteristic spread (*PCS*). It requires 99.73% coverage, which is limited in Gaussian distributions between 0.013% and 99.865%. The Gamma distribution density function can be defined as follows Gamma distribution is closely associated with exponential distribution because, when the shape parameter is equal to one, the exponential distribution will be achieved:

$$f(x) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda t} \quad , \quad x \geq 0, r > 0, \lambda > 0 \tag{2}$$

In the above formula, *r* is called the shape parameter, and λ is called the scale parameter. The cumulative Gamma distribution function is defined as follows. By integrating the density function, the distribution function is obtained, and according to the function parameters, the value of the function is between zero and one:

$$F(a) = 1 - \int_a^\infty \frac{\lambda}{\Gamma(r)}(\lambda t)^{r-1}e^{-\lambda t}\,dx \tag{3}$$

Different values have been stated for *LSL* and *USL* values, but Park has used the following method [23]:

$$F(0)=LSL \ , \ F(0.995)=USL \tag{4}$$

### *I. Median absolute deviation (MAD)*

Suppose that $x_1, x_2, \dots, x_n$ is a random sample, and *M* is the sample's median. Then, the median absolute deviation is defined as follows [24]:

$$MAD = b \times median\{|x_i - M|\} \quad i = 0,\dots,n \tag{5}$$

For symmetric distributions, it is worth noting that the median is equal to the mean, i.e., $Md = \mu$. For finite-state or an example, we have:

$$md_n = median(x_1, x_2, \dots, x_n) = x_{k+1:n} \, , n = 2k+1 \tag{6}$$

$$md_n = aX_{k:n} + (1-a)X_{k+1:n} \ ; 0 \leq a \leq 1, n = 2k \tag{7}$$

In general, $a$=1/2. Also, mo≤$M$≤$\mu$, that mo is the same mode and *Md* or *M* is the median [25]. If we follow a Gaussian distribution, the variable of *b* is a fixed value for better compatibility. It's equal to 1.4826. The median absolute deviation is directly related to the mean distance of the observations from the median of the distribution, and for variables with a normal distribution, this value is 0.6745 times the standard deviation and 0.8453 times the mean deviation. Thus, it is a robust estimator of the standard deviation of the sample, which calculates the deviation of the data from the median, and was first proposed by Hempel, who attributed it to Gauss [26]. The expression $b_n MAD$ can be considered a non - bias approximation of Sigma that the expression $b_n$ acts as a correction factor. When the number of data is large, this number tends to 1 [24]. For $n \leq 9$, the values of $b_n$ are obtained as follows:

TABLE 1
THE CORRECTION FACTORS FOR MAD

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $b_n$ | 1.192 | 1.495 | 1.363 | 1.206 | 1.1200 | 1.140 | 1.129 | 1.107 |

Moreover, numbers greater than nine are obtained from the following equation:

$$b_n = \frac{n}{n-0.8} \qquad (8)$$

When there is more number, the more this number goes to 1, and for data that follows a normal distribution, this number will be 1.4826. So, it can be said for the normal state that [2]:

$$\hat{\sigma} = 1.4826\ MAD \qquad (9)$$

As mentioned earlier, the above statement is a strong estimate of the standard deviation. Of course, if the problem is uncorrected, the value of b can be considered 1. Now for distributions whose data does not follow the Gaussian distribution, their value *b* is slightly different depending on the problem's circumstances. In the case of non-normal distribution, its value is equal to:

$$b=1/Q\ (0.75) \qquad (10)$$

In normal distributions, b = 1 / Q (0.75) = 1.4826 [27].

*II. Interquartile range (IQR)*

The interquartile range is another process capability estimator. To understand it, we must have sufficient knowledge of the concept of quartiles. The interquartile range is defined as follows:

$$IQR=Q_3 - Q_1 \qquad (11)$$

The above relation shows the difference between the upper and lower value. That is the first and third quartile. Moreover, it says that 75% and 25% probability of data are:

$$F_{r,\lambda}(Q_3) = 0.75 \qquad (12)$$

$$F_{r,\lambda}(Q_1) = 0.25 \qquad (13)$$

That:

$$\int_{-\infty}^{Q_1} f(x)dx = 0.25 \qquad (14)$$

$$\int_{-\infty}^{Q_3} f(x)dx = 0.75 \qquad (15)$$

According to the definition of interquartile range, the process capability indices can be defined based on the interquartile range as follows [12]:

$$C_P = \frac{USL-LSL}{2*IQR} \qquad (16)$$

Process capability indices based on *MAD* and *IQR* should not be directly compared. Because (2) ×*IQR* distribution range indicates 100 % nominal coverage, but for normal mode, (8.9) × *MAD*, shows 99.73 coverage [28].

*III. $Q_n$ estimator*

Another estimator that has many applications in the discussion of process capability is the $Q_n$ estimator, which has a breakpoint of 50% and an efficiency of 82% for Gaussian distributions and is defined as follows [2]:

$$Q_n = d\{|x_i - x_j|; 1 \le i < j \le n\}_l \qquad (17)$$

That:

$$l = \binom{h}{2}\ ; h = \left[\frac{n}{2}\right] + 1 \qquad (18)$$

That is, the state for " l " can be imagined. Moreover, *d* is a correction factor whose value for normal data is initially equal to 2.2219, which is later corrected and included in the software 2.21914. This expression is equal to ((1 / (*sqrt* (2) × *qnorm* (5 / 8)). If we consider an estimate of Sigma, due to the differences caused by the experiments, we have to use the correction factor $d_n$. Th correction factors are slightly different for the even and odd mode. So, we have [29]:

TABLE 2
THE CORRECTION FACTORS FOR $Q_n$ ESTIMATOR

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $d_n$ | 0.399 | 0.994 | 0.512 | 0.844 | 0.611 | 0.857 | 0.669 | 0.872 |

If 9 < $n$ and it is odd, then the above correction factor is defined as follows:

$$d_n = \frac{n}{n-1.4} \tag{19}$$

However, if n is even, the above expression is considered as follows:

$$d_n = \frac{n}{n-3.8} \tag{20}$$

$$Q_n = d_n \times 2.21914\{|x_i - x_j| \,; i < j\} \tag{21}$$

The process capability indices based on $Q_n$is defined as follows:

$$C_P = \frac{USL - LSL}{6 * Q_n} \tag{22}$$

The estimators that used the correction factor are in the corrected state because if their coefficients are considered one, it would be on unmodified form.

*IV. Jackknife method*

In 1950, Quenouille developed a method for estimating standard error and bias error, later known as the jackknife method. Assume that $\hat{\theta} = (X_1, X_2, \dots, X_n)$estimates the parameter $\theta$. In this method, one observation removes in each step, and then $\hat{\theta}$ calculates based on the remaining observations[4].

$$X_{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \quad i = 1,2, \dots, n \tag{23}$$

Therefore, the $i^{th}$sample of jackknife $X_{(i)}$ $i$ = 1, 2, …, $n$ is obtained by removing the $i^{th}$point.

$$\hat{\theta}_{(i)} = S(X_{(i)}) \qquad i = 1,2, \dots, n \tag{24}$$

For the substitution statistic $\hat{\theta} = t(\hat{F})$, $\hat{\theta}_{(i)}$ is equal to t ($\hat{F}_{(i)}$), where $\hat{F}_{(i)}$is the experimental distribution for $n$-1 points. The jackknife bias and standard estimates are defined as follows [15][16][30]:

$$\widehat{Bias}_{jack} = (n-1)(\hat{\theta}_{(0)} - \hat{\theta}_{(i)}) \tag{25}$$

$$\widehat{Se}_{jack} = \left[\frac{n-1}{n}\sum(\theta_{(i)} - \theta_{(0)})^2\right]^{\frac{1}{2}} \tag{26}$$

$$\hat{\theta}_{(0)} = \sum_{i=1}^{n} \hat{\theta}_{(i)} \tag{27}$$

## BOOTSTRAP CONFIDENCE INTERVALS

In this section, there are several methods for estimating the confidence interval for the target parameter in bootstrap. Methods include standard normal bootstrap *(SB)*, base bootstrap confidence interval *(BB)*, percentile bootstrap confidence interval *(PB)*, bootstrap confidence interval with bias-corrected accelerated *(BCa)*.Suppose $(x_1, x_2, \dots, x_n)$ are observations with random variables $(X_1, X_2, \dots, X_n)$from the distribution $F$, i.e., $F \sim X_1, X_2, \dots, X_n$. That is, $X = \{X_1, X_2, \dots, X_n\}$ show all sets. we select a random instance by placing it from the set $X$ to the size $n$, and we show them with $X^* = X_1^*, X_2^*, \dots, X_n^*$. There are $n^n$ resampling sets, it calculates $n^n$values of $\hat{\theta}^*$, and θ as efficiency indices. Each of them is an estimate of $\theta^*$, and the set of them all forms the bootstrap distribution $\hat{\theta}$. The bootstrap sample is equivalent to the placement sample of the experimental distribution function. Thus, the bootstrap distribution $\hat{\theta}$ is an estimate of the experimental distribution $\theta$ [31].

*I. Bootstrap percentile confidence interval (PB)*

The bootstrap percent confidence interval was first expressed by Efron[29]. Moreover, it is similar to the standard bootstrap confidence interval. Consider $X_1, X_2, \dots, X_n$ that contain *n* independent samples of a community with unknown parameter $\theta$. $\hat{\theta}$ is the estimator of $\theta$. In this method, the bootstrap samples $X_1^*, X_2^*, \dots, X_n^*$ are created randomly by placing the observed samples $x_1, x_2, \dots, x_n$. Then the bootstrap estimator $\hat{\theta}^*$is calculated and generated. By repeating the *B* step of the bootstrap estimators $\hat{\theta}_1^*, \hat{\theta}_2^*, \dots, \hat{\theta}_n^*$ are calculated. Finally, the bootstrap percentile confidence interval with a coefficient of 1-α is calculated as follows[5][30]:

$$(\hat{\theta}^*_{[\mathrm{B}(\frac{\hat{\alpha}}{2})]}; \hat{\theta}^*_{[\mathrm{B}(1-\frac{\hat{\alpha}}{2})]}) \tag{28}$$

*II. Bias Corrected accelerated bootstrap confidence interval (BCa)*

Efron introduced an edited version of the bootstrap percent confidence interval called the bias-corrected accelerated bootstrap confidence interval, which covered some of the problems of the previous version [29]. We denote by two values of bias-corrected $\hat{z}_0$ and the acceleration of $\hat{\alpha}$. The task of bias correction is to return the intended distribution. If we delete the $i^{th}$ point of the sample and $\hat{\theta}_i = t(X_{(i)})$the estimate of the parameter θ is by omitting the observation of $i^{th}$, and $U_i$ as $(\hat{\theta}_{(0)} - \hat{\theta}_{(i)})$ is defined as we have [30]:

$$\hat{\theta}_{(0)} = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_{(i)} \tag{29}$$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} \mathrm{U}_i^3}{6[\sum_{i=1}^{n} \mathrm{U}_\mathrm{i}^2]^{\frac{3}{2}}} \tag{30}$$

Finally, the Bias Corrected accelerated *(BCa)* bootstrap confidence interval with 1-*α* coefficient is calculated as follows [31]:

$$(\hat{\theta}^*_{[\mathrm{B}(\frac{\hat{\alpha}}{2})]}; \hat{\theta}^*_{[\mathrm{B}(1-\frac{\hat{\alpha}}{2})]}) \tag{31}$$

*III. Standard normal bootstrap confidence interval (SB)*

The standard normal bootstrap confidence interval is one of the simplest methods, but it is not necessarily the best option for this purpose. Assume that $\hat{\theta}$ is an estimate of the parameter $\theta$. Se($\hat{\theta}$) is the standard error estimator. If E($\hat{\theta}$) is the sample average and the sample size is also large, according to the central limit theorem, distribution

$$Z = \frac{\hat{\theta} - E(\hat{\theta})}{se(\hat{\theta})} \tag{32}$$

is the normal standard. Therefore, if $\hat{\theta}$ is non-bias for $\theta$, then an approximate confidence interval of 100 ($\alpha$ - 1) % for $\theta$ is [12]:

$$(\hat{\theta} - z_{\frac{\alpha}{2}} se(\hat{\theta}), \hat{\theta} + z_{\frac{\alpha}{2}} se(\hat{\theta})) \tag{33}$$

*IV. Basic Bootstrap confidence interval (BB)*

The bootstrap confidence intervals change the basis of the repeated distribution by reducing the observed statistics. Quantile of sample distributions provides samples of simulated samples. From the set of $\hat{\theta}_i^*$, the basic bootstrap confidence intervals are represented as follows [32]:

$$(2\hat{\theta} - z_{1-\frac{\alpha}{2}}, 2\hat{\theta} - z_{\frac{\alpha}{2}}) \tag{34}$$

The basic bootstrap confidence interval is based on axial quantity and can be easily calculated.

*V. Method of analysis and study of the behavior of estimators*

Using the *R* software, the analysis process begins for three values in the form ($a$, $b$). The first component is the scale parameter, and the second component is the shape parameter. After compiling and selecting the data, *MAD*, *IQR* and $Q_n$ were obtained for sizes 20, 50, and 100. By selecting boot from the package section, boot () can apply to each of the estimators. Then any kind of their confidence intervals can be easily calculated [32]. One can determine the bias error or the standard error using the jackknife method. Then, using the *Excel* software, the radar diagram of confidence intervals can be drawn and analyzed. On this basis, the behavior of the Gamma distribution density function for different parameters is present in Fig. 1.

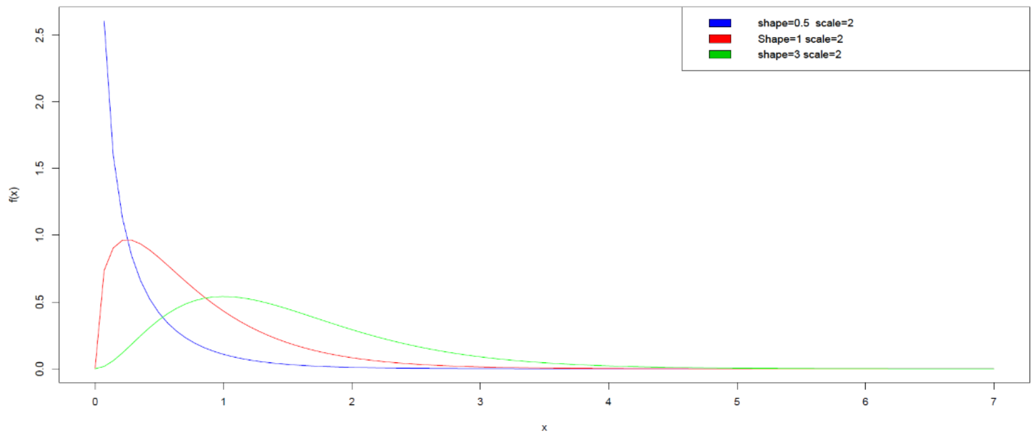


FIGURE 1
GAMMA DISTRIBUTION DENSITY FUNCTION DIAGRAM

It is noticed that the *R* and *Excel* softwares are among the simplest and most powerful software in the field of programming. Accordingly, the *R* software was used to write formulas and evaluate data. In additions, *Excel* software was used to analyze the data due to its suitable convenience.

## DISCUSSION

Three robust estimators were studied to estimate the process capability for stochastic data derived from the Gamma distribution. The discussion of robust estimators is very complex and could be the subject of further research. Three different types of curves with different sample sizes corresponding to Fig 1 considered for Gamma distribution. The values obtained for *MAD* and $Q_n$in Table 3 are regardless of the final correction factor. After multiplying by the correction factor, it can be considered as an approximation of the standard deviation, which is shown in Table 4. The standard error and the bias error values in Table 3 are given by the jackknife method in Table 5. After reviewing Table 3, it can be seen that when the number of data is small, the estimator values are small at first. As the number of data increases, it is observed that the mentioned values first increase and then decrease. As the shape values increase, the density value decreases. The value of $Q_n$ decreases as the shape values increase and the number of data increases. It can even be seen that when the shape values are large, the same happens for *IQR* as the number of data increases compared to the previous state. In general, the *IQR* value is higher than other estimators. Because it covers a wide range of information, which is 50%. The value of *MAD* is higher than $Q_n$, so in general, one can say that *IQR* has the highest value and $Q_n$has the lowest value.

We chose shape and scale values hypothetically and inspired by basic papers on Weibull distribution. Considering keeping the scale values constant, we fluctuated the shape values a little and then by increasing the sample, we tried to check the behavior of the 3 estimators in different states using the gamma distribution and see if there is any behavior with different shape values. They show. As we know that different situations may happen in the industry, we assumed that something similar to this research may happen in the industry. Therefore, we have briefly explained all the situations. According to Table 5, when the difference between the values of the shape and scale is high, that is, the shape values are small, the jackknife standard error value is generally lower than $Q_n$

TABLE 3
ESTIMATION OF *MAD*, *IQR*, $Q_n$ FOR THREE SAMPLES OF THE GAMMA DISTRIBUTION

| Shape | Scale | Sample size | Scale type | value |
|---|---|---|---|---|
| 0.5 | 2 | 20 | *MAD* | 0.2363 |
| | | | *IQR* | 0.5657 |
| | | | $Q_n$ | 0.1763 |
| | | 50 | *MAD* | 0.4438 |
| | | | *IQR* | 1.6069 |
| | | | $Q_n$ | 0.3242 |
| | | 100 | *MAD* | 0.2431 |
| | | | *IQR* | 1.1572 |
| | | | $Q_n$ | 0.1877 |
| 1.5 | 2 | 20 | *MAD* | 0.1596 |
| | | | *IQR* | 1.4118 |
| | | | $Q_n$ | 0.1244 |
| | | 50 | *MAD* | 0.2821 |
| | | | *IQR* | 2.2936 |
| | | | $Q_n$ | 0.1890 |
| | | 100 | *MAD* | 0.2739 |
| | | | *IQR* | 2.5147 |
| | | | $Q_n$ | 0.1690 |
| 3 | 2 | 20 | *MAD* | 0.2597 |
| | | | *IQR* | 5.1542 |
| | | | $Q_n$ | 0.1988 |
| | | 50 | *MAD* | 0.2823 |
| | | | *IQR* | 4.2206 |
| | | | $Q_n$ | 0.1613 |
| | | 100 | *MAD* | 0.2262 |
| | | | *IQR* | 3.4906 |
| | | | $Q_n$ | 0.1359 |

TABLE 4
ESTIMATION OF *MAD*, AND $Q_n$ BY CONSIDERING THE SECOND CORRECTION FACTOR

| Shape | Scale | Sample size | Scale type | value |
|---|---|---|---|---|
| 0.5 | 2 | 20 | *MAD* | 0.2462 |
| | | | $Q_n$ | 0.1481 |
| | | 50 | *MAD* | 0.4510 |
| | | | $Q_n$ | 0.3015 |
| | | 100 | *MAD* | 0.2470 |
| | | | $Q_n$ | 0.1809 |
| 1.5 | 2 | 20 | *MAD* | 0.1663 |
| | | | $Q_n$ | 0.1045 |
| | | 50 | *MAD* | 0.2866 |
| | | | $Q_n$ | 0.1758 |
| | | 100 | *MAD* | 0.2761 |
| | | | $Q_n$ | 0.1630 |
| 3 | 2 | 20 | *MAD* | 0.2706 |
| | | | $Q_n$ | 0.1670 |
| | | 50 | *MAD* | 0.2868 |
| | | | $Q_n$ | 0.15 |
| | | 100 | *MAD* | 0.228 |
| | | | $Q_n$ | 0.1311 |

As the shape approaches symmetry and the sample number increases, *MAD* jackknife standard error becomes more significant than $Q_n$. When the data number is low, the *IQR* standard error increases with increasing the shape values. If there is a significant difference between the scale values and shape values, when the data number increases, the standard error $Q_n$ increases and then decreases. If the shape values increase, these changes become smaller, and the standard error decreases. In all cases, the standard error of *MAD* and $Q_n$ is less than *IQR*. In general, *MAD* contains less standard error than $Q_n$, and is shown in the radar diagram.

For the *MAD* and $Q_n$ bias error, it can be said that it has fewer fluctuations than *IQR*. In each stage, *IQR* bias error has almost gone down by increasing the sample. Often, $Q_n$ bias error is less than *IQR*, but there is no specific order for comparing these two estimators. It can even be said that the $Q_n$ bias error is less than *MAD*. In the case of $Q_n$ and *IQR*, a typical trend for change can not be predicted, but in most cases, the $Q_n$ bias error is less than *IQR*. These bias error fluctuations are shown in Fig. 3.

TABLE 5
ESTIMATION OF BIAS ERROR AND STANDARD ERROR USING THE JACKKNIFE METHOD

| Shape | Scale | Sample size | Scale type | Standard error | ias error |
|---|---|---|---|---|---|
| 0.5 | 2 | 20 | *MAD* | 0.016 | 0 |
| | | | *IQR* | 0.2802 | 0.0225- |
| | | | $Q_n$ | 0.0223 | -0.0894 |
| | | 50 | *MAD* | 0.0019 | 0 |
| | | | *IQR* | 0.5055 | -0.0864 |
| | | | $Q_n$ | 0.0515 | -0.5771 |
| | | 100 | *MAD* | 0.0018 | 0 |
| | | | *IQR* | 0.2437 | 0.6298 |
| | | | $Q_n$ | 0.0295 | -0.3793 |
| 1.5 | 2 | 20 | *MAD* | 0.0039 | 0 |
| | | | IQR | 0.4869 | 0.4239 |
| | | | $Q_n$ | 0.0209 | -0.1683 |
| | | 50 | *MAD* | 0.0154 | 0 |
| | | | *IQR* | 0.4728 | -0.1076 |
| | | | $Q_n$ | 0.0413 | 0.4037 |
| | | 100 | *MAD* | 0.0327 | 0 |
| | | | *IQR* | 0.1596 | -1.1169 |
| | | | $Q_n$ | 0.0218 | -0.5129 |
| 3 | 2 | 20 | *MAD* | 0.0273 | 0 |
| | | | *IQR* | 1.5233 | 0.3688 |
| | | | $Q_n$ | 0.0617 | -0.4168 |
| | | 50 | *MAD* | 0.0374 | 0 |
| | | | IQR | 0.4344 | -0.1131 |
| | | | $Q_n$ | 0.0215 | -0.2919 |
| | | 100 | *MAD* | 0.0152 | 0 |
| | | | *IQR* | 0.5689 | 2.8626 |
| | | | $Q_n$ | 0.0114 | -0.3019 |

TABLE 6
UPPER AND LOWER BOUNDS OF BOOTSTRAP CONFIDENCE INTERVALS FOR *MAD* AND *IQR* AND $Q_n$

| Shape | Scale | Sample size | Scale type | Normal | | Basic | | Percentile | | BCa | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | LL | UL | LL | UL | LL | UL | LL | UL |
| 0.5 | 2 | 20 | *MAD* | 0.1528 | 0.3957 | 0.2065 | 0.3796 | 0.0929 | 0.2660 | 0.0923 | 0.2594 |
| | | | *IQR* | -0.217 | -1.193 | 0.5152 | 0.9359 | 0.1955 | 1.6466 | 0.2093 | 1.7230 |
| | | | $Q_n$ | 0.0760 | 0.2720 | 0.0624 | 0.2547 | 0.0980 | 0.2902 | 0.1245 | 0.3093 |
| | | 50 | *MAD* | 0.3499 | 0.5279 | 0.2967 | 0.5056 | 0.3819 | 0.5908 | 0.3814 | 0.5802 |
| | | | *IQR* | 0.064 | 2.680 | -0.239 | 2.231 | 0.592 | 3.062 | 0.617 | 4.7720 |
| | | | $Q_n$ | 0.1610 | 0.4801 | 0.1535 | 0.4595 | 0.1888 | 0.4948 | 0.2218 | 0.5621 |
| | | 100 | *MAD* | 0.2245 | 0.2636 | 0.2286 | 0.2656 | 0.2206 | 0.2577 | 0.2206 | 0.2577 |
| | | | *IQR* | 2.268 | 8.870 | 2.086 | 8.627 | 1.681 | 8.223 | 2.030 | 9.0230 |
| | | | $Q_n$ | 0.1105 | 0.2676 | 0.1081 | 0.2673 | 0.1081 | 0.2673 | 0.1197 | 0.2783 |
| 1.5 | 2 | 20 | *MAD* | 0.0854 | 0.2323 | 0.0908 | 0.2476 | 0.0717 | 0.2285 | 0.0686 | 0.2272 |
| | | | *IQR* | 0.878 | 2.259 | 0.821 | 2.062 | 1.151 | 2.393 | 1.203 | 2.451 |
| | | | $Q_n$ | 0.0725 | 0.1942 | 0.0576 | 0.1860 | 0.0628 | 0.1913 | 0.0786 | 0.2254 |
| | | 50 | *MAD* | 0.2203 | 0.3424 | 0.2129 | 0.3384 | 0.2258 | 0.3514 | 0.2236 | 0.3513 |
| | | | *IQR* | 1.560 | 3.094 | 1.141 | 2.943 | 1.644 | 3.446 | 1.776 | 3.806 |
| | | | $Q_n$ | 0.1374 | 0.2608 | 0.1309 | 0.2487 | 0.1293 | 0.2470 | 0.1405 | 0.2861 |
| | | 100 | *MAD* | 0.2327 | 0.3289 | 0.2335 | 0.3339 | 0.2139 | 0.3142 | 0.2116 | 0.3137 |
| | | | *IQR* | 2.579 | 5.711 | 1.974 | 5.412 | 3.030 | 6.464 | 3.056 | 6.585 |
| | | | $Q_n$ | 0.0281 | 0.0682 | 0.0275 | 0.0679 | 0.0280 | 0.0684 | 0.0305 | 0.0710 |
| 3 | 2 | 50 | *MAD* | 0.1184 | 0.3930 | 0.0906 | 0.4212 | 0.0983 | 0.4289 | 0.0747 | 0.3944 |
| | | | *IQR* | 0.600 | 1.0630 | 0.665 | 1.576 | 0.738 | 1.649 | 0.762 | 1.691 |
| | | | $Q_n$ | 0.0347 | 0.0931 | 0.0275 | 0.0896 | 0.0295 | 0.0915 | 0.0379 | 0.1092 |
| | | 50 | *MAD* | 0.2237 | 0.3420 | 0.2169 | 0.3464 | 0.2181 | 0.3475 | 0.2181 | 0.3453 |
| | | | *IQR* | 1.693 | 3.348 | 1.656 | 3.223 | 1.806 | 3.373 | 1.845 | 3.626 |
| | | | $Q_n$ | 0.1246 | 0.2091 | 0.1240 | 0.2075 | 0.1150 | 0.1985 | 0.1322 | 0.2173 |
| | | 100 | *MAD* | 0.1699 | 0.2819 | 0.1633 | 0.2760 | 0.1764 | 0.2889 | 0.1758 | 0.2812 |
| | | | IQR | 2.473 | 4.360 | 2.365 | 4.311 | 2.670 | 4.616 | 2.613 | 4.554 |
| | | | $Q_n$ | 0.1172 | 0.1617 | 0.1168 | 0.1617 | 0.1100 | 0.1550 | 0.1172 | 0.1626 |

TABLE 7
BOOTSTRAP CONFIDENCE INTERVALS DOMAIN FOR *MAD* AND *IQR* AND $Q_n$

| Shape | Scale | Sample size | Scale type | Normal | Basic | Percentile | BCa |
|---|---|---|---|---|---|---|---|
| 0.5 | 2 | 20 | *MAD* | 0.2494 | 0.1731 | 0.1731 | 0.1671 |
| | | | *IQR* | 1.41 | 1.4511 | 1.4511 | 1.5137 |
| | | | $Q_n$ | 0.196 | 0.1923 | 0.1992 | 0.1848 |
| | | 50 | *MAD* | 0.178 | 0.2089 | 0.2089 | 0.1988 |
| | | | *IQR* | 2.616 | 2.47 | 2.47 | 4.101 |
| | | | $Q_n$ | 0.3191 | 0.306 | 0.306 | 0.3403 |
| | | 100 | *MAD* | 0.0931 | 0.037 | 0.0371 | 0.0371 |
| | | | *IQR* | 6.602 | 6.541 | 6.542 | 6.993 |
| | | | $Q_n$ | 0.1571 | 0.1592 | 0.1592 | 0.1586 |
| 1.5 | 2 | 20 | *MAD* | 0.1469 | 0.1568 | 0.568 | 0.1586 |
| | | | *IQR* | 1.381 | 1.241 | 1.242 | 0.248 |
| | | | $Q_n$ | 0.1217 | 0.1284 | 0.1285 | 0.1468 |
| | | 50 | *MAD* | 0.1221 | 0.1255 | 0.1256 | 0.1277 |
| | | | *IQR* | 1.534 | 1.802 | 1.802 | 2.03 |
| | | | $Q_n$ | 0.1234 | 0.1178 | 0.1177 | 0.1456 |
| | | 100 | *MAD* | 0.0962 | 0.1004 | 0.1003 | 0.1021 |
| | | | *IQR* | 3.312 | 3.438 | 3.343 | 3.529 |
| | | | $Q_n$ | 0.0401 | 0.0404 | 0.0404 | 0.0405 |
| 3 | 2 | 20 | *MAD* | 0.2746 | 0.3306 | 0.3306 | 0.3197 |
| | | | *IQR* | 1.655 | 0.911 | 0.911 | 0.929 |
| | | | $Q_n$ | 0.0584 | 0.0621 | 0.062 | 0.0713 |
| | | 50 | *MAD* | 0.1183 | 0.1295 | 0.1294 | 0.1272 |
| | | | *IQR* | 1.658 | 1.556 | 1.567 | 1.781 |
| | | | $Q_n$ | 0.0845 | 0.0835 | 0.0835 | 0.0851 |
| | | 100 | *MAD* | 0.112 | 0.1125 | 0.1125 | 0.1056 |
| | | | *IQR* | 1.887 | 1.946 | 1.946 | 1.941 |
| | | | $Q_n$ | 0.0445 | 0.0449 | 0.045 | 0.0454 |

Figure 4 shows the nine-mode bootstrap confidence interval radar diagram for the two types of *MAD* and $Q_n$ estimators. Figure 5 shows the bootstrap confidence interval radar diagram with nine modes for *IQR*. As can be seen, *IQR* is wider than *MAD* and $Q_n$, and for *IQR* in the four bootstrap modes, this range is almost identical. In each step, by data increase, it is observed that the confidence interval moves away from the center, but it is noticeable that the closer the graph is to the symmetry state, the fluctuations of these confidence intervals occur near the origin.

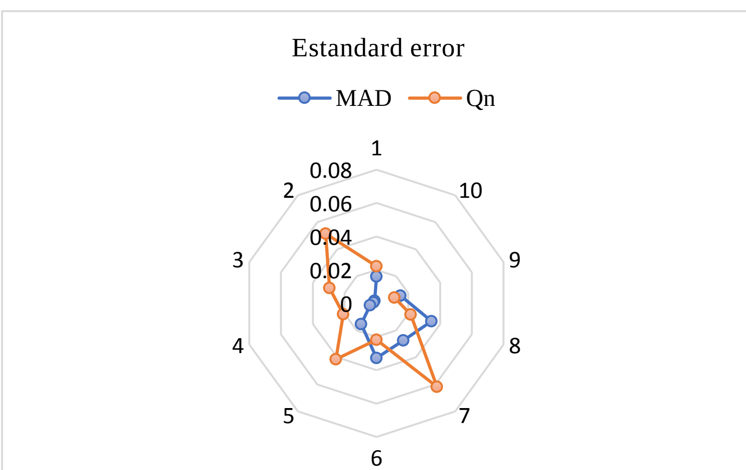


FIGURE 2
*MAD* AND $Q_n$ STANDARD ERROR RADAR DIAGRAMS

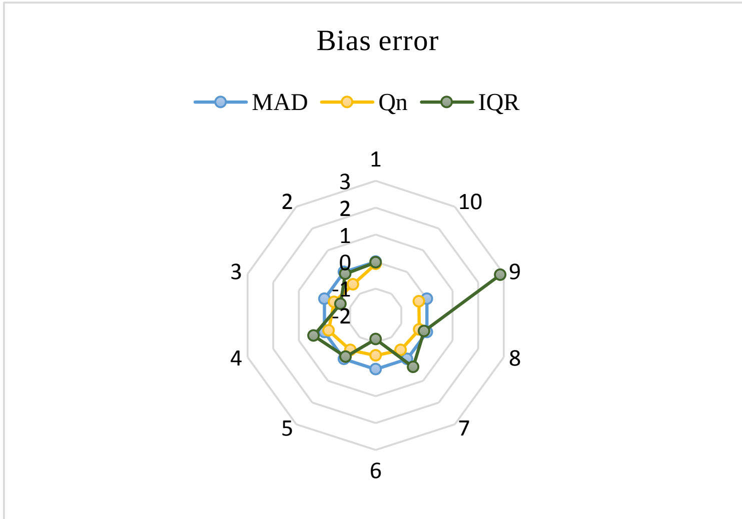


FIGURE 3
*MAD* , *IQR*, AND $Q_n$ BIAS ERROR RADAR DIAGRAM

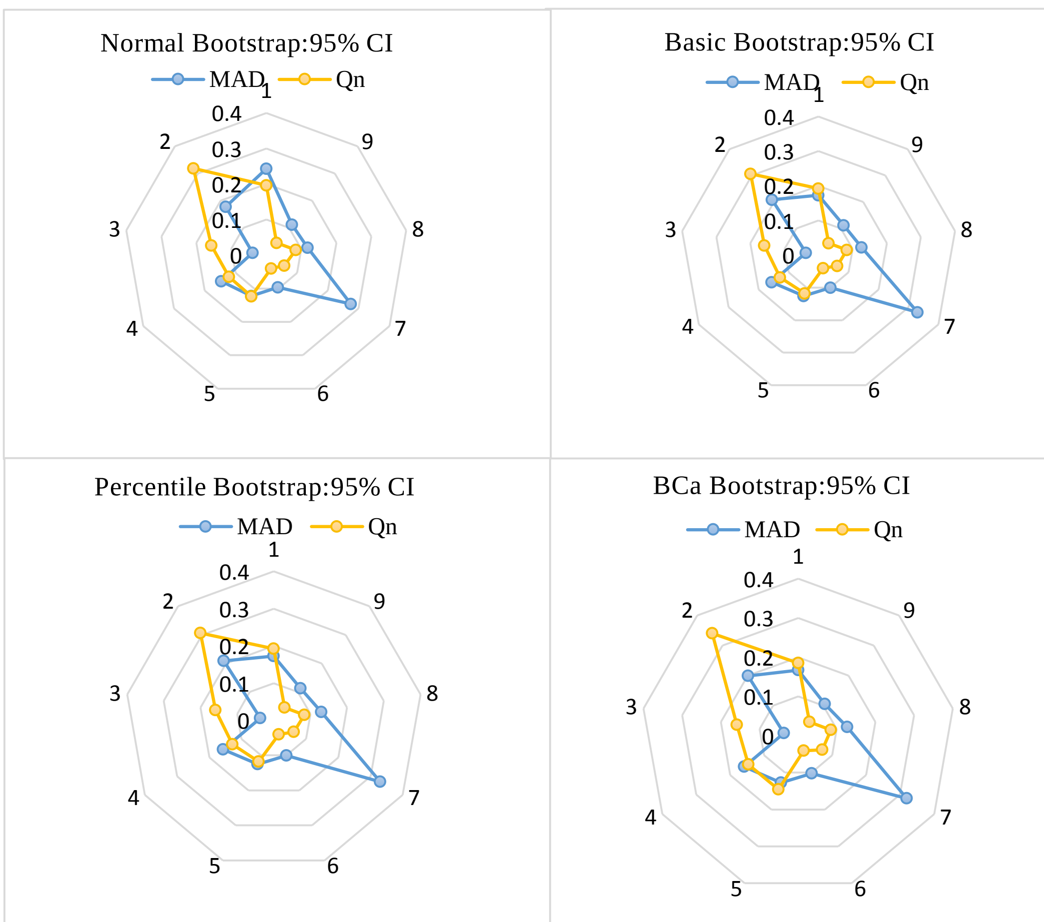


FIGURE 4
*MAD* AND $Q_n$ BOOTSTRAP CONFIDENCE INTERVAL RADAR DIAGRAMS

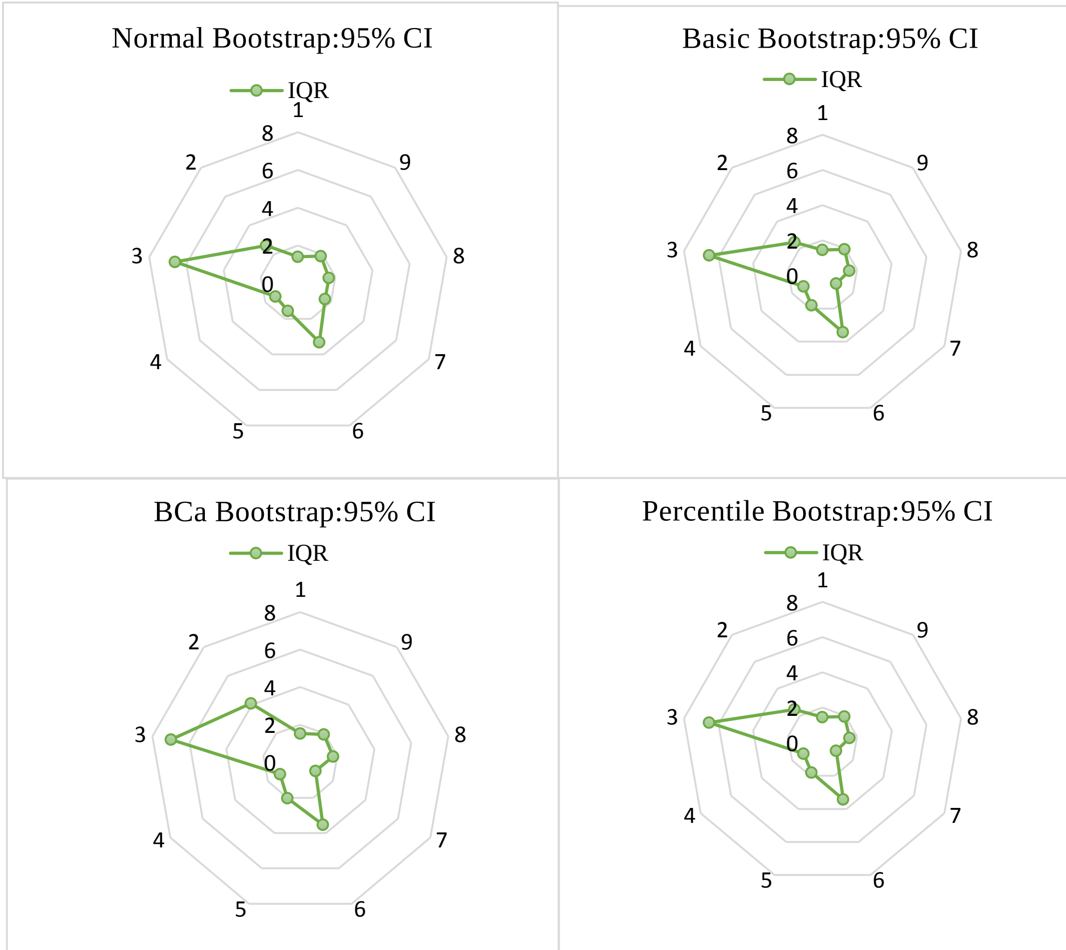


FIGURE 5
*IQR* BOOTSTRAP CONFIDENCE INTERVAL RADAR DIAGRAMS

When we study the confidence interval for a particular subject, we must ensure that when the confidence interval coverage is more focused, meaning that the product's reliability is higher than the produced products. Regarding *IQR*, one can say that according to the radar diagram of confidence intervals in Fig. 5, the bootstrap by *BCa* method is slightly wider than the other items in this figure. Regarding *MAD*, the standard bootstrap is slightly wider. So, the *BCa* bootstrap for $Q_n$it is a little better than the rest. So, we can point out that in our decisions, if we were going to use bootstrap and if the methods and data studied are following this research; it is better to use the *BCa* bootstrap method. Process capability indices have many applications, including composites and insulation between two liquids. Finally, if we want to produce such products, the quality of these products must be examined in the first step, which requires the use of process capability indices in controlling the quality of products and examining fluctuations. If we consider a non-normal process that follows the Gamma distribution, to calculating the process capability indices and evaluate them in their decisions, if $Q_n$ is used, it can be pointed out that they had a lower value than other cases. Moreover, it is worth noting that if one reduces the difference between shape values and scale, one can see that the bias error and jackknife error were acceptable compared to other cases. The *BCa* method can be used to check their confidence interval. When we increase the shape values, $Q_n$ behavior was better than the others.

## CONCLUSION

Process capability indicators are an effective tool for measuring process quality and efficiency. They are used in manufacturing industries and are a criterion for evaluating the accuracy, precision, and performance of production processes. They can be effective in analyzing reliability because they provide helpful information about reliability behavior and its improvement. *IQR* contains 50% of the information, and so it is clear that it covers a little more than the rest. From the data in the tables, it's clear that it always has higher values than the other two estimators. Also observed that $Q_n$is smaller than *MAD* due to the data selection type and its correction factor. To produce any product, we need to study and calculate the related formulas, which can have many errors. Therefore, it is better to consider a method for calculations with fewer errors. It's noticeable that every product produced has deviations that should be tried to calculate and improve. The study also noted that a product might don't follow a standard process. It makes calculating deviations a little more complicated. The method of calculating them with Gamma distribution and considering process capability indices was discussed. It was observed that the *MAD* standard error was more stable than $Q_n$. For *IQR*, it is clear that it was more unstable than the other two estimators and took on more value. If we consider the bias error, the *IQR* again shows unstable behavior. However, this time $Q_n$ seems more stable than *MAD* and *IQR*. According to the data of this research, it's noticeable that the *BCa* bootstrap method is better for $Q_n$and *IQR* and the standard bootstrap method are better for *MAD*. In other researches, no gamma function was used and they did not focus on the characteristics of q. If we consider all the characteristics of gamma and q distributions, we can reduce costs in the industry. Therefore, according to what was said, the topic of process capability indices has a wide scope and more features can be presented in relation to process capability indices. In this research, the characteristics of process capability index estimators and gamma distribution and their bootstrap confidence intervals have been combined. Therefore, considering the extent of statistical distributions and quality control topics, this research is considered as a small part of quality control topics in this field and there are many materials to fill the existing gaps. The following topics can be mentioned for future research.

- Examining the estimators of process capability indicators in multivariable mode by considering the error of the measurement tool.
- Communication between bootstrap confidence intervals of univariate and multivariate process capability indicators.
- Evaluation of process capability estimators in fuzzy mode; In the industry, many parameters.

It cannot be clearly obtained. And this issue cannot be ignored simply, this issue makes the analysis of the fuzzy set of great importance.

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