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Exact solutions for wave-like equations by differential transform method

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Abstract

Differential transform method has been applied to solve many functional equations so far. In this article, we have used this method to solve wave-like equations. Differential transform method is capable of reducing the size of computational work. Exact solutions can also be achieved by the known forms of the series solutions. Some examples are prepared to show the efficiency and simplicity of the method.

Key worda: Differential transform method; Wave-like equations

1. Introduction

The wave-like model is the integral part of applied sciences and arises in various physical phenomena. This paper investigates for the first time the applicability and effectiveness of differential transform method on wave-like equations. The basic idea of differential transform method (DTM) was initially introduced by Zhou [1] in 1986. Its main application therein was to solve both linear and nonlinear initial value problems arising in electrical circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive especially for high order equation. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years researchers have applied the method to various linear and nonlinear problems for example it was applied to partial differential equations [2], to integro-differential equations [3], to two point boundary value problems [4], to differential-algebraic equations [5], to the KdV and mKdV equations [6], to the Schrödinger equations [7] and to fractional differential equations [8].

2. Basic idea of differential transform method

The basic definitions and fundamental operations of the two-dimensional differential transform are defined as follows in [4, 9]. The differential transform function of the function $u(x,y)$ is defined as follows

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)}, \tag{1}$$

where $u(x,y)$ is the original function and $U(k,h)$ is the transformed function. The inverse differential transform of $U(k,h)$ is defined as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)(x-x_0)^k(y-y_0)^h. \tag{2}$$

In a real application, and when (x_0,y_0) are taken as $(0,0)$, then the function $u(x,y)$ is expressed by a finite series and Eq. (2) can be written as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]x^k y^h. \tag{3}$$

Eq. (3) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study we use the lower case letters to present the original functions and upper case letters stand for the transformed functions (T-functions). From the definitions of Eqs. (1) and (2), it is readily proved that the transformed functions comply with the following basic mathematical operations.

Similarity k-dimensional differential transform of $u(x_1,x_2,\ldots,x_m)$ is defined by

$$U(k_1,k_2,\ldots,k_m)=\frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\ldots+k_m}u(x_1,x_2,\ldots,x_m)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots,0)}, \tag{4}$$

where $u(x_1,x_2,\ldots,x_m)$ is the original and $U(k_1,k_2,\ldots,k_m)$ is the transformed function. The differential inverse transform of $u(x_1,x_2,\ldots,x_m)$ is defined

$$u(x_1,x_2,\ldots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}U(k_1,k_2,\ldots,k_m)x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}, \tag{5}$$

and from Eqs. (4) and (5) can be concluded

$$u(x_1,x_2,\ldots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\ldots+k_m}u(x_1,x_2,\ldots,x_m)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots,0)} x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}.$$

(6)

The following theorems can be easily stated proved

1. If $u(x_1,x_2,\ldots x_m)=f_1(x_1,x_2,\ldots x_m)\pm f_2(x_1,x_2,\ldots x_m)$, then

$U(k_1,k_2,\ldots k_m)=F_1(k_1,k_2,\ldots k_m)\pm F_2(k_1,k_2,\ldots k_m).$

2. If $u(x_1,x_2,\ldots x_m)=\lambda g(x_1,x_2,\ldots x_m)$, then

$U(k_1,k_2,\ldots k_m)=\lambda G(k_1,k_2,\ldots k_m)$, where, λ is a constant.

3. If $u(x_1,x_2,\ldots x_m)=\dfrac{\partial g(x_1,x_2,\ldots x_m)}{\partial x_i}$, then

$U(k_1,k_2,\ldots k_m)=(k_i+1)G(k_1,\ldots k_i+1,\ldots k_m),\quad 1\le i\le m.$

4. If $u(x_1,x_2,\ldots x_m)=\dfrac{\partial^{r+s}g(x_1,x_2,\ldots x_m)}{\partial x_i^{\,r}\partial x_j^{\,s}},\quad 1\le i\ne j\le m$ then

$$U(k_1,k_2,\ldots k_m)=(k_i+1)(k_i+2)+\ldots+(k_i+r)(k_j+1)(k_j+2)+\ldots(k_j+s)G(k_1,\ldots k_i+r,\ldots,k_j+s,\ldots k_m).$$

5. If $u(x_1,x_2,\ldots x_m)=x_1^{h_1}x_2^{h_2}\ldots x_m^{h_m}$ then

$U(k_1,k_2,\ldots k_m)=\delta(k_1-h_1)\delta(k_2-h_2)\ldots\delta(k_m-h_m)$, where $\delta(k_i-h_i)=\begin{cases}1, & k_i=h_i,\\ o & otherwise.\end{cases}$

6. If $u(x_1,x_2)=f_1(x_1,x_2)f_2(x_1,x_2)$, then

$$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}F_1(r,h-s)F_2(k-r,s).$$

7. If $u(x_1,x_2,x_3)=f_1(x_1,x_2.x_3)f_2(x_1,x_2,x_3)$, then

$$U(k_1,k_2,k_3)=\sum_{r=0}^{k_1}\sum_{s=0}^{k_2}\sum_{p=0}^{k_3}F_1(r,k_2-s,k_3-p)F_2(k_1-r,s,p).$$

3. Numerical Applications

In this section, we apply the differential transform method (DTM) for solving wave-like equations. The results reveal that the method is very effective and simple.

Example 1. We consider the one-dimensional initial and boundary value problem [10]

$$\frac{\partial^2 u}{\partial t^2}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}=0, \tag{7}$$

with initial condition

$$\begin{aligned} &u(x,0)=x,\\ &\frac{\partial u}{\partial t}(x,0)=x^2. \end{aligned} \tag{8}$$

Taking the differential transform of (7), leads to

$$(h+1)(h+2)U(k,h+2)-\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-2)\delta(h-s)(s+1)(s+2)U(k-r,s+2)=0. \tag{9}$$

From the initial condition given by Eq. (8) we obtain

$$\begin{aligned} &U(k,0)=\begin{cases}1 & k=1,\\ 0 & k=0,2,3,\ldots.\end{cases}\\ &U(k,1)=\begin{cases}1 & k=2,\\ 0 & k=0,1,3,\ldots.\end{cases} \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (9), if we generalize these coefficient we have

$$U(k,h)=\begin{cases}\frac{1}{h!} & k=2 \ and \ h \ is \ odd,\\ 1 & k=1 and \ h=0,\\ 0 & otherwise.\end{cases}$$

We have series solution for u and rearranging this solution yield closed from series solution as follows

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h=x+x^2\sum_{h=1,3,\ldots}\frac{1}{h!}t^h=x+x^2\sinh t,$$

which is the exact solution.

Example 2. Consider the two-dimensional wave-like equation with variable coefficients [10]

$$\frac{\partial^2 u}{\partial t^2}-\frac{1}{12}x^2\frac{\partial^2 u}{\partial x^2}-\frac{1}{12}y^2\frac{\partial^2 u}{\partial y^2}=0, \tag{11}$$

subject to the initial condition

$$\begin{aligned} &u(x,y,0)=x^4,\\ &\frac{\partial u(x,y,0)}{\partial t}=y^4. \end{aligned} \tag{12}$$

Taking the differential transform of Eq. (11) we have

$$\begin{aligned} &(m+1)(m+2)U(k,h,m+2)\\ &-\frac{1}{12}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta(r-2)\delta(h-s)\delta(m-p)(k-r+1)(k-r+2)U(k-r+2,s,p)\\ &-\frac{1}{12}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta(r)\delta(h-s-2)\delta(m-p)(s+1)(s+2)U(k-r,s+2,p)=0. \end{aligned} \tag{13}$$

From the initial condition given by Eq. (12) we have

$$\begin{aligned} &U(k,h,0)=\begin{cases}1 & k=4,h=0,\\ 0 & otherwise.\end{cases}\\ &U(k,h,1)=\begin{cases}1 & k=0,h=4,\\ 0 & otherwise.\end{cases} \end{aligned} \tag{14}$$

Substituting (14) in (13), all spectra can be found as

$$U(k,h,m)=\begin{cases}\dfrac{1}{m!} & k=4,h=0 \text{ and } m \text{ is even},\\ \dfrac{1}{m!} & k=0,h=4 \text{ and } m \text{ is odd},\\ 0 & otherwise.\end{cases}$$

Thus approximate solution of the given equation is obtained as

$$u(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U(k,h,m)x^k y^h t^m = x^4\sum_{m=0,2,\dots}\frac{1}{m!}t^m + y^4\sum_{m=1,3,\dots}\frac{1}{m!}t^m = x^4\cosh t + y^4\sinh t,$$

which is an exact solution.

Example 3. Consider the three-dimensional initial and boundary value problem [10]

$$\frac{\partial^2 u}{\partial t^2}-(x^2+y^2+z^2)-\frac{1}{2}(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2})=0, \qquad (15)$$

with initial condition,

$$u(x,y,z,0)=0$$
$$\frac{\partial u(x,y,z,0)}{\partial t}=x^2+y^2-z^2. \qquad (16)$$

The transformed version of Eq. (15) is

$$(n+1)(n+2)U(k,h,m,n+2)-(\delta(k-2)\delta(h)\delta(m)\delta(n)+\delta(h-2)\delta(k)\delta(m)\delta(n)+\delta(m-2)\delta(k)\delta(h)\delta(n))$$
$$-\frac{1}{2}\left((k-1)(k)U(k,h,m,n)+(h-1)(h)U(k,h,m,n)+(m-1)(m)U(k,h,m,n)\right)=0. \qquad (17)$$

The transformed version of Eq. (16) is

$$U(k,h,m,0)=0,$$
$$U(k,h,m,1)=\delta(k-2)\delta(h)\delta(m)+\delta(h-2)\delta(k)\delta(m)-\delta(m-2)\delta(k)\delta(h). \qquad (18)$$

Substituting (18) in (17), all spectra can be found as

$$U(k,h,m,n)=\begin{cases}\frac{(-1)^n}{n!} & \text{for } m=2 \text{ and } n=1,2,\ldots,\\ \frac{1}{n!} & \text{for } k=2 \text{ and } n=1,2,\ldots,\\ \frac{1}{n!} & \text{for } h=2 \text{ and } n=1,2,\ldots,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, the closed form of the solution can be easily written as

$$u(x,y,z,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}U(k,h,m,n)x^k y^h z^m t^n$$
$$=(x^2+y^2)\sum_{n=1,2,\ldots}\frac{1}{n!}+z^2\sum_{n=1,2,\ldots}\frac{(-1)^n}{n!}t^n=(x^2+y^2)(e^t-1)+z^2(e^{-t}-1),$$

which is an exact solution.

4. Conclusion

In this paper, we applied differential transform method (DTM) for solving wave-like equation. This method (DTM) can easily be applied to many linear and nonlinear problems and is capable of reducing the size of computational work. The fact that suggested technique solves problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method**.**

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