Linear Programming, the Simplex Algorithm and Simple Polytopes

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Abstract

In the first part of the paper we survey some far reaching applications of the basis facts of linear programming to the combinatorial theory of simple polytopes. In the second part we discuss some recent developments concurring the simplex algorithm. We describe sub-exponential randomized pivot roles and upper bounds on the diameter of graphs of polytopes.

Keywords: Simplex algorithm, Randomized Pivot rule complexity combinational theory of simple polytopes.

1. Introduction

A convex polyhedron is the intersection S of a finite number of closed half spaces in R^d. S is a d – dimensional polyhedron (briefly a d – polyhedron) If the points in S affinely span R^d a convex d-dimensional polytopes. (briefly, a d – polytope) is a bounded convex d – polyhedron. Alternatively a convex d – polytopes is the convex hull of a finite set of points which affinely spans R^d.

A (non – trivial) face F of a d-polyhedron S is the intersection of S with a supporting hyper plane. F it self is polyhedron of some lower dimension. If the dimension of F is K we call F a K face of S. The empty set and S itself are

regarded as trivial faces. o-faces of S are called vertices, I–faces are called edges and (d-1)-faces are called facets. For material on convex polytopes and for many references see Ziegler's recent book [32]. The set of vertices and (bounded) edges of S can be regarded as an abstract graph called the graph of S and denoted by G(S).

We will denote by f_K (S) the number of K-faces of S. The vector ($f_0(S)$, $f_1(S)$….. $f_d(S)$) is called the f-vector of S. Euler's fame formula V-E+F=2 given a connection between the number V, E, F of vertices, edges and 2-faces of every 3-polytope.

A convex d-polytopes (or polyhedron) is called simple if every vertex of S belongs to precisely d edges. Simple polyhedron correspond to non generate linear programming problems. When you cut a simple polytopes S near a vertex V with a hyper plane H which intersect the interior of S, the intersection S∩H is a (d-1) dimensional simplex S. The vertices of S are the intersections of edges of S which contain V with H and the (K-1) dimensional faces of S are the intersection of K faces of S with H. The following basic property of simple polytopes follows.

Let S be a simple d-polytopes and let V be a vertex of S Every set of K edges adjacent to v determines a K-dimensional faces of S which contains the vertex V. In Particular there're precisely $\binom{d}{k}$ K – faces in S containing V and altogether 2nd faces (of all dimensions) which contain V.

Linear programming and the simplex algorithm linear programming is the problems of maximizing a linear objective function ϕ subject to a finite set of linear inequalities. The relevance of convex polyhedral to linear programming problem is clear. The set S of feasible solution for a linear programming problem is a polyhedron. There are two fundamental facts concurring linear programming the reader should keep in mind.

 ❖ If ϕ is bounded from above on S then the maximum of ϕ on S   is attained at a face of S, in particular there is a vertex V for which the maximum is attained. If ϕ is not bounded from above on S then there is an edge of S on which ϕ is not bounded from above.
 ❖ A sufficient condition for V to be a vertex of S on which ϕ is maximal is that V is a local maximum namely ϕ (V) is bigger or equal than ϕ(W) for every vertex W which is a neighbour of V.

The simplex algorithm is a method to solve a linear programming problem by repeatedly from one vertex V to an adjacent vertex W of the feasible polyhedron so that in each step the value of the objective function is increased. The specific way to choose W given V is called the pivot rule.

The d-dimensional simplex and the d-dimensional cube .The d-dimensional simplex S_d is the convex hull of d+1 affinely independent points in R^d. The faces of S_d are themselves simplices. In fact, the convex hull of every subset of vertices of a simplex face and therefore $f_k(S_d) = \begin{pmatrix} d+1 \\ k+1 \end{pmatrix}$. The graph of S_d is the complete graph on d+1 vertices. The d-dimensional cube C_d is the set of all points $(x_1, x_2, x_3 \ldots \ldots x_d)$ in R_d such that for every i, $0 \le x_i \le 1$; The vertices of (c_d) are all the (0,1) vectors of length d and two vertices are adjacent (in the graph of (c_d) if they agree in all but one coordinates , $f_K ((C_d)) = 2^{d-k} \begin{pmatrix} d \\ k \end{pmatrix}$

2. *Applications of the fundamental properties of linear programming to the combinatorial theory of simple polytopes*

Let S be a simple d-polytopes, and ϕ be linear objective function which attains different values on different vertices of S. Call such a linear objective function generic. (Actually it will be enough to assume only that ϕ is not constant on any edges of S. The fundamental fact concerning linear programming is that the maximum of ϕ on S is attained at a vertex V and that a sufficient condition for V to be the vertex of S on which ϕ is maximal is that v is a local maximum, namely $\phi(V)$ is strictly bigger than $\phi(w)$ for every vertex W which is a neighborhood of V).

Every face F of S is itself a polytope and ϕ attains different values on distinct vertices of F. Among the vertices of F there is a vertex on which ϕ is maximal and again this vertex is the only vertex in F which is a local maximum of ϕ in the face F. These considerations have far reaching applications on the understanding of the combinatorial structures of simple polytopes. We refer the reader to Ziegler's Books [32] for historical notes and for reference to the original papers. Our presentation is also quite close to that in [26]. We hope that the theory of h-numbers described below will reflect back on linear programming but this is left to be seen.

3. *Degrees and h-numbers*

Let S be a simple d-polytopes and let ϕ be a generic linear objective function. For a vertex V of S define the degree V denoted by deg (v) to be the number of its neighboring vertices with smaller value of objective function . Clearly $0 \le deg(v) \le d$. Define now $h_K (S)$ to be the number of vertices of S of degree K. This number as we define it depends on the objective function ϕ but we will soon see that it is actually independent form ϕ. We can see one sign for this already nomatter what ϕ is there will always be precisely one vertex of degree d (on which ϕ attains the maximum) and one vertex of degree 0 (on which ϕ attains the minimum). This follows at once from the fact that local maximum = global maximum.

To continue will count pairs of the form (F, V) where F is a K face of S and V is vertex of F which is local maximum (hence a global maximum) of ϕ in F. on the other hand, let us compute how many pairs contain a given vertex V of S. This depends only on the degree of V. Assume that $\deg(v) = r$ and consider the set of edges of S.

$T = \{[V, W] : \phi(v) > \phi(w)\}$

Thus $|T| = r$. As we mentioned above every set B of K edges containing V determines a K-face F(B) containing V. In this face the set of edges containing V is precisely B. In order for V to be a local maximum in this face it is necessary and sufficient that for every edge [v,w] in B, $\phi(v) > \phi(w)$. This occurs if and only if $B \subset T$. Therefore, the number of K faces containing V for which V is a local maximum is precisely the number of subsets of T of size K, namely $\binom{r}{k}$ summing over all vertices V of S and recalling that $h_K(S)$ denote the number of vertices of degree K we obtain.

$$(*)\ \sum_{r=0}^{d} h_r(S)\binom{r}{k} = f_k(S), K = 0,1,2\ldots\ldots d$$

Note that this formula describe the f-vector of S ($f_0(S)$, $f_1(S)$, ………………. $f_d(S)$) as an upper triangular matrix (with ones on the diagonal) times the vector of $S(h_0(S), h_1(S)\ldots h_d(S))$. Therefore the h numbers are in fact linear combinations of the face numbers and in particular they do not depend on the linear objective function ϕ.

Put $F_p(x) = \sum_{k=0}^{d} f_k(S)x^k$, $H_p(x) = \sum_{k=0}^{d} h_k(S)x^k$

Relation (*) given

$$H_p(x+1) = \sum_{r=0}^{d} h_r(S)(x+1)^r$$

$$= \sum_{k=0}^{d} (\sum_{r=0}^{d} h_r(S)\binom{r}{k})x^k$$

$$= \sum_{k=0}^{d} f_k(S)x^k = F_p(x)$$

Therefore $H_P(x) = F_p(x-1)$ and

$$h_k(S) = \sum_{r=0}^{d} (-1)^{r-k} f_r(S) \binom{r}{k}$$

In particular

$h_0(S) = f_0(S) - f_1(S) + f_2(S) \ldots\ldots\ldots + (-1)^d f_d(S)$

$h_1(S) = f_1(S) - 2f_2(S) + 3f_3(S) \ldots\ldots\ldots + (-1)^{d-1} df_d(S)$

$h_2(S) = f_2(S) - 3f_3(S) + 6f_4(S) \ldots\ldots\ldots + (-1)^{d-2} \binom{d}{2} f_d(S)$

$h_d(S) = f_d(S) - 1$, $h_{d-1}(S) = f_{d-1}(S) - d$

$h_{d-2}(S) = f_{d-2}(S) - (d-1)f_{d-1}(S) + \binom{d}{2}$

For the simplex B_d, $h_K = 1$ for every K. The graph of B_d is the complex graph on d+ 1 vertices and for every generic objective function there will be precisely one vertex of degree K for $1 \le k \le d$. For the cube C_d, $h_k = \binom{d}{k}$. To see this consider the objective functions ϕ which is the sum of the co – ordinates . (This is nota generic objective function but it is not count on the edges of the polytopes and this is sufficient for our purposes). The vertices of degree K are precisely those having $\phi(V) = K$ and there are $\binom{d}{k}$ such vertices.

4. Euler formula and the Dehn-Sommerville relations

For a generic linear objective function there is a unique maximal vertex. Therefore, $h_0(S) = h_d(S)$ and by the formulas above we obtain.

$f_0(S) - f_1(S) + f_2(S) \ldots\ldots\ldots\ldots + (-1)^d f_d(S) = 1$ which is Euler formula usually written.

$f_0(S) - f_1(S) + f_2(S) \ldots\ldots\ldots\ldots + (1)^{d-1} f_{d-1}(S) = 1 - (-1)^d$

More generally, if ϕ is a generic linear objective function then so is -ϕ, However, if V is a vertex of a simple Polytope S and V has degree K w.r. to ϕ then V has degree d-k w.r.t -ϕ. This given the Dehn – Sommerville relation

$h_K(S) = h_{d-K}(S)$

The Dehn-Sommerville relations are the only linear equalities among face number of simple d-Polytopes.

Defination (Cyclic Polytopes) .

The cyclic d-Polytopes with n vertices denoted by C (d,n) is the convex hull of n distinct point on the moment curve x(t) =(t, t^2........ t^d) $\subset R^d$. This is a remarkable class of polytopes and the reader should consult (10, 26, 32) for their properties . C* (d,n) will denote a polar polytopes to C(d,n). (For the defination of polarity see [10, 26, 32] C* (d,n) is a simple d-polytope with n facets.

5. *The upper bound theorem*

Motzkin conjectured that the maximal number of vertices (and more generally of K – dimensional faces) for d-polytopes with n facets. This conjecture was proved by Mc mullen [23]. It is easy to reduce this conjecture to simple polytopes and to calculate the h-number of C* (d,n) see [32,26]. This gives

h_k (C* (d,n)) =h_{d-k} (C*(d,n)) = $\begin{pmatrix} n-d+k-1 \\ k \end{pmatrix}$

For 1$\leq$ K $\leq$ [d/2]

Since the face numbers are linear combination of h numbers with non – negative coefficients in upper bound theorem follows from the following relations (and the Dehnsomerville relation)

$$h_{d-k}(S) \leq \begin{pmatrix} n-d+k-1 \\ k \end{pmatrix}, \quad 1 \leq k \leq \left[\frac{d}{2}\right]$$

Proof. Consider a generic linear objective function ϕ which gives higher values to verities in a facets F than to all other vertices. (To construct such an objective function start with objective function whose maximum is attained precisely on the facet F and then make a slight perturbations to make it generic) Every vertex V of degree k -1 in F has precisely one neighborhood not in F and therefore the degree of V in k. This gives (*) h_{K-1} (F) $\leq$ h_K(S)

Next, (* *) $\sum h_k (f)$ =(k+1)h_{k+1} (S) + (d-k) h_k(S)

Where the sum is over all facets F of S

To prove (* *) consider a vertex V of degree k in S. The vertex V is adjacent to d edges and every subset of (d-1) out of them determine a facet. The degree of V is (k-1) in every facets determined by d-1 edges adjacent to V where one of the k edges pointing down (w.r.t ϕ) is deleted and there are K such facets. The degree of V is k in the remaining d-k facets (*) and (* *) gives the upper bound relations.

$$h_{\text{d-k}} (S) \leq \binom{n-d+k-1}{k} \quad \text{By induction on k.}$$

For k=1 we have equality $h_{\text{d-1}} = \text{n-d}$. For $k \geq 1$ we obtain

$(d-k+1)h_{\text{d-k+1}}(S)$ Further, assuming the upper bound relation to k-1 we obtain for k.

$$h_{\text{d-k}} (S) \leq \frac{n-d+k-1}{k}\binom{n-d+k}{k-1} = \binom{n-d+k-1}{k}$$

Abstract objective functions and telling the polytope form its graph consider an ordering α of the vertices of a simple d-polytope S for a non empty face F we say that a vertex V of F is a local maximum in F if V is larger w.r.t. the ordering α than all its neighboring vertices in F. An abstract objective function (AoF) of a simple d-polytpe and α is a linear ordering of the vertices we define, as if S is a simple d-poytope and α is a linear ordering of the vertices we define as before, the degree of a vertex V w.r.t the ordering as the number of adjacent vertices to V that are smaller than V w.r.t. α. Thus the degree of a vertex is a non negative number between 0 and d. Let h_k^α be the number of vertices of degree k. Finally, Put F(S) to be the total number of non empty faces of S.

Claim 1:

$$\sum_{r=0}^{d} 2^k h_k^\alpha \geq F(S)$$

And equality holds if and only if the ordering α is an AoF .

Proof. Count pair (F,v) were F is a non empty face of S (of any dimension) and v is a vertex which is local maximum is in F w.r.t the ordering α. On the one hand every vertex v of degree k contributes precisely 2^K pairs (F,v) corresponding to all subsets of edges from v leading to smaller vertices w.r.t. α. Therefore the number of pairs is precisely $\sum_{r=0}^{d} 2^k h_k^\alpha$ on the other hand the number of such pair is atleast F(S) (every face has atleast one local maximum) and it is equal to F(S) if every face has exactly one local maximum i.e if the ordering is an AoF.

Claim 2: A connected k-regular sub graph H of G(S) is the graph of a k-face, if and only if there is an AoF in which all vertices in H are smaller than all vertices not in H.

Proof. It H is the graph of a k-face F of S then consider a linear objective function ϕ which attains its minimum precisely at the point in F. (By definition for every non trivial face such a linear objective function exists) Now perturb Ψ a little to get a generic linear objective function ϕ in which all vertices of H have smaller values than all other vertices. On the other hand if there is an AoF, in X which all vertices in H smaller than all vertices not in H, consider the vertex v of H which the largest w.r.t. α There is a face F of S determined by the k edges in H adjacent to v and v is a local maximum in this face. Since the ordering is an AoF, v must be larger than all vertices of F hence the vertices of F are contained in H and the graph of F is a sub graph of H. But the only k-regular sub graph of a connected k-regular graph is the graph itself and therefore k is the graph of F.

Claims-1 . and 2: provide a proof to a theorem of Blind and Mari [3]

Theorem 2.1.: The combinatorial structure of a simple polytope is determined by its graph.

Indeed, claim 1 allows us to determine just form the graph all the ordering which are AoF's using this claim 2 allows to determine which sets of vertices form the vertices of some k-dimensional face. Let us mention that the proof gives a very poor algorithm (exponential in the number of vertices) and it is an open problems to find better algorithms.

Further facts without such simple geometric proofs one of the most important developments in the theory of convex polytopes is the complete descriptions of h-vertices of simple d-polytopes, conjectured by McMullen and proved by Stanley and Billena and Lee see.[2,30,24].

Crucial part of this characterization is the following. For every simple d-polytopes $h_1(S) \leq h_2(S) \leq \ldots \ldots \leq h_{[d/2]}^{(S)}$

In words the number of vertices of degree k is smaller or equal than the number of vertices of degree k+1, when $k \leq [d/2]$. It is a challenging problem to find a direct geometrical proof for this inequality. (The existing proofs have algebraic ingredients and are very difficult).

One possible measure for the progress of a certain pivot rule of the simplex algorithm would be via the degree of the vertices. Unfortunately, it seems difficult to predict how the degree of vertices will behave in a path of vertices given by some pivot rule. Starting with a random vertex of a simple polytope it is possible to say what will be the effect on the degree in a single random pivot step. By a random pivot step we mean the following. Starting with a vertex v we choose at random one of the d neighboring vertices W. It $\phi(w) > \phi(v)$. We move to w and otherwise we stay at v. The average degree $E_o(S)$ of vertices in a simple d-polytopes (which is the excepted degree of a

random vertex) is by the Dehn-Sommerville relations d/2. The average degree $E_1(s)$ of a vertex of s obtained by a single random pivot step (as described above) starting from a random vertex v is $1+2f_2(s)/f_1(s)$. For example, for the d-cube = 1/2d+1/2. (Simian formulas exist it we choose at random an r-containing v and move from v to its higest vertex. The above formula for $E_1(s)$ note that the probability that after one random pivot step we reach a (specific) vertex w of degree k is $\left[\dfrac{1}{f_0(s)}\right]\dfrac{2k}{d}$. Indeed, if we start at w (this occurs with probability $\left[\dfrac{1}{f_0(s)}\right]$ then with probability $\dfrac{k}{d}$ we stay at w. If we start with one of the k lower neighbors of w (altogether this occur with probability $\left[\dfrac{k}{f_0(s)}\right]$. then we reach w after one step with probability $\dfrac{1}{d}$. It follows that

$$E_1(s) = \frac{1}{f_0(s)}\sum_{k=0}^{d}\left(\frac{2k^2}{d}\right)h_k(s)$$

Which equals $1+\dfrac{2f_2(s)}{f_1(s)}$ by the formulas above . Note that $E_1(S)$ does not depend on the objective function. This is no longer true if we are interested in $E_2(s)$ the average degree after two random pivot steps. The following problem (of independent interest) naturally arises.

Problem 2.1 Let S be a simple d-polytopes and ϕ be generic linear objective function. Let h_{ij} be the numbers of pair of adjacent vertices v,w such that $\phi(v) < \phi(w)$ and deg (v) = i, deg (w)=j What can be said about the collection of numbers (h_{ij} $1\leq i$, $j\leq d$) . This array of numbers depends on the objective function and not only on the polytopes. It will be interesting to describe the possible h_{ij} numbers even for the special case when the Polytopes is combinatorialy isomorphic to the d-dimensional cube (The question is interesting also for abstract objective function)

6. Arrangements

We would like to close this section with the following remarks. Consider an arrangement of n hyper planes in general position in R^d, and a generic linear objective function ϕ . This arrangement divides R^d into simple d- polyhedral.

The average value of $h_k(s)$ over all these polyhedra is $\binom{d}{k}$. To see this just note that every vertex v in the arrangement belongs to 2^d , d-polyhedra and has

degree k in $\begin{pmatrix} d \\ k \end{pmatrix}$ of these polyhedra. Similarly, the average h-vector over r

dimensional faces of the arrangement is the h-vector of the r-dimensional cube.

7. *Hirsch conjecture and sub exponential randomized pivot also for the simplex algorithm*

In this section we describe recent developments concerning the simplex algorithm. We describe sub exponential randomized pivot rules and recent upper bounds for the diameter of graphs of polytopes. The algorithm we consider should be regarded in the general context of LP algorithms discovered by Megiddo [25] Clarkson [5] seidel [28]. Dyer and Frieze [7] and many others . But we will not attempt to prove this, but we give this general picture here. For the use of randomized algorithm in computational geometry the reader is referred to Mulmuley books [26]. Another word of warning is that the language we use is quite different than the usual LP terminology and we leave it to the interested reader to make the translation.

8. *The Complexity of linear Programming*

Given a linear program max (b,x) subject to Ax ≤ c with n inequalities in d variables, we denote L as the total input size of the problem when the co-efficient are described in binary. We denote $C_A(d,n,L)$ as the number of arithmetic operations needed in the worst case by an algorithm A to solve a linear programming problem with d variables , n inequalities and input size L . The (worst case) complexity of linear programming is (roughly) the function C(d,n, L) which described for every value of d,n, L the smallest possible value of $C_A(d,n,L)$ over all possible algorithms.

Khachiyan's breakthrough result [12] was that the complexity of the ellipsoid method E is a polynomial function of d,n and L namely $C_E (d,n, L) \leq S(d,n)$ L. Other algorithms which improve on Khachiyan's original bound (and also had immense practical impact on the subject) were found by Karmarkar and many others. By considering solutions to all subsets of d from the n inequalities we can easily see that $C(d,n, L) \leq f(d,n)$ i.e. linear programming can be solved by a number of arithmetic operation which is a function of d and n and independent of the input size L It is an outstanding open problem to find a strongly polynomial algorithm for linear programming, that is to find an algorithm which requires a polynomial number in d and n of arithmetic operations which is independent from L. Denote $C(d,n) = max_L C(d,n,L)$ Klee and Minty [18] and subsequently others have shown that several common pivot rules for the simplex algorithm are exponential in the worst case. Explaining the excellent performance of the simplex algorithm in practice (especially in view of the exponential worst case behavior on various Pivot rules) is a major challenge of the simplex algorithm. The result on the average case behaviour provide one such explanation. (see Borgwardt's book [4] for a

description of his work and for references to otherworks in [29]. The fact that the complexity of linear programming is a polynomial (by Khachiyan's result) even if not via the simplex algorithm provide another practical explanation.

Of course, finding a pivot rule which requires a polynomial number of steps in the worst case or even proving that there are always a polynomial number of Pivot steps leading to the optimal vertex (without prescribing an algorithm to find these steps) are very desirable.

9. *Using randomness for Pivot Rules*

We will consider now randomized algorithms. Namely, algorithm which depend on internal random choices .Given such a randomized algorithm A we denote by C_A^R (d, n) the excepted number of arithmetic operation needed – in the worst case – by A on a LP-Problem will d variables and n inequalities. $C^R(d,n) \leq C(d,n)$. (Note we are interested in a worst case analysis of the average running time where the randomization is internal to the algorithm. This is in contrast with average case analysis where the LP problem itself is random. Perhaps the simplest random pivot rule is to choose at each step at random with equal probabilities a neighboring vertex with a higher value of the objective function. Unfortunately it seems very difficult to analyses this rule for general problem. Recently Gartner, Henk and Ziegler [9] managed to analyze the behavior of random pivoting on the Klee – Minty cube.

10. Hirsch Conjecture

Let Δ (d,n) denote the maximal diameter of the graphs of d-polyhedra S with n facets and Let $\Delta_b(d,n)$ denote the maximal diameter of the graphs of d-polytopes with n-vertices. Given a d-polyhedron S, a linear objective function ϕ which is bounded from above on S and a vertex v of S, denote by m(v) the minimal length of a monotone path in G(S) from v to a vertex of S on which ϕ attains its maximum. Let H(d,n) be the maximum of m(v) overall d-polyhedral S with n facets, all linear functionals ϕ on R^d and all vertices v of S (A monotone path is a path in G(S) on which ϕ is increasing). Let M(d, n) be the maximal number of vertices in a monotone path in G(S) over all d-polyhedra S with n facets and all linear functional ϕ on R^d. Clearly.

$$\Delta (d, n) \leq H (d,n) \leq M(d,n)$$

Here H (d,n) can be regarded as the number of pivot steps needed by the simplex algorithm when the pivots are chosen by an oracle in the best possible way. M(d,n) can be regarded as the number of pivot steps needed when pivots are chosen by an adversary in the worst possible way. In 1957 Hirsch conjecture is false for unbounded polyhedra. The Hirsch conjecture for polytopes is still open. The special case asserting that $\Delta_b(d,2d) = d$ is called the

d-step conjecture and it was shown by Klee and walkup to imply the general case.

Theorm 3.1 (Klee and walkup [19], 1967)

$\Delta (d,n) \geq$ n-d +min $\{[d/4],[(n-d)/4]\}$

Theorem 3.2 (Holt and Klee [11], 1997) for all d $\geq$14 and n>d

Δ_b (d,n) $\geq$ n-d

Theorem 3.3 (Larman [20], 1970)

Δ (d,n) $\leq$ n2^{d-3}

Theorem 3.4 (Kalai and Kleitmann [17], 1992

$$\Delta(\text{d,n}) \leq n \binom{\log n + d}{\log n} \leq n^{\log d + 1}$$

Klee and Minty [18] considered a certain geometric realization of the d-cube (called now the Klee – Minty cube) to show that Klee and Minty [18] considered a certain geometric realization of the d-cube (called now the "Klee – Minty cube") to shown that Theorem 3.5

Theorem 3.5 (Klee and Minty [18], 1972, M(d,2d) $\geq$2^d.

Subex potential randomized pivot rules.

We will assume (and thre is no loss of generality assuming this) that the LP problem is non-degenerate (i.e the feasible polyhedron is simple) and that a vertex v of the feasible polyhedron is given with a slight change of terminology all the algorithms and results we describe apply to the degenerate case. Several years ago the author [16] and independently Matousek, Shanier and Welzel [22] found a randomized sub exponential pivot rule for LP thus proving that C^R(d,n) $\leq \exp(K \sqrt{d \log n}$). Slightly sharper bounds are described below). In my paper various variants of the algorithm were presented and we will see here two variants. The first and simplest variant is one of our originals and is equivalent (in a dual – setting) to the sharier – Welzel algorithm (27) on which (22) is based. The second variant presented here is a joint work with Martin Dyer and Nimrod Negiddo. It is a better and simplified version of other variants from [16]. All these algorithms apply to abstract objective functions and even more general's settings see also Gantner's paper [8]. Consider an LP problem of optimizing a linear objective function ϕ over a d-polyhedron S and a vertex v of S. our aim is to reach top (S) which is a vertex of S on which the objective function is maximal or an edge of S on which the objective functions is unbounded from above.

Algorithm -1

Given a vertex v ∈ S choose a facets F containing v at random.

Apply the algorithm on F until reaching w=top (F)

Repeat the algorithm from w

Remark : The algorithm terminates if v = top (S). It v = top (F$^/$) for some facet F containing v (in which case v has only one improving edge) we choose F at random from the other d-1 facets containing F.

(Unless v= tops (S) there is at must one such facets F$^/$)

Algorithm –II Chose at random an ordering of the facets $F_{\pi(1)}$, $F_{\pi(2)}$ …… $F_{\pi(n)}$.

Phase – 1 Apply the algorithm until you reach a vertex in $F_{\pi(1)}$ (on reach Top (S)).

Phase –II : Apply the algorithm recursively inside F_π until reaching w=top($F_{\pi(1)}$)

Phase –III Delete the facts $F_{\pi(1)}$ from the ordering and continue turn the algorithm from w.

Phase I and phase –III are performed w.r.t. initial random order of the n inequalities but in phase II you have to find again a new random a ordering of the facets.

11. Analysis of the rules

We say that a facet F of S is active w.r.t v if ϕ (v) < max {ϕ(x);x∈ F}. We will study the number of pivot steps as a functions of the number of variables d and the number of active facet. The number of pivot step avail not depend on the total number of facets n. However, we do not assume that we know while running the algorithms which facts are active and the number of arithmetic operations per pivot step depends therefore (poly nominally) also on N note that in Algorithm II only the ordering of the active facets matters.

For a linear programming problems U with d variables and N inequalities and a feasible vertex v of U such that there are n active facets v, we denote by f(U,v) the excepted number of pivot steps needs by algorithm I on the problem U starting with the vertex v. f(d,n) denote the maximal value of f(U,v) over all problems U and vertices v. The function f(d,n) is not decreasing with n. Similarly, g(d,n) will be the average number of pivot steps in the worst case problem for Algorithm – II

Analysis of Algorithm 1 we start with a situation where there are n active facets let F_1, F_2, F_3…………. F_d be the facets containing v, ordered such that φ (top (F_1) ≤ φ ((top (F_2)) ≤ …….. φ (top (F_d).Note that (unless v = top (S) at most one namely only F_1) of there facets can be non – active. The average number of steps needed to reach top (F) from v is at most f(d-1, n-1)

If F_1 is active then with probability 1/d the chosen random facet F equals F_i for i = 1, 2, 3 ………….. d and then after reaching w = top (F) there are at most n-I active facts remaining and the average number of steps needed to reach top (S) from w is at most f(d,n-i+1). Aver aging over i we get that the average number of steps needed to reach top (S) from w is at most $\frac{1}{d}\sum_{i=1}^{d} f(d, n-i)$

If F_1 is not active the F = F_i with probability $\frac{1}{d-1}$ for i = 2, 3…d and by the same taken the average number of steps needed to reach top (S) from w is at most $\frac{1}{d-1}\sum_{i=1}^{d-1} f(d, n-i)$. This is (slightly) higher than the previous expression by the montonicity of f(d,n) as a function of n. In sum

$$F(d,n) \leq f(d-1, n-1) + \frac{1}{d-1}\sum_{i=1}^{d-1} f(d, n-i)$$

This given f(d,n) $\leq \exp(K\sqrt{n\log d})$ see [22]

Analysis of Algorithm – II

For phase –II we need at most G(d-1), n-1) steps on the average. For phase III we can repeat the argument of the previous algorithm with probability 1/n there are (at most) n-I active facets let after reaching top ($F_{\pi(i)}$) for i = 1, 2, …. n so the average number of pivot step for this phase is at most $\frac{1}{n}\sum_{i=1}^{n} g(d-1, i)$. We claim now that the average number of pivot steps for phase 1 is also at most $\frac{1}{n}\sum_{i=1}^{n} g(d-1, i)$ To see this note.

As long as we run the algorithm from v meeting only vertices in r active facets we can regard our self running the algorithm from v in the LP problem obtained by deleting the inequalities corresponding to the other active facets. This LP problem has only r active facets. Since the average number of pivot steps needed for this problem is at most g(d,r) we conclude that after an average number of g(d,r) pivot steps teaking running the algorithm while meeting vertices on r active facets.

The pivot steps taken running the algorithm white meeting vertices on r active facets do not depend on the ordering of the remaining active facets. Therefore the identity of the active facets to be the next we meet. (which is a probability distribution on the remaining active facets) does not depend on the ordering of the remaining n-r active facets. It follows that with probability 1/n the facts $F_{\pi(i)}$. will be the ith active facet to be met i= 1,2,………..

So we get g(d,n) $\leq g(d-1, n-1) + \dfrac{2}{n}\sum_{i=1}^{n} g(d, n-i)$

This relation implies the following.

1. g(d,n) $\leq$ exp $(k\sqrt{\log n})$

2. If d and n are comparable we get a better estimate $g(d, T_d) \leq$ exp [k(T) √d] [K(T) is a constant depending on T]

3. The following estimates are useful when t =n-d is small w.r.t n

   g(d, d+t) $\leq K(\left(\dfrac{1+\in}{\in}\right)^{t} d^{\in}$, g(d,d+1)$\leq$ k(logd)$^{t\text{-}1}$ These bounds apply to f(d, d+t) as well

4. The following estimates are useful when d is small w.r.t n

   g(d,n) $\leq$ K$^{k}\left(\dfrac{2+\in}{\in}\right)^{d} n^{1+\in}$ for every $\in$ >0 and g(d, n) $\leq$ k(logn)$^{d\text{-}1}$n

   It is possible to use generating function techniques to get a precise asymptotic for f(d,n) and g(d,n). It follows from the recession that n! g(d,n) is bounded above by t(d,n) – the number of permutations of {1,2, ….n} such that each cycle in the permutation (considered as a product of disjoint cycles) is decorated by a nonnegative integer an by a plus or minus sign such that the sum of the integers is d. For t(d,n) there is the closed formula.

   t(d,n) = $\sum 2^{k} C(n,k)\begin{pmatrix} d+k-1 \\ k-1 \end{pmatrix}$

   When C(n,k) is the number of permutation of {1, 2..n} with k cycles. (C(n,k) is the absolute value of the stirling number of first kind) However, for the asymptotic facts describe above (without getting the precise constants) the simplest proofs are by direct estimations.

Remark.   matousek [21] found remarkable classes of abstract objective functions on the d-dimensional cube for which the except number of pivot steps for Algorithm I described above is indeed exp (c√d). Further understanding of similar examples may give impression on some of the problems described in this section.

Lp duality to LP duality allows us to move form a problem with d variables and n inequalities to the dual problems with n-d variable and n inequalities.

12.Conclusion

The situation develop due to Hirsch conjecture and the complexity of the simplex algorithm is rather frustrating. Again we are short of Polynomial bounds for the diameter and despite the simplicity of the proofs for the known bounds we can not modify them any further. For n=2d we can not find a randomized pivot rule which will require $\exp(d^{1/2-\epsilon})$ pivot steps for some $\epsilon > 0$, even if the feasible polytope is combinatorially equivalent to to a d − dimensional cube. And we can not find a deterministic pivot rule (without randomization) which is not exponential. We leave these tasks for the reader.

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