The cost and income efficiency models with MOLP structure

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Abstract

In this paper, the cost and income efficiency models have been considered with regard to the multiple objective programming structures. For finding the efficient points in purposed MOLP problem, some various methods like Lexicography & the weighted sum can be used. So by introducing the MOLP problem the cost & income efficiencies will be achieved. In the present study, the MOLP problem is converted to an objective function by the weighted sum of the objective function. So we determine the efficient hyperplanes gradient by finding the dual linear programming (LP) problem.

Keywords: DEA, Multiple objective programming, efficiency

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1.Introduction

Charnes introduced a nonparametric method for measuring the efficiency of decision making units (DMUs) in [2]. In the case of multiple inputs/multiple outputs, efficiency is defined as the ratio of the weighted sum of inputs to the weighted sum of outputs, and the maximum efficiency score is 1. In other words, if the DMU lies on the frontier, it is considered efficient, otherwise inefficient. In

DEA, we use a linear programming problem to evaluate each DMU. Therefore, by considering each DMU in its best condition, we obtain the best weights for it, and then we determine its efficiency.In the second section MOLP and DEA said nutshell, the third section, the cost & income efficiency model is introduced. In the fourth section , the multipurpose programming problem is offered . At the end of the present research, we also have a conclusion.

2.MOLP and DEA

In this section, a brief description of multiple objective linear programming and data envelopment analysis is provided. A close study of the Pareto optimal solution of MOLP and the efficient units of DEA can be useful in understanding the relation between MOLP and DEA, considering their similar structures.

2.1 Multiple objective linear programming

We consider the following multiple objective linear programming problem:

$$Max \quad (f_1(x), f_2(x), ..., f_l(x))$$
$$s.t. \quad Ax = b \tag{1}$$
$$x \geq 0$$

where $f_i(x) = c^i x$ for $i = 1, ..., l$ are the objective functions, $A \in R^{m \times n}$ is the constraint matrix, $b \in R^m$ is the right-hand side vector and $x \in R^n$ is the vector of variables. We shall denote the feasible set of the (1) by X. In the following we assume, without loss of generality, that X is non empty. The objective function can be written as $C^T x$, where $C \in R^{n \times l}$ has columns c^i . A solution $x^* \in X$ of (1) is (weakly) Pareto optimal if there is no $x \in X$ such that $C^T x \geq C^T x^*$ and $C^T x \neq C^T x^*$. A fundamental result of multiple objective linear programming states that (weakly) Pareto optimal solutions of (1) can be characterized as optimal solutions of single-objective linear programs with a convex combination of objectives $c^i, \quad i = 1, ..., l$.

Let $\hat{\Lambda} = \{\lambda \in R^l : \quad \lambda \geq 0, \quad \sum_{i=1}^{l} \lambda_j = 1\}$ and $\Lambda^0 = \{\lambda \in R^l : \lambda > 0, \sum_{j=1}^{n} \lambda_j = 1\}$.

Theorem 1. A feasible point $x \in X$ is (weakly) Pareto optimal for (1) if and only if there exists $\lambda \in \Lambda^0$ $(\lambda \in \hat{\Lambda})$ such that x is an optimal solution of

$$Max \quad \sum_{i=1}^{l} \lambda_i (c^i)^T (x)$$
$$s.t.$$
$$Ax = b$$
$$x \geq 0 - 0.3 \tag{2}$$

The proof can be found in Steuer[4].

2.2 Data Envelopment Analysis

Consider n $DMUs$ with m inputs and s outputs. The input and output vectors of DMU_j $(j=1,\dots,n)$ are $X_j = (x_{1j},\dots,x_{mj})^t, Y_j = (y_{1j},\dots,y_{sj})^t$, respectively, where $X_j \geq 0, X_j \neq 0, Y_j \geq 0, Y_j \neq 0.$ We define the most general production possibility set T as follows:

$$T = \left\{ (X,Y) : X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda \in \Lambda \right\}$$

where Λ is one of the following:

$$\Lambda_c = \{\lambda \mid \lambda \geq 0\},$$

$$\Lambda_v = \{\lambda \mid \sum_{j=1}^{n} \lambda_j = 1, \lambda \geq 0\},$$

$$\Lambda_{NI} = \{\lambda \mid \sum_{j=1}^{n} \lambda_j \leq 1, \lambda \geq 0\},$$

$$\Lambda_{ND} = \{\lambda \mid \sum_{j=1}^{n} \lambda_j \geq 1, \lambda \geq 0\} \text{ where } \lambda = (\lambda_1,\dots,\lambda_n) \in \mathsf{R}^n.$$

Therefore, we obtain four production possibility sets, in which we denote T by T_c, T_v, T_{NI}, T_{ND}, when $\lambda \in \Lambda_c$, $\lambda \in \Lambda_v$, $\lambda \in \Lambda_{NI}$ and $\lambda \in \Lambda_{ND}$, respectively. When a DMU_o, $o \in \{1,2,\dots,n\}$, is under evaluation, we use the input-oriented DEA model proposed by Banker [1] as follows:

$$Min \quad \theta - \varepsilon[\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+]$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta_{io}, \qquad i=1,\dots,m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \qquad r=1,\dots,s \qquad (3)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j=1,\dots,n$$

$$s_i^- \geq 0, s_r^+ \geq 0, \quad for\ all\ i, r$$

where $\varepsilon > 0$ is a so-called non-Archimedean element defined to be smaller than any positive real number. This is equivalent to solving (3) in two stages by first minimizing θ, then fixing $\theta = \theta^*$, where the slacks are to be maximized without altering the previously determined value of $\theta = \theta^*$. Clearly, the evaluated DMU_o is efficient if and only if $\theta^* = 1$ and all slack variables in the optimal solution are zero in problem (3).

3.The cost and income efficiency model

Consider n DMUs with m inputs and s outputs. The input and output vectors of DMU_j $(j = 1,\ldots,n)$ $X_j = (x_{1j},\ldots,x_{mj})^t$, $Y_j = (y_{1j},\ldots,y_{sj})^t$ where $X_j \geq 0$, $X_j \neq 0$, and $Y_j \geq 0, Y_j \neq 0$.

If the objective of the production unit(s), or the objective assigned by the analyst, is cost minimization, then the input prices C $\geq$ 0 must be known. The overall minimum cost of producing output vector y_o is obtained by solving the following:

$$\underset{x,\lambda}{Min} c^T X$$

$$s.t.$$

$$X \geq \sum_{j=1}^{n} X_j \lambda_j,$$

$$Y_o \leq \sum_{j=1}^{n} Y_j \lambda_j, \tag{4}$$

$$\sum_{j=1}^{n} 1^T \lambda_j = 1,$$

$$\lambda_j \geq o, \ j = 1,\ldots,n.$$

for a VRS frontier. Overall cost efficiency (OEi) is determined by dividing overall minimum cost $c^T x^*$ by observed cost:

$$OE_i = \frac{c^T x^*}{c^T x_o}$$

where $0 \leq OEi \leq 1$. Overall cost efficiency can be decomposed into two measures: input radial technical efficiency (TEi) and input allocative efficiency (AEi) where

$$OEi = TEi \ . \ AEi. \tag{5}$$

Once technical efficiency is obtained by solving a model such as CCR model, input allocative efficiency can be derived from (5).

Alternatively, if the objective of the DMUs is known or assumed to be revenue maximization, then actual revenue can be calculated for each observation

provided the output prices $r \geq 0$ are known. See [3]. The overall maximum revenue for input vector $x_\circ$ is obtained from the following:

$$\underset{Y,\lambda}{Max}\, r^T Y$$

$$s.t.$$

$$X_\circ \geq \sum_{j=1}^{n} X_j \lambda_j \,,$$

$$Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \tag{6}$$

$$\sum_{j=1}^{n} 1^T \lambda_j = 1,$$

$$\lambda_j \geq \circ, \; j=1,\ldots,n.$$

and overall revenue efficiency is defined as

$$OE_0 = \frac{r^T Y^*}{r^T Y_\circ} \geq 1. \tag{7}$$

4.The suggestive MOLP problem:

With regard to models (4),(5), the MOLP problem is regarded as follow:

$$Min \; \left\{ \frac{C^T X}{C^T X_\circ}, -\frac{r^T Y}{r^T Y_\circ} \right\}$$

$$s.t.$$

$$\sum_{j=1}^{n} \lambda_j X_j \leq X \,,$$

$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y \,, \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j = 1 \,,$$

$$\lambda_j \geq 0 \,, \; j=1,\ldots,n$$

$$X \geq 0, Y \geq 0.$$

For having solution, the Lexicography method can be used. In this method, we can optimize the first objective function then by using its answer, we can optimize the second objective function. See[4]. Now by using the weighted sum of objective function & with regard to the

$$W = \left\{ (w_1, w_2) \,|\, w_1 + w_2 = 1 \,, \qquad w_1, w_2 > 0 \right\}. \tag{9}$$

The MOLP problem (8) is converted as follow:

$$Min \; (\frac{w_1}{c^T X_\circ} c^T X - \frac{w_2}{r^T Y_\circ} r^T Y)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j X_j \le X \quad ,$$

$$\sum_{j=1}^{n} \lambda_j Y_j \ge Y \quad ,$$ (10)

$$\sum_{j=1}^{n} \lambda_j = 1 \; ,$$

$$\lambda_j \ge 0 \; , \; j = 1,...,n$$

$$(w_1, w_2) \in W,$$

$$X \ge 0, Y \ge 0.$$

With regard to the u_0, U, V variables dual model (10) is regarded as follow:

$$\max \; u_0$$

s.t.

$$U^T Y_j - V^T X_j + u_0 \le 0, \qquad j = 1,...,n,$$ (11)

$$-U \le -\frac{w_2}{r^T Y_0} r \; ,$$

$$V \le \frac{w_1}{c^T X_O} c.$$

Now with regard to the conditions $Y^* > 0, X^* > 0$ and by using the additional complement conditions.

$$If \quad X^* > 0 \longrightarrow V^* = \frac{w_1}{c X_o} c$$ (12)

$$If \; Y^* > 0 \longrightarrow U^* = \frac{w_2}{r Y_o} r$$ (13)

so the efficient hyperpelans is written as follow:

$$U^* Y - V^* X + u^*_0 = 0$$ (14)

By selecting the various weights from the W set , there are many different functional infra papers.

5.Numerical example

We consider eight DMUs with one input and one output in Table 1 as follows:

Table 1: The data of inputs and output

DMU	A	B	C	D	E	F	G	H
I_1	3	3	4	6	7	9	7	7
O_1	2	3	4	6	6	6	4	2

We suppose w=(1,1), c=1 and r=1 in the all of cases. Solving (11) numerically yields to the following results:

Table 2: The results from model (11)

DMU	V_1^*	U_1^*	u_0^*
A	0.3	0.5	-1.2
B	0.3	0.3	0
C	0.25	0.25	0
D	0.16	0.16	0
E	0.14	0.16	-0.12
F	0.11	0.16	-0.3
G	0.14	0.25	-0.66
H	0.14	0.5	-2.16

6.Conclusion

As a result of finding the cost efficiency & the income efficiency, the two problems in the linear programming will be solved. But by using one MLP program, the cost efficiency & the income efficiency will be achieved. By using the weighted sum of the objective function, we determine the efficient hyperplanes gradient by finding the dual linear programming (LP) problem.

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