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Chebyshev Acceleration Technique for Solving Fuzzy Linear System

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Abstract

In this paper, Chebyshev acceleration technique is used to solve the fuzzy linear system (FLS). This method is discussed in details and followed by summary of some other acceleration techniques. Moreover, we show that in some situations that the methods such as Jacobi, Gauss-Sidel, SOR and conjugate gradient is divergent, our proposed method is applicable and the acquired results are illustrated by some numerical examples.

Keywords: Acceleration techniques; Chebyshev acceleration technique; Fuzzy system of linear equations; Iterative methods.

1. Introduction

Equations involving fuzzy numbers are the most important ingredients in many fields such as mathematics, physics, statistics, etc. Since in many applications at least some of the system's parameters and concepts are represented by fuzzy numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by Zadeh [16], and in [6]. A general model for solving an $n \times n$ FLS which coefficient matrix is crisp and the right-hand side is arbitrary fuzzy number vector was first proposed by Friedman et al. [9]. Afterwards, in the literature on fuzzy linear system of equations various methods were proposed to solve such systems, see [1-4, 8, 12-15]. In this paper we propose a new method based on Chebyshev acceleration technique to deal with FLS problems.

This paper is organized in 6 sections. In Section 2 we introduce fuzzy linear systems. In Sections 3, 4, we respectively give the Chebyshev acceleration (CA) technique and some convenient iterative methods. In Section 5, we examine the advantage of the CA technique for solving fuzzy linear systems. Finally, we conclude in Section 6 based on the obtained results from numerical examination as given in Section 5.

2. *Fuzzy linear system*

Following [9] we represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$.
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number a is simply represented by $\underline{u}(r) = \overline{u}(r) = a, 0 \leq r \leq 1$. By appropriate definitions, the fuzzy number space $\{\underline{u}(r), \overline{u}(r)\}$ becomes a convex cone E^1 which is then embedded into a Banach space.

Definition 2.1 For two arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$ and a real number k, equality, summation and scalar multiplication on fuzzy numbers are defined as

1. $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.
2. $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
3. $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Definition 2.2: The $n \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n, \end{aligned} \qquad (1)$$

(1)

where the coefficient matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in E^1, 1 \leq i \leq n$ is a called fuzzy linear system (FLS).

Definition 2.3 A fuzzy number vector $(x_1, x_2, \dots, x_n)^t$ given by

$$x_i = \left(\underline{x}_i(r), \overline{x}_i(r)\right), 1 \leq i \leq n, \ 0 \leq r \leq 1,$$

is called a solution of the fuzzy system if

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i . \end{aligned} \qquad (2)$$

(2)

In general, however, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ may include a linear combination of $\underline{x}_j$'s and $\overline{x}_i$'s. Consequently, in order to solve the system given by Eq. (1) one must solve a $(2n) \times (2n)$ crisp linear system where the right-hand side column is the function vector $(\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n, -\overline{y}_1, -\overline{y}_2, \dots, -\overline{y}_n)^t$.

Let us now rearrange the linear system of Eq. (2) so that the unknowns are $\underline{x}_j, (-\overline{x}_i), 1 \le i \le n$ and the right-hand side column is

$Y = (\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n, -\overline{y}_1, -\overline{y}_2, \dots, -\overline{y}_n)^t$.

We get the $(2n) \times (2n)$ matrix $S = (s_{ij}), 1 \le i, j \le 2n$, where s_{ij} are determined as follows:

$$\begin{aligned} &\text{if } a_{ij} \ge 0 \text{ then } s_{ij} = s_{i+n, j+n} = a_{ij} \\ &\text{if } a_{ij} < 0 \text{ then } s_{i+n, j} = s_{i, j+n} = -a_{ij} \end{aligned} \qquad (3)$$

and any s_{ij} which is not determined by Eq. (3) is zero. Using matrix notation we get

$$SX = Y, \qquad (4)$$

where

$$X = \begin{pmatrix} \underline{x}_1 \\ \cdot \\ \cdot \\ \underline{x}_n \\ -\overline{x}_1 \\ \cdot \\ \cdot \\ -\overline{x}_n \end{pmatrix}, Y = \begin{pmatrix} \underline{y}_1 \\ \cdot \\ \cdot \\ \underline{y}_n \\ -\overline{y}_1 \\ \cdot \\ \cdot \\ -\overline{y}_n \end{pmatrix}.$$

The structure of S implies that $s_{ij} \ge 0, 1 \le i, j \le n$ and that $S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ where B contains the positive entries of A, C contains the absolute values of negative entries of A and $A = B - C$.

Theorem 2.4 The matrix S is nonsingular if and only if the matrices $A = B - C$ and $B + C$ are both nonsingular.

Theorem 2.5 If S^{-1} exists it must have the same structure as S.

Theorem 2.6 The unique solution X, that is $X = S^{-1}Y$ is a fuzzy vector for arbitrary Y if and only if S^{-1} is nonnegative, i.e.

$$(S^{-1})_{ij} \ge 0, 1 \le i, j \le 2n.$$

Theorem 2.7 The inverse of a nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix.

We now restrict the discussion to triangular fuzzy numbers, i.e. $\underline{y}_i(r), \overline{y}_i(r)$ and consequently $\underline{x}_i(r), \overline{x}_i(r)$ are all linear functions of r. Having calculated X which solves Eq. (4) we now define the 'fuzzy solution' to the original system given by Eq. (1).

Definition 2.8: Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n\}$ denote the unique solution of Eq. (4). The fuzzy number vector $u = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \le i \le n\}$ defined by

$$\underline{u}_i(r) = \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\},$$
$$\overline{u}_i(r) = \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}, \quad (5)$$

Is called the fuzzy solution of *SX=Y*.

The use of $\underline{x}(1)$ in Eq. (5) is meant to eliminate the possibility of fuzzy numbers whose associated triangle possess an angle greater than $90°$. If $(\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n$, are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r), \overline{u}_i(r) = \overline{x}_i(r), 1 \le i \le n$ and *U* is called a strong fuzzy solution. Otherwise, *U* is a weak fuzzy solution.

3. *Chebyshev acceleration technique:*

In the SOR method [7], a parameter ω is adjusted to make spectral radius of the iteration matrix $M = R^{-1}T$ as small as possible. We now consider acceleration technique where the iteration matrix is fixed but the structure of the iteration is altered to increase the convergence speed. An iterative method to solve *SX=Y* where *S* is extended matrix corresponding to original matrix *A*, has the following form:

$$Rx_{k+1} = Tx_k + Y, \quad (6)$$

where $S = R - T$ is a splitting for the extended coefficient matrix *S*. Following [10], we wish to determine coefficients $v_j(k), j = 0, \ldots k$ such that

$$w^{(k)} = \sum_{j=0}^{k} v_j(k) x^{(j)},$$

represents an improvement over x^k. If $x^{(0)} = \cdots = x^{(k)} = x$, then it is reasonable to insist that $w^{(k)} = x$. Hence, we require

$$\sum_{j=0}^{k} v_j(k) = 1.$$

subject to this constraint, we consider how to choose the $v_j(k)$ so that the error in $w^{(k)}$ is minimized.

We know that $x^{(k)} - x = (R^{-1}T)^k e^{(0)}$ where $e^{(0)} = x^{(0)} - x$, we see that

$$w^{(k)} - x = \sum_{j=0}^{k} v_j(k)(R^{-1}T)^j e^{(0)}.$$

Working in the 2-norm we therefore obtain

$$\left\| w^{(k)} - x \right\|_2 \le \|p(M)\|_2 \left\| e^{(0)} \right\|_2,$$

Where $p_k(z) = \sum_{j=0}^{k} v_j(k) z^j$ such that $p_k(1) = 1$.

At this point we assume that *M* is symmetric with eigenvalues λ_i that satisfy

$$-1 < s \le \lambda_n \le \cdots \le \lambda_1 \le l < 1.$$

It follows that

$$\|p_k(M)\|_2 = \max_{\lambda_i \in \lambda(A)} |p_k(\lambda_i)| \le \max_{s \le \lambda \le l} |p_k(\lambda)|.$$

Thus, to make the norm of $p_k(M)$ small, we need a polynomial $p_k(z)$ that is small on $[s, l]$ subject to the constraint that $p_k(1) = 1$.

Consider the Chebyshev polynomials $C_j(z)$ generated by the recursion

$$C_j(z) = 2zC_{j-1}(z) - C_{j-2}(z)$$

where $C_0(z) = 1$ and $C_1(z) = z$. These polynomials satisfy $|C_j(z)| \leq 1$ on $[-1,1]$ but grow rapidly off this interval. As a consequence, the polynomial

$$p_k(z) = \frac{C_k\left(-1 + 2\frac{z-s}{l-s}\right)}{C_k(\mu)}$$

where

$$\mu = -1 + 2\frac{1-s}{l-s} = 1 + 2\frac{1-l}{l-s}$$

Satisfies $p_k(1) = 1$ and tends to be small on $[s, l]$. From the definition of $p_k(z)$, then we see

$$\left\|w^{(k)} - x\right\|_2 \leq \frac{\left\|x - x^{(0)}\right\|_2}{|C_k(\mu)|}.$$

Thus, the larger μ is, the greater the acceleration of convergence. It is possible to derive a three-term recurrence among the $w^{(k)}$ by exploiting the three-term recurrence among the Chebyshev polynomials.

In particular, it can be shown that the three-term acceleration scheme based on iteration (6) has the following form:

$$Rx_{k+1} = \alpha_k[\beta_k(Tx_k + y) + (1-\beta_k)Rx_k] + (1-\alpha_k)Rx_{k-1}, \qquad (7)$$

where α_k and β_k are parameters that characterize the acceleration scheme, $x_{-1} = 0$, and x_0 is the starting guess.

From Section 2, we have that the extended matrix S has the following form:

$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ where $A = B - C$.

Thus, we assume the following splitting form of S for using the above scheme.

$S = R - T$, where $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_1 \end{pmatrix}$ and $T = \begin{pmatrix} T_1 & -C \\ -C & T_1 \end{pmatrix}$, where $B = R_1 - T_1$. Thus, we have the following equations as an iterative method:

$$R_1\underline{x}_i^{k+1} = \alpha_k\left[\beta_k\left(T_1\underline{x}_i^k - C\overline{x}_i^k + \underline{y}_i\right) + (1-\beta_k)R_1\underline{x}_i^k\right] + (1-\alpha_k)R_1\underline{x}_i^{k-1},$$
$$R_1\overline{x}_i^{k+1} = \alpha_k\left[\beta_k\left(T_1\overline{x}_i^k - C\underline{x}_i^k + \overline{y}_i\right) + (1-\beta_k)R_1\overline{x}_i^k\right] + (1-\alpha_k)R_1\overline{x}_i^{k-1}.$$

Finally, in matrix form, we have:

$$x_{k+1} = \alpha_k[\beta_k(R^{-1}Tx_k + R^{-1}y) + (1-\beta_k)x_k] + (1-\alpha_k)x_{k-1}. \qquad (8)$$

supposing that R^{-1} exists.

For Chebyshev acceleration, the parameters α_k and β_k are

$$\alpha_{k+1} = 2\frac{2-l-s}{l-s}\frac{C_k(\mu)}{C_{k+1}(\mu)}, \text{for } k \geq 0,$$
$$\beta_k = \frac{2}{2-l-s}, \text{for } k \geq 0,$$

as mentioned above, l denotes the most positive eigenvalue of M and s denotes the most negative eigenvalue of M.

4. A survey of some other acceleration techniques

4.1. Jacobi's iteration method

If we split S to $S = R - T$ where R is a diagonal matrix that includes the diagonal entries of S and $-T$ is remained entries of S, and $\alpha_k = \beta_k = 1$, we obtain Jacobi's iteration method [7, 10].

4.2. Gauss-Sidel's iteration method

By choosing R and T where R is lower triangular sub matrix of S and $-T$ is strictly upper triangular submatrix of S that $S = R - T$, when $\alpha_k = \beta_k = 1$, the Eq. (8) reduces to Gauss-Sidel's iteration method [7, 10].

4.3. SOR iterative method

To acquire successive over relaxation (SOR) iterative equation, we should select $R = L + \frac{1}{\omega}D$ and $T = \frac{1-\omega}{\omega}D - U$, where L, D and U are strictly lower triangular submatrix, diagonally submatrix and upper triangular submatrix of S, respectively, where $S = L + D + U$ and $\alpha_k = \beta_k = 1$. ω is parameter of SOR method [7, 10].

4.4. Conjugate gradient method

Although, the theoretical basis for the conjugate gradient method is quite different from the theoretical basis for Chebyshev acceleration, and actually has its roots in optimization theory and it is guaranteed to converge only when S is symmetric and positive definite [11], but it can be considered as Chebyshev acceleration. The conjugate gradient method corresponds to taking $M = I - S$ and making the following choices for acceleration parameters: $\alpha_0 = 1$,

$$\beta_k = \frac{\|r_k\|^2}{{r_k}^t S r_k} \text{ for } k \geq 0,$$
$$\alpha_k = \left(1 - \frac{\beta_k}{\beta_{k-1}}\frac{\|r_k\|^2}{\|r_{k-1}\|^2}\frac{1}{\alpha_{k-1}}\right)^{-1} \text{ for } k \geq 1,$$

Where $r_k = y - Sx_k$ is the residual at step k and $\|.\|$ denotes the Euclidean norm.

5. Numerical Example

In this section, we give some numerical examples to demonstrate the method.

***Example 5**.1* Consider the following 3×3 fuzzy system

$$4x_1 + x_2 + 1.5x_3 = (54 + 35r, 119 - 30r),$$
$$x_1 + 4x_2 + 2.5x_3 = (116 + 43r, 191 - 32r),$$
$$1.5x_1 + 2.5x_2 + 4x_3 = (123 + 37r, 203 - 43r).$$

the extended matrix S is

$$S = \begin{pmatrix} 4 & 1 & 1.5 & 0 & 0 & 0 \\ 1 & 4 & 2.5 & 0 & 0 & 0 \\ 1.5 & 2.5 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 1.5 \\ 0 & 0 & 0 & 1 & 4 & 2.5 \\ 0 & 0 & 0 & 1.5 & 2.5 & 4 \end{pmatrix}.$$

We define

$$R_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, T_1 = \begin{pmatrix} 0 & -1 & -1.5 \\ -1 & 0 & -2.5 \\ 1.5 & -2.5 & 0 \end{pmatrix},$$

and

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_1 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix},$$

where $S = R - T$.

The exact solution is

$x_1 = (2 + 6r, 12 - 4r), x_2 = (16 + 8r, 26 - 2r), x_3 = (20 + 2r, 30 - 8r),$

The exact and approximated solutions are plotted and compared in Figure 1. Note that we computed the approximated solution by Chebyshev acceleration technique

With $\|x - x_k\|_2 < 10^{-15}$ for $k = 47$.

Example 5.2 Let us treat the following fuzzy linear system

$$2x_1 + 5x_2 + 20x_3 = (15 + 5r, 25 - 5r),$$
$$30x_1 - 7x_2 + 4x_3 = (20 + 5r, 35 - 10r),$$
$$5x_1 + 20x_2 - x_3 = (40 + 10r, 60 - 10r).$$

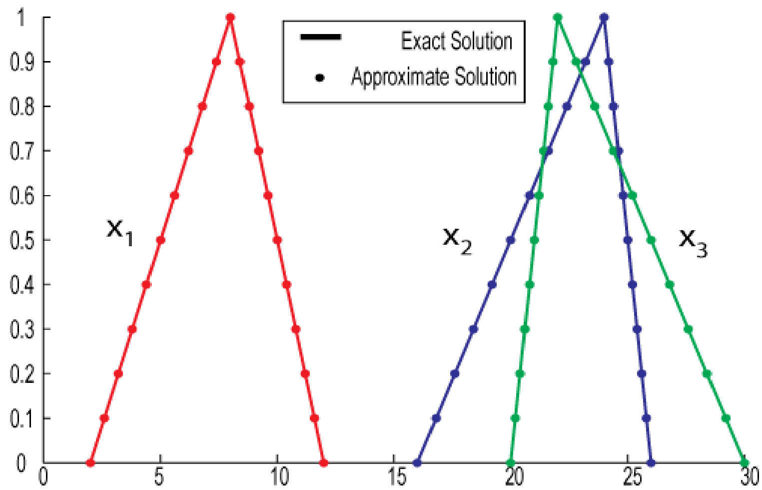

Figure 1: Comparison of the exact and approximate solution.

In accord with previous sections, we obtain that the extended matrix S has the following form

$$S = \begin{pmatrix} 2 & 5 & 20 & 0 & 0 & 0 \\ 30 & 0 & 4 & 0 & 7 & 0 \\ 5 & 20 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 5 & 20 \\ 0 & 7 & 0 & 30 & 0 & 4 \\ 0 & 0 & 1 & 5 & 20 & 0 \end{pmatrix}.$$

Clearly, all of the Jacobi, Gauss-Sidel and SOR methods are divergent because of singularity of the corresponding obtained matrix R by splitting S. Unfortunately, S is not symmetric and positive definite, therefore the conjugate gradient method is also divergent.

By definition of R and T as

$$R = \begin{pmatrix} 2 & 5 & 20 & 0 & 0 & 0 \\ 30 & 0 & 4 & 0 & 0 & 0 \\ 5 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 5 & 20 \\ 0 & 0 & 0 & 30 & 0 & 4 \\ 0 & 0 & 0 & 5 & 20 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

where $S = R - T$, the acquired results by utilizing Chebyshev's method are illustrated in Figure 2, with the same precision as previous example for $k = 14$.

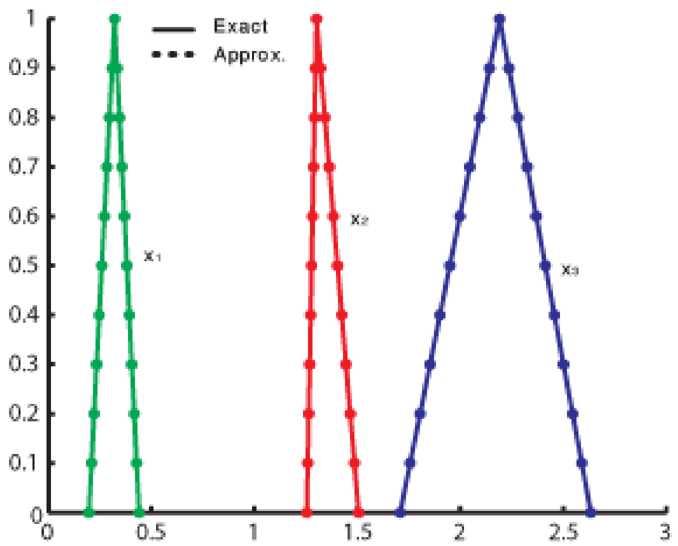

Figure 2: Comparison of the exact and approximate solution.

6. *Conclusion*

In this paper, we used Chebyshev acceleration technique for solving an FLS in extended form [9], and compared it by some other famous iterative techniques. As seen in Example 5.2, utilizing of such methods as Jacobi, Gauss-Sidel and SOR because of their specific splitting of the coefficients matrix and also the conjugate gradient method that is convergent only for symmetric and positive definite matrices is unsuccessful. Therefore, using appropriate numerical methods such as our proposed method is a suitable approach to associate with FLS problems.

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