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Application of Exp-function method to the (2 + 1)-dimensional Calogero Bogoyavlanskii

Schiff equation.

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Abstract

In this paper, the Exp-function method, with the aid of a symbolic computation system such as Maple, is applied to the (2 + 1)-dimensional Calogero Bogoyavlanskii Schiff equation. Exact and explicit generalized solitary solutions are obtained in more general forms. The free parameters can be determined by initial or boundary conditions. The method is straightforward and concise, and its applications are promising. It is shown that the Exp-function method, with the help of symbolic computation, provides a very effective and powerful mathematical tool for solving Calogero Bogoyavlanskii Schiff equation.

Keywords: Exp-function method; Calogero Bogoyavlanskii Schiff equation; partial differential equation.

1 Introduction

The investigation of exact solutions of nonlinear equations plays an important role in the study of nonlinear physical phenomena. In recent years, several powerful methods have been proposed to obtain approximate solution of nonlinear partial differential equations, such as the tanh method [1-2], sine–cosine method [3], homotopy perturbation method [4-5], variational iteration method [6-7], Adomian decomposition method [8], and many others. Most recently, a novel approach called Exp-function method [9-13] has been developed to obtain solitary solutions and periodic solutions of various nonlinear equations. The solution procedure of

this method, by the help of Matlab, Maple or any mathematical packages, is of utter simplicity.

In this paper Exp-function method has been applied to obtain the exact solution of the Calogero Bogoyavlanskii Schiff equation (CBS equation). This equation has the following form;

$$u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z = 0. \tag{1}$$

The CBS equation was first constructed by Bogoyavlanskii and Schiff in different ways. Bogoyavlanskii used the modified Lax formalism, where as schiff derived the same equation by reducing the Self-dual-yang-mills equation. The equation is used to describe the interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. Wazwaz applied tanh-coth method and Hirota's bilinear method to the CBS equation [14, 15].

2 The Exp-function method

To illustrated the Exp-function method let's consider a general nonlinear PDE, in the following form

$$p(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0. \tag{2}$$

Using the transformation

$$\xi = kx + wt, \tag{3}$$

where k and w are constants we can rewrite Eq. (2) as the following nonlinear ODE:

$$G(u, u', u'', u''', \ldots) = 0, \tag{4}$$

where the superscripts denotes the derivatives with respect to ξ .

According to Exp-function method, we assume that the solution can be expressed in the following form, [9].

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}, \tag{5}$$

where c, d, p and q are positive integers which could be freely chosen, a_m for $m = -d, \ldots, c$ and b_n, for $n = -q, \ldots, p$ are unknown constants and should be determined. To determine the values of c and p, we balance the linear terms of the highest order in Eq. (4) with the highest order nonlinear terms.

Similarly for determining the values of d and q, we balance the linear terms of the lowest order in Eq. (4) with the lowest order nonlinear terms.

3 Solution of the equal-width equation by Exp-function method

Using the transformation

$$u = U(\xi), \qquad \xi = kx + wt + sz,$$

where k, w and s are constants to be determined later, Eq.(1) converts to an ordinary differential equation, as follows:

$$wU'' + k^2 s U^{(4)} + 6ks U'U'' = 0. \tag{6}$$

Where the super scripts denotes the differential with respect to ξ. By integrating the last equations we find:

$$wU' + k^2 sU''' + 3ksU'^2 = 0. \tag{7}$$

We assume that the solution can be expressed in the form (5). In order to determine c and p, we balance the linear term of the highest order in Eq. (1) with the highest order nonlinear term. By simple calculation, we derive to

$$U''' = \frac{c_1 \exp[(7p+c)\xi] + \cdots}{c_2 \exp[8p\xi] + \cdots}, \tag{8}$$

and

$$U'^2 = \frac{c_3 \exp[(2p+2c)\xi] + \cdots}{c_4 \exp[4p\xi] + \cdots} = \frac{c_3 \exp[(6p+2c)\xi] + \cdots}{c_4 \exp[8p\xi] + \cdots}. \tag{9}$$

By balancing the highest order of Exp-function in Eqs. (8) and (9), we have:

$$c + 7p = 2c + 6p, \tag{10}$$

which leads to the following result:

$$p = c. \tag{11}$$

The values of d and q can be determined in a similar way. We balance the linear term of the lowest order in Eq. (6)

$$U''' = \frac{\cdots + d_1 \exp[-(7q+d)\xi]}{\cdots + d_2 \exp[-8q\xi]}, \tag{12}$$

with the nonlinear term

$$U'^2 = \frac{\cdots + d_3 \exp[-(2q+2d)\xi]}{\cdots + d_4 \exp[-4q\xi]} = \frac{\cdots + d_3 \exp[-(6q+2c)\xi]}{\cdots + d_4 \exp[-8q\xi]}. \tag{13}$$

c_i's and d_i's are identified symbolically, so that for the sake of simplicity of the form of equations (8), (9), (12) and (13). By balancing the lowest order of Exp-function in Eqs. (12) and (13), we derive;

$$-(7q+d) = -(2d+6q),$$

which leads to the following result:

$$d = q. \tag{14}$$

It is possible to choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d.

3.1 The choice of p = c = 1 and q = d = 1

For the sake of simplicity, we choose $p = c = 1$ and $q = d = 1$, the trials function, Eq. (5) converts to as:

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{15}$$

In case $b_1 \neq 0$ Eq. (15) can be simplified as

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{16}$$

Substituting Eq. (16) into Eq. (7), by the help of the symbolic computation in the Maple package, leads to

$$\frac{1}{A}[C_3 \exp(3\xi) + C_2 \exp(2\xi) + C_1 \exp(\xi) + C_0$$
$$+ C_{-1} \exp(-\xi) + C_{-2} \exp(-2\xi) + C_{-3} \exp(-3\xi)] = 0,$$

Where A and C_i are constants.

Equating the coefficients of $\exp(n\xi)$ in each term with zero yield to a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, w, s$, and k .

$$\begin{cases} C_4 = 0, \ C_3 = 0, \ C_2 = 0, \ C_1 = 0, \\ C_0 = 0, \\ C_{-4} = 0, \ C_{-3} = 0, \ C_{-2} = 0, \ C_{-1} = 0. \end{cases}$$

The solution of this system of algebraic equations are as follows,

$$a_1 = a_1, \ a_0 = 0, \ a_{-1} = a_{-1}, \ b_0 = 0, \ b_1 = b_1,$$
$$w = -\frac{(-a_{-1} + a_1 b_{-1})^2 s}{4 b_{-1}^2}, \ k = -\frac{a_{-1} + a_1 b_{-1}}{4 b_{-1}}. \tag{17}$$

Where s, a_{-1}, b_{-1} and a_1 are free parameters. Substituting Eq. (17) into Eq. (16), we obtain the following exact solution

$$u_1(x, z, t) = \frac{a_1 \exp(\xi) + a_{-1} \exp(-\xi)}{\exp(\xi) + b_{-1} \exp(-\xi)}. \tag{18}$$

Where

$$\xi = -\frac{a_{-1} + a_1 b_{-1}}{4 b_{-1}} x - \frac{(-a_{-1} + a_1 b_{-1})^2 s}{4 b_{-1}^2} t + sz.$$

If we set $b_{-1} = -1$ and $a_1 = a_{-1},$ Eq. (18) reduces to

$$u(x, z, t) = \coth(\frac{1}{2} a_1 x - a_1^2 st + sz), \tag{19}$$

For $b_{-1} = 1$ and $a_1 = -a_{-1},$ we get:

$$u(x, z, t) = \tanh(\frac{1}{2} a_1 x - a_1^2 st + sz), \tag{20}$$

where in Eqs. (19) and (20), s and a_1 are free parameters.

3.2 The choice of p = c =2 and q = d = 1
If we choose $p = c = 2$ and $q = d = 1$, Eq. (5) takes the following form:

$$U(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(2\xi) + b_1 + \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{21}$$

Proceeding in a similar way as illustrated in 3.1, we can identify parameters, $a_2, a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, w, s$, and k in Eq. (21) as follows

Case1:

$$\begin{cases} a_2 = \dfrac{4kb_0 + a_0}{b_0}, \ a_1 = 0, \ a_0 = a_0, \ b_0 = b_0, s = s, \\[2mm] w = -4k^2 s, \ b_1 = 0, \ k = k, \ a_{-1} = 0, \ b_{-1} = 0. \end{cases} \tag{22}$$

Case2:

$$\begin{cases} a_2 = \dfrac{a_{-1} + 6kb_{-1}}{b_{-1}}, \ a_1 = 0, \ a_0 = a_0, \ b_0 = 0, s = s, \\[2mm] w = -9k^2 s, \ b_1 = 0, \ k = k, \ a_{-1} = a_{-1}, \ b_{-1} = b_{-1}. \end{cases} \tag{23}$$

And

Case3:

$$\begin{cases} a_{-1} = a_{-1}, \ a_0 = \dfrac{a_{-1} b_0}{b_{-1}}, \ a_1 = -\dfrac{4kb_{-1} + a_{-1}}{b_0}, \ a_2 = \dfrac{4kb_{-1} + a_{-1}}{b_{-1}}, \\[2mm] b_{-1} = b_{-1}, \ b_0 = b_0, \ b_1 = \dfrac{b_{-1}}{b_0}, \ s = s, \ w = -4k^2 s, \ k = k. \end{cases} \tag{24}$$

Substituting Eq. (22) into Eq. (21), we obtain following exact solution

$$u_2(x, z, t) = \frac{\dfrac{4kb_0 + a_0}{b_0} \exp 2(kx - 4k^2 st + sz) + a_0}{\exp 2(kx - 4k^2 st + sz) + b_0} = \frac{4k \exp 2(kx - 4k^2 st + sz)}{\exp 2(kx - 4k^2 st + sz) + b_0} + \frac{a_0}{b_0}$$

(25)

If we set $b_0 = 1$, Eq. (25) reduces to

$$u(x, z, t) = a_0 + 2k \exp(\xi) \operatorname{sech}(\xi), \tag{26}$$

For $b_0 = -1$ we get:

$$u(x, z, t) = -a_0 + 2k \exp(\xi) \operatorname{csch}(\xi), \tag{27}$$

Where

$$\xi = kx - 4k^2 st + sz.$$

If we set $b_0 = 1$, $a_0 = 0$, $k_1 = 2k$, and $m_1 = 2s$, Eq. (25) reduces to

$$u_2(x, z, t) = \frac{2k_1 \exp(k_1 x + m_1 z - k_1^2 m_1 t)}{1 + \exp(k_1 x + m_1 z - k_1^2 m_1 t)}.$$

This result is exactly the same as that obtained by wazwaz [15].

By Eq. (23), we obtain the following solution:

$$u_3(x,z,t) = \frac{\dfrac{a_{-1}+6kb_{-1}}{b_{-1}}\exp 2(\xi)+a_{-1}\exp(-\xi)}{\exp 2(\xi)+b_{-1}\exp(-\xi)} = \frac{6k\exp 2(\xi)}{\exp 2(\xi)+b_{-1}\exp(-\xi)}+\frac{a_{-1}}{b_{-1}},$$

(28)

where

$$\xi = kx - 9k^2st + sz.$$

And from Eq. (24), we get

$$u_4(x,z,t) = \frac{\dfrac{4kb_{-1}+a_{-1}}{b_{-1}}\exp(2\xi)-\dfrac{4kb_{-1}+a_{-1}}{b_0}\exp(\xi)+\dfrac{a_{-1}b_0}{b_{-1}}+a_{-1}\exp(-\xi)}{\exp(2\xi)+\dfrac{b_{-1}}{b_0}\exp(\xi)+b_0+b_{-1}\exp(-\xi)}$$

(29)

$$= \frac{a_{-1}}{b_{-1}}+4k\;\frac{\exp(2\xi)-\dfrac{b_{-1}}{b_0}\exp(\xi)}{\exp(2\xi)+\dfrac{b_{-1}}{b_0}\exp(\xi)+b_0+b_{-1}\exp(-\xi)}.$$

Where

$$\xi = kx - 4k^2st + sz.$$

3.3 The choice of p = c = 1 and q = d = 2

As mentioned earlier the values of c and d can be freely chosen, now we set $p = c = 1$ and $q = d = 2$, so Eq. (5) turns to the following form;

$$U(\xi) = \frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{\exp(\xi)+b_0+b_{-1}\exp(-\xi)+b_{-2}\exp(-2\xi)}.$$

(30)

By the same manipulation as illustrated earlier, we obtain

Case1:

$$\begin{cases} a_1 = a_1,\; a_0 = 0,\; a_{-1} = 0,\; a_{-2} = a_{-2},\; b_0 = 0,\; b_{-1} = 0, b_{-2} = b_{-2}, \\ w = -\dfrac{(-a_{-2}+a_1b_{-2})^2 s}{4b_{-2}^2},\; k = \dfrac{-a_{-2}+a_1b_{-2}}{6b_{-2}^2},\; s = s. \end{cases}$$

(31)

We, therefore, obtain the solution of Eq. (1) which reads

$$u_5(x,z,t) = \frac{a_1\exp(\xi)+a_{-2}\exp(-2\xi)}{\exp(\xi)+b_{-2}\exp(-2\xi)}.$$

(32)

Where

$$\xi = \frac{-a_{-2}+a_1b_{-2}}{6b_{-2}^2}x - \frac{(-a_{-2}+a_1b_{-2})^2 s}{4b_{-2}^2}t + sz.$$

Case2:

$$\begin{cases} a_1 = a_1, \ a_0 = \dfrac{a_1 b_{-2}}{b_{-1}}, \ a_{-1} = \dfrac{a_{-2} b_{-1}}{b_{-2}}, \ a_{-2} = a_{-2}, \ b_0 = \dfrac{b_{-2}}{b_{-1}}, \ b_{-1} = b_{-1}, \ b_{-2} = b_{-2}, \\ w = -\dfrac{(-a_{-2}+a_1 b_{-2})^2 s}{4 b_{-2}^2}, \ k = \dfrac{-a_{-2}+a_1 b_{-2}}{4 b_{-2}^2}, \ s = s. \end{cases} \qquad (33)$$

The solution of Eq. (1) is

$$u_6(x,z,t) = \dfrac{a_1 \exp(\xi) + \dfrac{a_1 b_{-2}}{b_{-1}} + \dfrac{a_{-2} b_{-1}}{b_{-2}} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(\xi) + \dfrac{b_{-2}}{b_{-1}} + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)}. \qquad (34)$$

Where

$$\xi = \dfrac{-a_{-2}+a_1 b_{-2}}{4 b_{-2}^2} x - \dfrac{(-a_{-2}+a_1 b_{-2})^2 s}{4 b_{-2}^2} t + sz.$$

Case3:

$$\begin{cases} a_1 = a_1, \ a_0 = 0, \ a_{-1} = 0, \ a_{-2} = 0, \ b_0 = 0, \ b_{-1} = b_{-1}, \ b_{-2} = 0, \\ w = -\dfrac{(-a_{-1}+a_1 b_{-1})^2 s}{4 b_{-1}^2}, \ k = \dfrac{-a_{-1}+a_1 b_{-1}}{4 b_{-1}^2}, \ s = s. \end{cases} \qquad (35)$$

By Eq. (35), we obtain the following solution;

$$u_7(x,z,t) = \dfrac{a_1 \exp(\xi) + a_{-1} \exp(-\xi)}{\exp(\xi) + b_{-1} \exp(-\xi)}. \qquad (36)$$

Where

$$\xi = -\dfrac{a_{-1}+a_1 b_{-1}}{4 b_{-1}} x - \dfrac{(-a_{-1}+a_1 b_{-1})^2 s}{4 b_{-1}^2} t + sz.$$

Which $u_7(x,z,t)$ is the same as $u_1(x,z,t)$.

3.4 The choice of $p = c = 2$ and $q = d = 2$

Under such case, the trial function (5) can be expressed as follows

$$U(\xi) = \dfrac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)}. \qquad (37)$$

Proceeding in a similar way as illustrated in 3.1, we can identify parameters, $a_2, a_1, a_0, a_{-1}, a_{-2}, b_1, b_0, b_{-1}, b_{-2}, w, s$, and k in Eq. (37) as the following

Case1:

$$\begin{cases} a_2 = \dfrac{4k b_0 + a_0}{b_0}, \ a_1 = 0, \ a_0 = a_0, \ a_{-1} = 0, \ a_{-2} = 0, \ b_1 = 0, \ b_0 = b_0, \ b_{-1} = 0, \ b_{-2} = 0, \\ w = -4k^2 s, \ k = k, \ s = s. \end{cases}$$

(38)

Substituting Eq. (38) into Eq. (37), we obtain following exact solution

$$u_8(x,z,t) = \frac{\dfrac{4kb_0 + a_0}{b_0}\exp(2\xi) + a_0}{\exp(2\xi) + b_0}.$$ (39)

and

$$\xi = kx - 4k^2t + sz.$$

Which is the same as $u_2(x,z,t)$.

Case2:

$$\begin{cases} a_2 = \dfrac{6kb_{-1} + a_{-1}}{b_{-1}}, \ a_1 = 0, \ a_0 = 0, \ a_{-1} = a_{-1}, \ a_{-2} = 0, \ b_1 = 0, \ b_0 = 0, \\[3mm] b_{-1} = b_{-1}, \ b_{-2} = 0, w = \dfrac{6kb_{-1} + a_{-1}}{b_{-1}}, \ k = k, \ s = s. \end{cases}$$ (40)

Inserting Eq. (40) into Eq. (37) yields the following generalized solution

$$u_9(x,z,t) = \frac{\dfrac{6kb_{-1} + a_{-1}}{b_{-1}}\exp(2\xi) + a_{-1}\exp(-\xi)}{\exp(2\xi) + b_{-1}\exp(-\xi)}.$$ (41)

Where

$$\xi = kx - 9k^2t + sz.$$

And that is the same as $u_3(x,z,t)$.

Case3:

$$\begin{cases} a_2 = \dfrac{4kb_{-1} + a_{-1}}{b_{-1}}, \ a_1 = \dfrac{4kb_{-1} + a_{-1}}{b_0}, \ a_0 = \dfrac{a_{-1}b_0}{b_{-1}}, \ a_{-1} = a_{-1}, \ a_{-2} = 0, \\[3mm] b_1 = \dfrac{b_{-1}}{b_0}, \ b_0 = b_0, \ b_{-1} = b_{-1}, b_{-2} = 0, w = -4k^2s, \ k = k, \ s = s. \end{cases}$$

(42)

Substituting Eq. (42) into Eq. (37) yields to

$$u_{10}(x,z,t) = \frac{\dfrac{4kb_{-1} + a_{-1}}{b_{-1}}\exp(2\xi) + \dfrac{4kb_{-1} + a_{-1}}{b_0}\exp(\xi) + \dfrac{a_{-1}b_0}{b_{-1}} + a_{-1}\exp(-\xi)}{\exp(2\xi) + \dfrac{b_{-1}}{b_0}\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}.$$

(43)

Where

$$\xi = kx - 4k^2t + sz.$$

Which is the same as $u_4(x,z,t)$.

Case4:

$$\begin{cases} a_2 = \dfrac{8kb_{-2}+a_{-2}}{b_{-2}},\ a_1 = 0,\ a_0 = 0,\ a_{-1} = 0,\ a_{-2} = a_{-2}, \\ b_1 = 0,\ b_0 = 0,\ b_{-1} = 0,\ b_{-2} = b_{-2}, w = -16k^2 s,\ k = k,\ s = s. \end{cases}$$

(44)

By means of Eqs. (37) and (44), we have

$$u_{11}(x,z,t) = \frac{\dfrac{8kb_{-2}+a_{-2}}{b_{-2}}\exp(2\xi) + a_{-2}\exp(-2\xi)}{\exp(2\xi) + b_{-2}\exp(-2\xi)}$$

(45)

$$= \frac{a_{-2}}{b_{-2}} + 8k\,\frac{\exp(2\xi)}{\exp(2\xi) + b_{-2}\exp(-2\xi)}.$$

Where

$$\xi = kx - 16k^2 t + sz.$$

If we set $k = \dfrac{K}{2}$ and $s = \dfrac{S}{2}$, Eq. (45) will be the same as Eq. (25).

Case5:

$$\begin{cases} a_2 = \dfrac{6kb_{-1}+a_{-2}}{b_{-2}},\ a_1 = \dfrac{6kb_{-2}+a_{-2}}{b_{-1}},\ a_0 = 0,\ a_{-1} = \dfrac{a_{-2}b_{-1}}{b_{-2}},\ a_{-2} = a_{-2}, \\ b_1 = \dfrac{b_{-2}}{b_{-1}},\ b_0 = 0,\ b_{-1} = b_{-1},\ b_{-2} = b_{-2},\ w = -9k^2 s,\ k = k,\ s = s. \end{cases}$$

(46)

By Eq. (46), we obtain the following solution:

$$u_{12}(x,z,t) =$$

$$\frac{\dfrac{6kb_{-1}+a_{-2}}{b_{-2}}\exp(2\xi) + \dfrac{6kb_{-2}+a_{-2}}{b_{-1}}\exp(\xi) + \dfrac{a_{-2}b_{-1}}{b_{-2}}\exp(-\xi) + a_{-2}\exp(-2\xi)}{\exp(2\xi) + \dfrac{b_{-2}}{b_{-1}}\exp(\xi) + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}$$

(47)

$$= \frac{a_{-2}}{b_{-2}} + \frac{6k\exp(2\xi) + \dfrac{6kb_{-2}}{b_{-1}}\exp(\xi)}{\exp(2\xi) + \dfrac{b_{-2}}{b_{-1}}\exp(\xi) + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}.$$

Where $\xi = kx - 9k^2 t + sz.$

4 Conclusions

In this article, we have obtained the exact solution of CBS equation. The solution has been achieved by applying Exp-function method. We predict that these solutions will be of great importance for analyzing the nonlinear phenomena arising in applied physical sciences. The results show that Exp-function method is a powerful tool for obtaining solitary solution. It may be concluded that, the Exp-function method can be easily extended to all kinds of nonlinear equations. The

advantage of this method over other methods is that we can obtain the exact solution by using a simple computer program. Applications for exp-function method are under study in our research group. The computations associated in this work were performed by using Maple 11.

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