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Sensitivity Analysis in Linear-Plus-Linear Fractional Programming Problems

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Abstract

In this paper, we study the classical sensitivity analysis when the right - hand – side vector, and the coefficients of the objective function are allowed to vary.

Keywords: Sensitivity analysis, Linear fractional programming

1 Introduction

In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by some methods although are approximations of the solution of the problem but they could be considered as the exact results of the corresponding perturbed problem and this is the motivation to investigate the sensitivity analysis. We would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. Generally, independent and simultaneous perturbations are investigated. The following problem is a general case of two problems such as, linear programming (LP)[3], linear fractional programming (LFP) [6, 7].

A general linear-plus-linear fractional programming problem is stated as

$$\begin{aligned} \max Z &= cx + \frac{px}{qx+d} \\ s.t: \quad Ax &= b \qquad\qquad (LLFP) \\ x &\geq 0, \end{aligned}$$

where $c, p\ and\ q$ are row vectors with n component ,b is a column vector with m component, A

is an $m \times n$ matrix and d is a scalar.
Major applications of the problem (LLFP) can be found in transportation, problems of optimizing enterprise capital, the production development fund and social, cultural and construction fund [5]. Several authors studied the problem (LLFP) and its variants and have discussed their solution properties [2, 4].
In this paper, we study classical sensitivity analysis when the coefficients of the objective function and the right-hand-side are parameterized. Therefore, we consider the problem (LLFP) with the following assumptions:

(1) The set of feasible solutions is regular; i.e., non-empty and bounded
(2) The objective function is pseudo-convex on the feasible solutions set [1]
(3) $qx + d$ is positive over all feasible solutions
(4) Our problem is non-degenerate and has an optimal basic feasible solution.

Due to the assumptions, the optimality criterion for the problem (LLFP) using the simplex type algorithm given by Teterev [8] is stated as follows:
Let B denotes the optimal basis matrix and let $\bar{x} = (\bar{x}_B, 0)$ be the corresponding basic feasible solution of (LLFP). This solution will be optimal if for all j

$$\Delta_j = \left(z_j^c - c_j\right) + \frac{z''\left(z_j^p - p_j\right) - z'\left(z_j^q - q_j\right)}{z''^2} \geq 0,$$

Where, $z' = \mathrm{p_B x_B}$, $z'' = \mathrm{q_B x_B}$, $z_j^c = \mathrm{c_B B^{-1} a_j}$, $\mathrm{z_j^p = p_B B^{-1} a_j}$ and $z_j^q = q_B B^{-1} a_j$.
Here $c_B, \mathrm{p_B}$ and q_B are the sub-vectors of c, p and q respectively that corresponded to the basis B and a_j is the jth column of A.

2 Formulation of the problem and sensitivity analysis

To study the problem (LLFP) when the right-hand-side vector and the coefficients of the objective function are perturbed, we consider

$$\begin{aligned} \max Z = (c + \lambda\Delta c)x + \frac{(p + \lambda\Delta p)x}{(q + \lambda\Delta q)x + d} \\ s.t.: \quad Ax = b + \lambda\Delta b \qquad (LLFP(\lambda)) \\ x \geq 0, \end{aligned}$$

Where $\Delta\mathrm{c}, \Delta\mathrm{p}$, $\Delta\mathrm{q}$ and $\Delta\mathrm{b}$ are perturbation vectors, and λ is a non-negative real parameter.
In the problem, the new coefficients are $\tilde{\mathrm{c}}_\mathrm{j} = \mathrm{c_j} + \lambda\Delta\mathrm{c_j}$, $\tilde{\mathrm{p}}_\mathrm{j} = \mathrm{p_j} + \lambda\Delta\mathrm{p_j}$ and $\tilde{\mathrm{q}}_\mathrm{j} = \mathrm{q_j} + \lambda\Delta\mathrm{q_j}$. Thus, we have

$$\tilde{z}_j^c - \tilde{c}_j = (c_B + \lambda \Delta c_B)B^{-1}a_j - \left(c_j + \lambda \Delta c_j\right) = (z_j^c - c_j) + \lambda\left(\Delta c_B B^{-1} a_j - \Delta c_j\right) \stackrel{\text{def}}{=} \alpha_j + \lambda \alpha_j',$$

$$\tilde{z}_j^p - \tilde{p}_j = (p_B + \lambda \Delta p_B)B^{-1}a_j - \left(p_j + \lambda \Delta p_j\right) = (z_j^p - p_j) + \lambda\left(\Delta p_B B^{-1} a_j - \Delta p_j\right) \stackrel{\text{def}}{=} \beta_j + \lambda \beta_j',$$

$$\tilde{z}_j^q - \tilde{q}_j = (q_B + \lambda \Delta q_B)B^{-1}a_j - \left(q_j + \lambda \Delta q_j\right) = (z_j^q - q_j) + \lambda\left(\Delta q B^{-1} a_j - \Delta q_j\right) \stackrel{\text{def}}{=} \gamma_j + \lambda \gamma_j',$$

$$\tilde{z}'' = \tilde{q}_B\, \tilde{x}_B + d = (q_B + \lambda \Delta q_B)(B^{-1}b + \lambda B^{-1}\Delta b) = (q_B B^{-1} b + d) +$$

$$(\Delta q_B B^{-1} b + q_B B^{-1} \Delta b)\lambda + (\Delta q_B B^{-1} \Delta b)\lambda^2 \stackrel{\text{def}}{=} z'' + M\lambda + F\lambda^2,$$

$$\tilde{z}' = \tilde{p}_B\, \tilde{x}_B = (p_B + \lambda \Delta p_B)(B^{-1}b + \lambda B^{-1}\Delta b) \stackrel{\text{def}}{=} z' + \overline{M}\lambda + \overline{F}\lambda^2.$$

Thus, in the case of simultaneous perturbations, for any basis B, we find that $\tilde{z}_j^c - \tilde{c}_j, \tilde{z}_j^p - \tilde{p}_j$ and $\tilde{z}_j^q - \tilde{q}_j$ are linear functions while $\tilde{z}''$ and $\tilde{z}'$ are quadratic functions of λ. We discuss the following cases.

2.1 The vectors p and c are perturbed simultaneously

In this case, $\Delta b = \Delta q = 0,\ \tilde{z}'' = z''$ and $\tilde{z}' = (p_B + \lambda \Delta p_B)B^{-1}b = z' + \lambda \Delta p_B B^{-1} b \stackrel{\text{def}}{=} z' + \lambda M_1$.

To satisfy optimality condition, for all j the new values $\overline{\Delta}_j$ are computed as:

$$\overline{\Delta}_j = \left(\alpha_j + \lambda \alpha_j'\right) + \frac{z''(\beta_j + \lambda \beta_j') - (z' + \lambda M_1)\gamma_j}{z''^2} = \left(\alpha_j + \frac{z''\beta_j - z'\gamma_j}{z''^2}\right) + \left(\alpha'_j + \frac{Z''\beta'_j - M_1\gamma_j}{z''^2}\right)\lambda = \Delta_j + \left(\alpha'_j + \frac{z''\beta'_j - M_1\gamma_j}{z''^2}\right)\lambda \geq 0,$$

Thus, the basis invariance interval is as follows:

$$0 \leq \lambda \leq min\left\{\frac{-\Delta_j}{\alpha'_j + \frac{z''\beta'_j - M_1\gamma_j}{z''^2}}:\ \alpha'_j + \frac{z''\beta'_j - M_1\gamma_j}{z''^2} < 0\right\}$$

2.2 The vector q is perturbed

In this case, we have $\Delta b = \Delta p = \Delta c = 0, \tilde{z}' = z',\ \tilde{z}_j^c - \tilde{c}_j = \alpha_j,\ \tilde{z}_j^p - \tilde{p}_j = \beta_j$ and $\tilde{z}'' = z'' + \lambda \Delta q_B B^{-1} b \stackrel{\text{def}}{=} z'' + \lambda F_1$.

To satisfy optimality condition, the new values $\overline{\Delta}_j$ are computed as:

$$\bar{\Delta}_j = \alpha_j + \frac{(z'' + \lambda F_1)\beta_j - z'(\gamma_j + \lambda\gamma_j')}{(z'' + \lambda F_1)^2}$$

$$= \frac{(\alpha_j z''^2 + z''\beta_j - z'\gamma_j) + (2z''F_1\alpha_j + F_1\beta_j - z\gamma_j')\lambda + (\alpha_j F_1{}^2)\lambda^2}{(z'' + \lambda F_1)^2}$$

$$\stackrel{\text{def}}{=} \frac{P_j\lambda^2 + Q_j\lambda + R_j}{(z'' + \lambda F_1)^2} \geq 0. \qquad (2.1)$$

By the assumption $qx + d > 0$ for any feasible solution x, to preserve this condition we need to have

$$z'' + \lambda F_1 > 0, \qquad (2.2)$$

Which the following restriction is obtained from (2.2):

$$\lambda \begin{cases} < \dfrac{-z''}{F_1}, & if\ F_1 < 0, \\ \geq 0, & if\ F_1 > 0. \end{cases} \qquad (2.3)$$

From (2.2), the relation (2.1) is satisfied if for all , we have

$$P_j\lambda^2 + Q_j\lambda + R_j \geq 0. \qquad (2.4)$$

As already mentioned, we have an optimal basic feasible solution $\bar{x} = (\bar{x}_B,\ 0)$ with an optimal basis B for $\lambda = 0$. Thus the inequalities (2.4) imply $R_j \geq 0$ for all j, which shows that inequalities (2.4) are consistent for all j. Now, we consider the equations

$$P_j\lambda^2 + Q_j\lambda + R_j = 0,\ \ \forall j \qquad (2.5)$$

In which an equation can be appeared only one of the following forms.

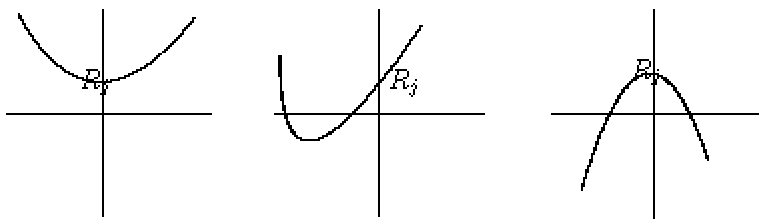

Fig. (a) Fig. (b) Fig. (c)

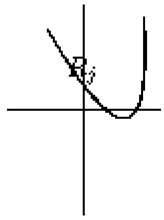

Fig. (d)

Figures (a) and (b) show that inequalities (2.4) hold for any $\lambda \geq 0$.
Figure (c) shows that inequalities (2.4) hold for $\lambda \in [0, \lambda_M^j]$, where λ_M^j is the positive root of the equation (2.5).
Figure (d) shows that inequalities (2.4) hold for $\lambda \in [0, \lambda_m^j]$,and $\lambda \geq \lambda_M^j$, where λ_m^j and λ_M^j are the smallest and greatest positive roots of the equation (2.5).

Now by considering Figures (a), (b), (c) and (d) together, we obtain

$$0 \leq \lambda \leq \min(\lambda^j), \qquad (2.6)$$

Where λ^j are the positive roots of the equations (2.5) for some j.
It is obvious that if λ satisfies (2.3) and (2.6), then $\bar{x}$ is an optimal solution of the perturbed LLFP(λ) problem.

2.3 Perturbations on c, p and q vectors are considered

In this case, $\Delta b = 0$, we have $\tilde{z}'' = z'' + \lambda \Delta q_B B^{-1} b \stackrel{\text{def}}{=} z'' + \lambda F_1$ and $\tilde{z}' = z' + \lambda \Delta p_B B^{-1} b \stackrel{\text{def}}{=} z' + \lambda F_2$.
To satisfy optimality condition, the new values $\bar{\Delta}_j$ are computed as:

$$\begin{aligned}
\bar{\Delta}_j &= (\alpha_j + \lambda\alpha'_j) + \frac{(z'' + \lambda F_1)(\beta_j + \lambda\beta'_j) - (z' + F_2\lambda)(\gamma_j + \lambda\gamma'_j)}{(z'' + \lambda F_1)^2} \\
&= \frac{\left(\alpha_j z''^2 + z''\beta_j - z'\gamma_j\right) + \left[z''\left(2F_1\alpha_j + \beta'_j + z''\alpha'_j\right) + F_1\beta_j - z'\gamma'_j - F_2\gamma_j\right]\lambda}{(z'' + \lambda F_1)^2} \\
&\quad + \frac{\left[F_1^2\alpha_j + 2z''F_1\alpha'_j + F_1\beta'_j - F_2\gamma'_j\right]\lambda^2 + (F_1^2\alpha'_j)\lambda^3}{(z'' + \lambda F_1)^2} \\
&\stackrel{\text{def}}{=} \frac{\bar{P}_j\lambda^3 + \bar{Q}_j\lambda^2 + \bar{S}_j\lambda + \bar{R}_j}{(z'' + \lambda F_1)^2} \geq 0. \qquad (2.7)
\end{aligned}$$

From (2.2), inequalities (2.7) hold when for all j, we have

$$\bar{P}_j\lambda^3 + \bar{Q}_j\lambda^2 + \bar{S}_j\lambda + \bar{R}_j \geq 0. \qquad (2.8)$$

As already mentioned, for $\lambda = 0$ we have an optimal basic feasible solution $\bar{x} = (\bar{x}_B,\ 0)$ with an optimal basis B. Thus for $\lambda = 0$ the inequalities (2.8) imply $\bar{R}_j \geq 0$ for all j, which show that inequalities (2.8) for all j are consistent.

Now for all j, we consider the equations

$$\bar{P}_j\lambda^3 + \bar{Q}_j\lambda^2 + \bar{S}_j\lambda + \bar{R}_j = 0. \qquad (2.9)$$

Where an equation can appears only one of the following shapes.

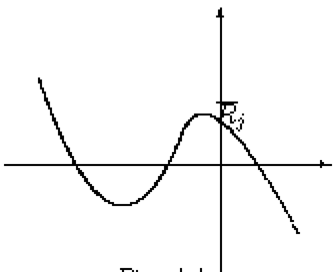

Fig. (a)

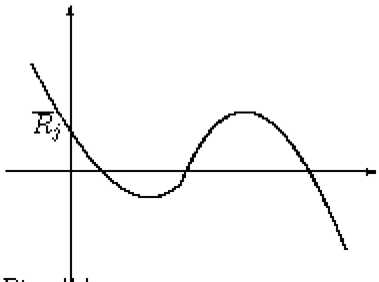

Fig. (b)

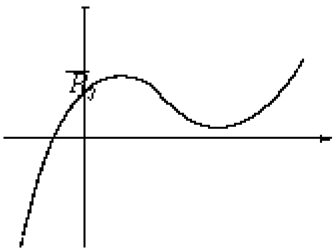

Fig. (d)

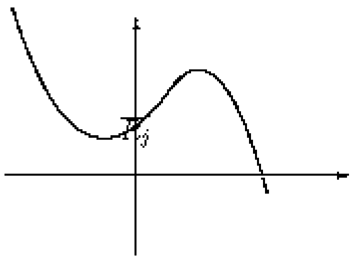

Fig. (c)

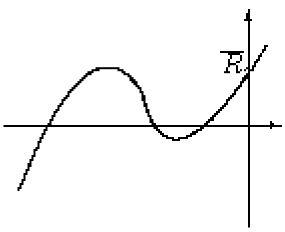

Fig. (e)

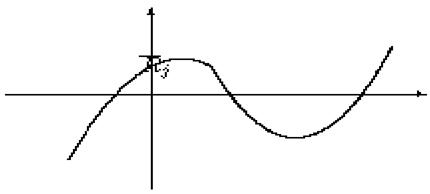

Fig. (f)

By similar discuss with (2.2), to satisfy (2.8), the admissible interval for λ is as follows

$$0 \le \lambda \le \min(\lambda^j), \qquad (2.10)$$

Where λ^j are the positive roots of the equations (2.9) for some j.

Thus, it is clear that if λ satisfies (2.3) and (2.10), then $\bar{x}$ is an optimal solution of the LLFP(λ).

2.4 The case where the vector b is perturbed

In this case, $z_j^c - c_j$, $z_j^p - p_j$ and $z_j^q - q_j$ are not changed, thus we have

$$\tilde{z}'' = q_B \tilde{x}_B + d = z'' + (q_B B^{-1} \Delta b)\lambda \stackrel{\text{def}}{=} z'' + F_3\lambda,$$

$$\tilde{z}' = p_B \tilde{x}_B = z' + (p_B B^{-1} \Delta b)\lambda \stackrel{\text{def}}{=} z' + E_1\lambda.$$

To satisfy optimality condition, the new values $\bar{\Delta}_j$ are computed as:

$$\bar{\Delta}_j = \alpha_j + \frac{(z'' + \lambda F_3)\beta_j - (z' + E_1\lambda)\gamma_j}{(z'' + \lambda F_3)^2}$$

$$= \frac{(\alpha_j z''^2 + z''\beta_j - z'\gamma_j) + [2z'' F_3 \alpha_j + F_3\beta_j - E_1\gamma_j]\lambda + (F_3^2 \alpha_j)\lambda^2}{(z'' + \mu F_3)^2}$$

$$\stackrel{\text{def}}{=} \frac{\tilde{P}_j \lambda^2 + \tilde{Q}_j \lambda + \tilde{R}_j}{(z'' + \lambda F_3)^2} \ge 0. \qquad (2.11)$$

In accordance with assumption which $qx + d > 0$ is for any feasible solution $\bar{x}$, we need to have

$$z'' + \lambda F_3 > 0, \qquad (2.12)$$

Which the following restriction is obtained from (2.12):

$$\lambda \begin{cases} < \dfrac{-z''}{F_3}, & if\ F_3 < 0, \\ \ge 0, & if\ F_3 > 0. \end{cases} \qquad (2.13)$$

From (2.12), the relation (2.11) is satisfied if for all j, we have

$$\tilde{P}_j \lambda^2 + \tilde{Q}_j \lambda + \tilde{R}_j \ge 0. \qquad (2.14)$$

By similar discuss with (2.2), for any $\lambda \in [0,\ \lambda_2]$, inequalities (2.14) hold, where $\lambda_2 = \min(\lambda^j)$ and λ^j are the positive roots of the equations $\tilde{P}_j \lambda^2 + \tilde{Q}_j \lambda + \tilde{R}_j = 0$

for some j. On the other hand, to satisfy feasibility condition, it is necessary that $\tilde{x}_B = B^{-1}(b+\lambda\Delta b)$. Thus

$$0 \le \lambda \le \min\left\{\frac{(-B^{-1}b)_i}{(B^{-1}\Delta b)_i}: (B^{-1}\Delta b)_i < 0\right\} = \lambda_3.$$

Therefore, it is obvious that if λ satisfies (2.13) and $0 \le \lambda \le \min\{\lambda_2,\ \lambda_3\}$, then $\bar{x}$ is an optimal solution of the LLFP(λ).

2.5 When perturbations are on p, c and b vectors

In this case, optimality condition leads to inequalities

$$\widehat{\mathrm{P}}_{\mathrm{j}}\lambda^3 + \widehat{\mathrm{Q}}_{\mathrm{j}}\lambda^2 + \widehat{\mathrm{S}}_{\mathrm{j}}\lambda + \widehat{\mathrm{R}}_{\mathrm{j}} \ge 0, \quad \forall j.$$

Inequalities hold for

$$0 \le \lambda \le \min\{\hat{\lambda}_j\} = \hat{\lambda},$$

Where $\hat{\lambda}_j$ are the positive roots of the equations

$$\widehat{\mathrm{P}}_{\mathrm{j}}\lambda^3 + \widehat{\mathrm{Q}}_{\mathrm{j}}\lambda^2 + \widehat{\mathrm{S}}_{\mathrm{j}}\lambda + \widehat{\mathrm{R}}_{\mathrm{j}} = 0, \qquad \text{for some j.}$$

On the other hand, feasibility condition holds for

$$0 \le \lambda \le \lambda_1.$$

It is obvious that if λ satisfies (2.13) and $0 \le \lambda \le \min\{\lambda_1,\ \hat{\lambda}\}$, then $\bar{x}$ is an optimal solution of the LLFP(λ).

3 Example

Let us consider the problem

$$\max \mathrm{Z} = 2\mathrm{x}_1 + 6\mathrm{x}_2 + 2\mathrm{x}_3 + \frac{3\mathrm{x}_1 + 5\mathrm{x}_2 + 6\mathrm{x}_3}{\mathrm{x}_1 + 3\mathrm{x}_2 + \mathrm{x}_3 + 2}$$

$$\begin{aligned} \mathrm{s.t:}\quad & 3\,\mathrm{x}_1 - \mathrm{x}_2 + 2\mathrm{x}_3 + \mathrm{x}_4 = 7 \\ & -2\mathrm{x}_1 + 4\mathrm{x}_2 + \mathrm{x}_5 = 12 \\ & -4\mathrm{x}_1 + 3\mathrm{x}_2 + 8\mathrm{x}_3 + \mathrm{x}_6 = 10 \\ & \mathrm{x}_1, \mathrm{x}_2,\ \mathrm{x}_3,\ \mathrm{x}_4,\ \mathrm{x}_5,\ \mathrm{x}_6 \ge 0. \end{aligned}$$

By using the simplex type procedure of Teterev [8], the optimal solution is $x^* = (x_B^*, 0)$, where $x_B^* = (x_1^*,\ x_2^*, x_6^*) = (4, 5, 11)$. Consider the following perturbation vectors:

$$\Delta\mathrm{c} = (1, -2, 3), \quad \Delta\mathrm{p} = (2, 1, 0), \quad \Delta\mathrm{q} = (3, -1, 1).$$

In this case, we have

$$\tilde{z}'' = 21 + 7\lambda, \qquad \tilde{z}' = 37 + 13\lambda, \qquad \tilde{z}_N^c - \tilde{c}_N = \left(2 - 3\lambda, 2, 2 - \frac{1}{2}\lambda\right),$$
$$\tilde{z}_N^p - \tilde{p}_N = \left(\frac{-8}{5} + 2\lambda, \frac{11}{5} + \lambda, \frac{9}{5} + \frac{1}{2}\lambda\right), \qquad \tilde{z}_N^q - \tilde{q}_N = (1 - \lambda, 1 + \lambda, 1).$$

Therefore, for $j \in N = \{3, 4, 5\}$ inequalities (2.8) are as follows

$$811.4 - 680.2\lambda - 785\lambda^2 - 147\lambda^3 \geq 0, \qquad (3.1)$$
$$891.5 + 574.4\lambda + 92\lambda^2 \geq 0, \qquad (3.2)$$
$$882.7 + 377.6\lambda - 14\lambda^2 - 24.5\lambda^3 \geq 0. \qquad (3.3)$$

Thus, the admissible interval for λ is

$$0 \leq \lambda \leq \min\{0.648536, 4.55246\} = 0.648536.$$

4 Conclusion

The sensitivity analysis of optimal solutions has been presented in this note. The different cases of the perturbation in the coefficients objective and RHS are considered. In each case, the underlying theory for sensitivity analysis has been presented to obtain bounds for each perturbation.

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