LIE SYMMETRIES, SELF-ADJOINTNESS, AND CONSERVATION LAWS OF THE MONG-AMPERE EQUATION

ABSTRACT. This paper addresses an extended 2-dimensional Mong-Ampere equation By The Lie method. The symmetries of Mong-Ampere equation are found and the method of non-linear self-adjointness is applied to the considered equation. By applying Ibragimov's method and Noether operators the infinite set of conservation laws associated with the finite algebra of Lie point symmetry of the Mong-Ampere equation is extracted. The corresponding conserved quantities are computed from their respective densities.

1. INTRODUCTION

Partial differential equations form the basis of many mathematical models of the physical, chemical and engineering, and more recently their use has spread into economics, financial forecasting, image processing and other fields. Lie group transformation technique, initially advocated by Norwegian mathematician Sophus Lie, is an effective and useful tool to derive an exact solution, symmetries and conservation laws of PDEs. Symmetries and conservation laws belong to the central studies of non-linear evolutional equations. Especially, one nonlinear partial differential equation is believed to be integral in the sense that it possesses an infinite number of symmetries or conservation laws. Ibragimov's method by applying Noether's operators and formal Lagrangian and lie point symmetries gives a useful and effective algorithm for obtaining the conservation laws of the non-liner self-adjointness PDEs [1, 2, 4, 8, 10, 11]. In this paper, we use the Lie point symmetry method for solving a non-linear partial differential equation. In fact, some linear and most non-linear differential equations are virtually impossible to solve using exact solutions, so it is often possible to find numerical or approximate solutions for such type of problems. The main goal of this paper is symmetry analysis of the 2-dimensional Mong-Ampere equation:

$$u_{xy}^2 - u_{xx}u_{yy} = f(x, y), \tag{1.1}$$

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where u is a dependent variable of x and y, $f(x, y)$ is an arbitrary function of x and y and subscripts denote the partial derivatives. By substituting $f(x, y) = u^3$ in the Eq. (1.1) this eqution equation is converted to the following eqution:

$$u_{xy}^2 - u_{xx}u_{yy} - u^3 = 0. \tag{1.2}$$

The main purpose of this paper is to expand a generalized Lie symmetry approach for determining the Lie point symmetries, conserved vectors and conservation laws for Mong-Ampere equation.

2. Preliminaries and basic definitions

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations are solved with the aid of this method [7]. There are still many authors who use this method to find the exact solutions of non-linear differential equations. Since this method has powerful tools to find exact solutions of non-linear problems. For example, when we are confronted with a complicated system of PDEs or ODEs, it is interesting to find a vast set of exact solutions for the given system via a systematic method with no limitation, this would be done by using Lie's symmetry as an analytic applicable method.

Here we recall some basic notions, definitions and theorems of the theory of Lie group.

Definition 2.1. An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such away that both the group operation and the inversion

$$m : G \times G \to G, \qquad i : G \to G,$$
$$m(g.h) = g.h, \quad g, h \in G \qquad i(g) = g^{-1}, \quad g \in G,$$

are smooth maps between manifolds[11].

Definition 2.2. Let G be a local group of transformations acting on a manifold M. G is called a symmetry group of a subset $\varphi \subset M$, if whenever $x \in \varphi$ and $g \in G$ is such that $g.x$ is defined then $g.x \in \varphi$ [10, 11].

Definition 2.3. Let G be a local group of transformations acting on a manifold M. a function $F : M \to N$ where N is another manifold is called a G invariant function if for all $x \in M$ and all $g \in G$ such that $g.x$ is defined $F(g.x) = F(x)$ [10].

Definition 2.4. Given a smooth real valued function $f(x) = f(x^1, \cdots, x^p)$ of p independent variables there are $p_k = \binom{p+k-1}{k}$ different k-th order partial derivatives of f we employ the multi-index notation

$$U_J = \partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1}, \ldots, \partial x^{j_k}}$$

for these derivatives. In this notation, $J = (j_1, \cdots, j_k)$, $1 \leq j_k \leq p$. Then the n-th prolongation of U is given as:

$$pr^{(n)} f(x) = U^{(n)} = u \times U_1 \times U_2 \times \cdots \times U_n.$$

Theorem 2.5. *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u) \frac{\partial}{\partial u^\alpha} \tag{2.1}$$

be a vector field on a open subset $M \subset x \times U$ the n-th prolongation of $\mathbf{v}$ *is the vector field*

$$\mathbf{v}^{(n)} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}, \tag{2.2}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$ the coefficient function ϕ_α^J of $pr^{(n)}\mathbf{v}$ are given by the following formula

$$\phi_\alpha^J(x, u^{(n)}) = D_J\big(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\big) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha\,, \tag{2.3}$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$ and $u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}$ and $J = (j_1, \cdots J_k) \quad 1 \leq J_k \leq p$ and

$$\phi_\alpha^{J.k} = D_k \phi_\alpha^J + \sum_{i=1}^{p} D_k \xi^i u_{Ji}^\alpha.$$

that

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u_{J,i}^\alpha \frac{\partial F}{\partial u_J^\alpha}, \tag{2.4}$$

is the total derivative of $F(x,u)$ where $i = 1, \cdots, p$ and $j = 1, \cdots, q$ denote independent variables and dependent variables respectively [10].

3. Lie Symmetry Analysis of The Mong-Ampere Equation

In this section, we obtain the Lie point symmetries of the Eq. (1.2). Based on the Lie group theory, if Eq.(1.2) be invariant under a one parameter Lie group of point transformation it also satisfies following invertible transformations on variables x, y, u and derivatives of u in terms of independent variables

$$\begin{aligned}
\bar{x} &= x + \epsilon\xi(x, y, u) + O(\epsilon^2),\\
\bar{y} &= y + \epsilon\eta(x, y, u) + O(\epsilon^2),\\
\bar{u} &= u + \epsilon\zeta(x, y, u) + O(\epsilon^2),\\
\bar{u}_{xx} &= u_{xx} + \epsilon\zeta^{xx}(x, y, u) + O(\epsilon^2),\\
\bar{u}_{xy} &= u_{xy} + \epsilon\zeta^{xy}(x, y, u) + O(\epsilon^2),\\
\bar{u}_{yy} &= u_{yy} + \epsilon\zeta^{yy}(x, y, u) + O(\epsilon^2).
\end{aligned} \tag{3.1}$$

where ξ, η and ζ are infinitesimals and ζ^{xx}, ζ^{xy} and ζ^{yy} are extended infinitesimals of order 2 and ϵ is the group parameter. Lie algebra to this group is spanned by the following vector fields

$$\mathbf{v} = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \zeta(x, y, u)\frac{\partial}{\partial u}. \tag{3.2}$$

where

$$\frac{d\bar{x}}{d\epsilon}\Big|_{\epsilon=0} = \xi(x, y, u), \qquad \frac{d\bar{y}}{d\epsilon}\Big|_{\epsilon=0} = \eta(x, y, u), \qquad \frac{d\bar{u}}{d\epsilon}\Big|_{\epsilon=0} = \zeta(x, y, u). \tag{3.3}$$

According to the infinitesimal invariance criterion $\mathbf{v}$ is an infinitesimal generator of transformation group on it's solution manifold if and only if

$$pr^{(2)}\mathbf{v}(F)\Big|_{F=0} = 0. \tag{3.4}$$

where F is Eq.1.2 and $pr^{(2)}$ is the 2-th prolongation of infinitesimal generator $\mathbf{v}$. Thus we prolong the generator $\mathbf{v}$ to the derivatives involved in Eq. (1.2) and obtain

$$pr^{(2)}\mathbf{v} = \mathbf{v} + \zeta^{xx}\frac{\partial}{\partial u_{xx}} + \zeta^{xy}\frac{\partial}{\partial u_{xy}} + \zeta^{yy}\frac{\partial}{\partial u_{yy}}. \tag{3.5}$$

The explicit expression for ζ^x, ζ^y and ζ^{xx} are

$$\begin{aligned}
\zeta^x &= D_x(\zeta) - u_y D_x(\eta) - u_x D_x(\xi)\\
&= \zeta_x + \zeta_u u_x - \eta_x u_y - \xi_x u_x - \xi_u u_x^2 - \eta_u u_y u_x,\\
\zeta^y &= D_y(\zeta) - u_y D_y(\eta) - u_x D_y(\xi)\\
&= \zeta_y + \zeta_u u_y - \eta_y u_y - \xi_y u_x - \eta_u u_y^2 - \xi_u u_y u_x,\\
\zeta^{xx} &= D_x(\zeta^x) - u_{xy} D_x(\eta) - u_{xx} D_x(\xi)\\
&= \zeta_{xx} + 2\zeta_{xu} u_x - \eta_{xx} u_y - \xi_{xx} u_x + \zeta_{uu} u_x^2 - 2\eta_{xu} u_y u_x - 2\xi_{xu} u_x^2 - \xi_{uu} u_x^3\\
&\quad -\eta_{uu} u_y u_x^2 + \zeta_u u_{xx} - 2\eta_x u_{xy} - 2\xi_x u_{xx} - \eta_u u_y u_{xx} - 3\xi_u u_x u_{xx} - 2\eta_u u_x u_{xy}.
\end{aligned} \tag{3.6}$$

Similarity one can obtain the coefficients prolongation ζ^{xy} and ζ^{yy}. By solving the above invariant Equation (3.4), one can determine ξ , η and ζ explicitly. To determine the Lie symmetries admitted by Eq. (1.2), we split the obtained relation by independent variables. We equate the coefficients to zero and solve the obtained over-determined system of linear PDEs. Then we compute the coefficients of vector field (3.2) as

$$\xi = C_1 x + C_2 x + C_3$$
$$\eta = C_4 x + C_5 y + C_6$$
$$\zeta = -2u(C_1 + C_2)$$

Hence the symmetry Lie algebra of the Mong-Ampere equation is spanned by the following generators:

$$\begin{aligned} \mathbf{v}_1 &= x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \qquad \mathbf{v}_2 = y\frac{\partial}{\partial x}, \qquad \mathbf{v}_3 = \frac{\partial}{\partial x}, \\ \mathbf{v}_4 &= x\frac{\partial}{\partial y}, \qquad \mathbf{v}_5 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}, \qquad \mathbf{v}_6 = \frac{\partial}{\partial y}. \end{aligned} \tag{3.7}$$

The one-parameter groups G_i generated by the above $\mathbf{v}_i$. The groups G_3 and G_6 demonstrate the space-invariance of the equation, reflecting the fact that the Mong-Ampere equation has constant coefficients. The well-known scaling symmetry turns up in G_1, G_2, G_4 and G_5 [6, 9, 10].

4. Non-linear Self-Adjointness and Conservation Laws

The concept of a conservation law in physics is a proposition that states a particular measurable property of an isolated physical system dose not change as the system evolves over time. In the study of systems of differential equations, the concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on, plays an important role in the analysis of basic properties of the solutions. Noether's theorem provides us with a systematic method for finding conservation laws of PDEs provided a Noether symmetry associated with a Lagrangian is known for Euler–Lagrange equations [3] . However, there exist some approaches in the literature for obtaining the conservation laws of the PDEs, which do not have a Lagrangian. On the basis of new conservation law theorem firstly proposed by Ibragimov.In the present paper, the classical definition of conservation laws for PDEs was used according to which a conservation law is understood as the vanishing of the divergence of some vector. This definition is a natural multidimensional generalization of the concept of a first integral for ODEs.

A vector $C = (C^x, C^y)$ is called a conserved vector for Equation (1.2) if it satisfies the conservation equation

$$\left[D_x(C^x) + D_y(C^y)\right]\Big|_{(1.2)} = 0, \tag{4.1}$$

The new conservation theorem given by Ibragimov provides a method to construct conservation laws for differential equations without classical Lagrangians. We suggest here the general concept of nonlinear self-adjointness. Let us consider a system of $m-$differential equations (linear or non-linear)

$$F_\alpha\left(x, u, u_{(1)}, \cdots, u_{(s)}\right) = 0, \quad \alpha = 1, \cdots, m, \tag{4.2}$$

with m-dependent variables $u = (u^1, \cdots, u^m)$. Eqs. (4.2) involve the partial derivatives $u_{(1)}, \cdots, u_{(s)}$ up to order s.

Definition 4.1. The adjoint equations to Eqs. (4.2) are given by

$$F^*_\alpha\left(x, u, v, u_{(1)}, v_{(1)}, \cdots, u_{(s)}, v_{(s)}\right) = \frac{\delta \mathcal{L}}{\delta u^\alpha}, \quad \alpha = 1, \cdots, m, \tag{4.3}$$

where $\mathcal{L}$ is the formal Lagrangian for Eqs. (4.2) defined by

$$\mathcal{L} = v^\beta F_\beta \equiv \sum_{\beta=1}^{m} v^\beta F_\beta. \tag{4.4}$$

Here $v = (v^1, \cdots, v^m)$ are new dependent variables, $v_{(1)}, \cdots, v_{(s)}$ are their derivatives, e.g. $v_{(1)} = v_i^\alpha$, $v_i^\alpha = D_i(v^\alpha)$. We use $\delta/\delta u^\alpha$ for the Euler-Lagrange operator

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \cdots, m. \tag{4.5}$$

Equation F is said to be non-linearly self-adjoint if the Equation F and F^* can be related by the equation $F^* = \lambda F$ after the substitution $v = \varphi(x, y, u)$ with a certain function $\varphi \neq 0$ [5].

It can easily be seen that the Mong-Ampere equation (1.2) has the following formal Lagrangian:

$$\mathcal{L} = vF = v(x, y)\left(u_{xy}^2 - u_{xx}u_{yy} - u^3\right). \tag{4.6}$$

We obtain the following adjoint equation to Eq. (1.2)

$$F^* \equiv -3u^2 v - v_{xx}u_{yy} - v_{yy}u_{xx} + 2v_{xy}u_{xx}. \tag{4.7}$$

Thus, the non-linear self-adjointness condition is written

$$-3u^2 v - v_{xx}u_{yy} - v_{yy}u_{xx} + 2v_{xy}u_{xx} = \lambda\left(u_{xy}^2 - u_{xx}u_{yy} - u^3\right), \tag{4.8}$$

where one makes the following replacements of v and its derivatives:

$$v = \varphi(x, y, u), \quad v_x = D_x(\varphi), \quad v_y = D_y(\varphi), \quad v_{xx} = D_x^2(\varphi).$$

After this replacement Eq. (4.8) should be satisfied identically in the variables $x, y, u, u_{xy}, u_{yy}, u_{xx}$. Let us express the derivatives of v involved in the adjoint equation (4.7) in the expanded form,

$$\begin{aligned}
&v_x = \varphi_u u_x + \varphi_x, \quad v_y = \varphi_u u_y + \varphi_y,\\
&v_{xx} = \varphi_{uu} u_x^2 + \varphi_u u_{xx} + 2\varphi_{xu} u_x + \varphi_{xx},\\
&v_{yy} = \varphi_{uu} u_y^2 + \varphi_u u_{yy} + 2\varphi_{yu} u_y + \varphi_{yy},\\
&v_{xy} = \varphi_{uu} u_x u_y + \varphi_u u_{xy} + \varphi_{yu} u_x + \varphi_{xu} u_y + \varphi_{xy},
\end{aligned}$$

and substitute them in the left side of Eq. (4.8). The comparison of the coefficients for u and its derivatives in both side of Eq. (4.8) yields

$$v = C_1 x + C_2. \tag{4.9}$$

According to the Ibragimov theorem if a differential equation be a nonlinear self-adjointness equation then any Lie point symmetry of it gives the conserved vector with components defined by the following explicit formulae

$$C^i = N^i(\mathcal{L}) \tag{4.10}$$

where C^i are components of conserved vector and N^i is the Noether operator given by below formula

$$N^i = W\left(\frac{\delta}{\delta u_i}\right) + \sum_{s=1}^{\infty} D_{i_1} \cdots D_{i_s}(w) \frac{\delta}{\delta u_{ii_1\cdots i_s}} \tag{4.11}$$

where W is the characteristic of the symmetries. Thus the components of conserved vector $C = (C^x, C^y)$ is given by:

$$\begin{aligned}
C^x &= W\frac{\partial \mathcal{L}}{\partial u_x} + D_x(W)\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) + D_y(W)\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right),\\
C^y &= W\frac{\partial \mathcal{L}}{\partial u_y} + D_x(W)\left(\frac{\partial \mathcal{L}}{\partial u_{xy}}\right) + D_y(W)\left(\frac{\partial \mathcal{L}}{\partial u_{yy}}\right).
\end{aligned} \tag{4.12}$$

we let $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, in the above formula. The operator $\mathbf{v}_1 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$ with characteristic $W = -2u - xu_x$ gives the conserved vector $C = (C^1, C^2)$ such as

$$\begin{aligned}
C^x &= (-2u - xu_x)(v_x u_{yy} - v u_{xyy} - 2v_y u_{xy}) + (3u_x + xu_{xx})(v u_{yy})\\
&\quad -(2u_y + xu_{xy})(2v u_{xy}),\\
C^y &= (-2u - xu_x)(-2v_x u_{xy} - v u_{xxy} + v_x u_{xx}) + (2u_y + xu_{xy})(v u_{xx})\\
&\quad -(3u_x + xu_{xx})(2v u_{xy}).
\end{aligned}$$

The symmetry $\mathbf{v}_2 = y\frac{\partial}{\partial x}$ with characteristic $W = -yu_x$ of the Eq. (1.2) provides the conserved vector $C = (C^1, C^2)$ with components

$$C^x = -yu_xv_xu_{yy} + yu_xu_{xyy} + 2yu_xv_yu_{xy} + vyu_{xx}u_{yy} - 2vu_xu_{xy},$$
$$C^y = 2yu_xv_xu_{xy} + vyu_xu_{xxy} - yu_xv_xu_{xx} + vu_xu_{xx} - 2vyu_{xx}u_{xy}.$$

Similarly one can obtain the other conserved vector with the symmetries (3.7)

5. Conclusion

In this paper, a Lie group analysis for an important PDE called Mong-Ampere equations is given. The Lie algebra of symmetries was found by a useful algorithm. Also, these operators are applied for finding conservation laws of the system due to Noether's method.

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