Optimization
Iranian Journal of Optimization
Volume 13, Issue 2, 2021, 93-101
Research Paper

Online version is available on: www.ijo.rasht.iau.ir

Islamic Azad University
Rasht Branch
ISSN: 2588-5723
E-ISSN:2008-5427

Convergence of Triple Accelerated Over-Relaxation (TAOR) Method for M-Matrix Linear Systems

Khadeejah James Audu*, Yusuph Amuda Yahaya, Rufus Kayode Adeboye and Usman Yusuf Abubakar

Department of Mathematics, School of Physical Sciences, Federal University of Technology, Minna, Nigeria

Revise Date: 04 February 2021
Accept Date: 19 September 2021

Keywords:
TAOR method
M-matrix
linear systems
Spectral radius
Convergence conditions

Abstract

In this paper, we propose some necessary conditions for convergence of Triple Accelerated Over-Relaxation (TAOR) method with respect to $M-$ coefficient matrices. The theoretical approach for the proofs is analyzed through standard procedures in the literature. Some numerical experiments are performed to show the efficiency of our approach, and the results obtained compared favourably with those obtained through the existing methods in terms of spectral radius of their iteration matrices.

***Correspondence E-mail**: k.james@futminna.edu.ng

INTRODUCTION

The study of large sparse system of linear equations with coefficient $M-$ matrices often appears in many physical problems of engineering and sciences, such as in solutions to computational fluid dynamics, oil and gas development, circuit simulators and thermal structural problems (Wu et al., 2007; Xue et al., 2013). In the field of numerical linear algebra, theory of $M-$ matrices play an essential role in obtaining solutions for linear systems that are usually computed by various iterative methods. Some stationary iteration methods like Gauss-Seidel, Successive Over-relaxation (SOR) and AOR (Accelerated Over-Relaxation) methods utilized in solving linear systems have shown to be convergent for $M-$matrix linear systems (Hadjidimos, 1980; Liu & Chen, 2010; Shi-Guang & Ting, 2017).

We seek to solve a class of $m \times m$ large linear systems in the form

$$Az = b \tag{1}$$

Where $A = [a_{ij}]$ belongs to a class of $M-$ matrices (matrix that satisfies $a_{ii} > 0$, $a_{ij} \leq 0 \ (i \neq j)$, $\det A \neq 0$ and $A^{-1} \geq 0$), z is the vector of unknowns and b is the known vectors.

The usual splitting of A is defined as:

$$A = D - L_T - U_T \tag{2}$$

Where $-U_T$ and $-L_T$ are the strictly upper and strictly lower segments of A and $D = \text{diag}\,[a_{11}, a_{22}, \ \cdots, a_{\text{mm}}]$ is the diagonal segment of A. Suppose the coefficient matrix (A) has non-vanishing diagonal elements ($a_{ii} \neq 0$) for $i = 1,2,\cdots,m$, then (1) can be multiplied by D^{-1} to obtain $D^{-1}A = I - L - U$, thus equation (1) can be transformed into

$$(I - L - U\,)z = \bar{b} \tag{3}$$

Where $L = D^{-1}L_T$, $U = D^{-1}U_T$, $\bar{b} = D^{-1}b$ and $I = D^{-1}D$. A regular splitting of the matrix A into

$$A = M - N \tag{4}$$

With M nonsingular, is required to solve the linear system (1) iteratively. By employing the splitting in (4), a linear stationary iterative method denoted by

$$z^{(s+1)} = M^{-1}Nz^{(s)} + M^{-1}b, \quad s = 0, 1, \cdots, m \tag{5}$$

Is constructed, where $T = M^{-1}N$ is referred to as the iteration matrix of the scheme and $C = M^{-1}b$ represents the column vector. The iteration 5 converges to the true solution $z = A^{-1}b$ for any initial guess $z^{(0)}$ provided the spectral radius $\rho(T) < 1$. It is a well- known fact that spectral radius of any iteration matrix decides the stability and convergence of the particular method. Meanwhile, convergence rate of iterative methods are usually high when their spectral radius is near zero and low when it is near one. This implies that the smaller the spectral radius the faster the method converges. Several researchers have been interested in constructing new iteration methods 5 for solutions of linear systems, to magnify the rate of convergence.

The well-known AOR method introduced by Hadjidimos (1978) for solution of (1) is governed by the expression:

$$z^{(s+1)} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U\,]z^{(s)} + (I - rL)^{-1}\omega\bar{b} \tag{6}$$

In recent years, the AOR iterative methods has been modified in different versions and established for solutions of linear systems. These AOR-type methods includes the QAOR (Quasi Accelerated Over-relaxation) method developed by Wu & Liu (2014), who studied the convergence of the method for $L-$, positive definite, irreducibly diagonally dominant and $H-$ matrices, though not as efficient as the AOR iterative method. Youssef & Farid (2015) introduced the KAOR method and proved the convergence of the method by considering $L-$, irreducibly diagonally dominant and consistently ordered matrices. The above methods with two parameters attempted to increase con-vergence rate of the AOR method. By introduction of a new parameter into the

AOR scheme, Vatti et al. (2019) developed the PAOR (Parametric Accelerated Over-Relaxation) method and studied the convergence of the method for consistently ordered matrices.

Recently, Audu et al. (2020) embarked on improvement of the AOR method with three parameters and developed the TAOR (Triple Accelerated Over-Relaxation) method. They studied convergence properties of the method for some special matrices such as irreducibly diagonally dominant, $-$, and $H-$matrices. The TAOR method for solving the linear system 1 is defined as

$$z^{(s+1)} = T_{v,r,\omega} z^{(s)} + (I - rL - vL)^{-1}\omega\bar{b} \quad (7)$$

Where $T_{v,r,\omega}$ represents the TAOR iteration matrix expressed as;

$$T_{v,r,\omega} = (I - rL - vL)^{-1}[(1-\omega)I + \omega U + (\omega - r - v)L] \quad (8)$$

The spectral radius of the TAOR method is represented as $\rho(T_{v,r,\omega})$. Interestingly, options of $T_{v,r,\omega} = T_{0,\omega,\omega}, T_{0,r,\omega}, T_{0,1,1}$ and $T_{0,0,1}$ in 8 corresponds to iterative matrices of SOR, AOR, Gauss-Seidel and Jacobi methods.

The Triple AOR (TAOR) method have been shown to be convergent for $L-$matrix, irreducibly diagonally dominant matrix, and $H-$ matrix linear systems. However, convergence of the method has not been established for $M-$ matrix. This research work is focused on investigating the convergence of the TAOR method with respect to $M-$matrix linear systems through some theoretical proofs and validated by numerical experiment.

PRELIMINARIES

Convergence conditions for $M-$ matrix will be examined in the next section, but prior to that, we need the following basic definitions and lemmas in our analysis.

Definition 2.1: A matrix $A = [a_{ij}]$ is said to be non-negative (positive) if and only if each element of matrix A is positive, denoted as $A \geq 0$ or $a_{ij} \geq 0$ for all $i,j = 1,2,\cdots,N$ (Varga, 2000).

Definition 2.2: A square matrix $A = [a_{ij}]$ is considered an $M-$matrix if the following;

I. $a_{ii} > 0$
II. $a_{ij} \leq 0$, $i \neq j$
III. Determinant $A \neq 0$ and
IV. $A^{-1} \geq 0$ for all $i,j = 1,2,\cdots,N$

are satisfied (Axelsson, 1996).

Definition 2.3: A square matrix $A = [a_{ij}]$ is an $L-$matrix if it has positive diagonal entries $(a_{ii} > 0)$ and its off-diagonal entries are less than or equal to zero $(a_{ij} \leq 0)$ for $i \neq j$, $i = 1,2,\cdots,n$.

Definition 2.4: A matrix $A = [a_{ij}]$ is said to be an $H-$ matrix if and only if its comparison matrix, that is, the matrix $\langle A \rangle$ with elements $\gamma_{ii} = |a_{ii}|$, $i = 1,2,\cdots,n$ and $\gamma_{ii} = -|a_{ii}|$, $i \neq j$ is an M-matrix.

Definition 2.5: A matrix $A = [a_{ij}]$ is said to be irreducible if and only if its directed graph associated with A i.e $G[A]$ is strongly connected (Berman and Plemmon, 1994).

Definition 2.6: The decomposition of any given matrix A into the form $A = M_T - N_T$ where M_T is a non-singular M matrix called a splitting of A. And such splitting is:

I. Nonnegative if ${M_T}^{-1} \geq 0$
II. Regular if ${M_T}^{-1} \geq 0$ and $N \geq 0$
III. Weak regular if ${M_T}^{-1} \geq 0$ and ${M_T}^{-1}N_T \geq 0$
IV. M splitting if M_T is a non-singular $M-$matrix and $N_T \geq 0$ (Young, 2014).

Definition 2.7: The Spectral radius of matrix $A = [a_{ij}]$ is the greatest value among the absolute values of the eigenvalues λ_k of matrix A, denoted as $\rho[A] = \max_{\lambda_k \in A}|\lambda_k|$.

Lemma 2.1: Let $A \geq 0$ be an irreducible square matrix. Then

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To the spectral radius of A denoted as $\rho(A)$, there corresponds an eigenvector $z > 0$.

iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A (Varga, 2000).

Lemma 2.2: Let $A = [a_{ij}]$ and $C = [c_{ij}]$, be two matrices such that $A \leq C$, where $c_{ij} \leq 0$ for all $i \neq j$, then if A is an $M-$matrix so also is matrix C. (Saad, 1995)

Lemma 2.3: Suppose matrix A is an $M-$matrix and the splitting $A = M_T - N_T$ is a weak regular or regular splitting of A, then $\rho\left({M_T}^{-1}N_T\right) < 1$ (Wang & Song, 2009).

MAIN RESULTS

The Triple Accelerated Over-relaxation (TAOR) method is a new class of iterative methods that is closely related to AOR method for solving linear systems, though with some benefits over the AOR method. Although the method is applicable to any coefficient matrix with non-zero entries on the diagonals, but convergence of the method is only guaranteed for H, irreducibly diagonally dominant, and $L-$matrices. Some basic results associated with the TAOR method are as follows:

Theorem 3.1: If matrix A is an $L-$matrix, then $\rho\left(T_{v,r,\omega}\right) < 1$ where $T_{v,r,\omega}$ is expressed in Eq. 8 and so the sequence $z^{(s)}$ generated by the method 7 converges to the true solution for any arbitrary initial approximation $z^{(0)}$ (Audu et al., 2020).

Theorem 3.2: Suppose an matrix A is of the linear system 1 is an $H-$ matrix, then $\rho\left(T_{v,r,\omega}\right) < 1$ where $T_{v,r,\omega}$ is defined in equation (8) and thereby the sequence $z^{(s)}$ generated by the method 7 converges to the true solution for any arbitrary initial guess $z^{(0)}$ (Audu et al., 2020).

Theorem 3.3: If a square matrix A of 1 is an irreducible diagonally dominant matrix, then $\rho\left(T_{v,r,\omega}\right) < 1$ where $T_{v,r,\omega}$ is defined in equation (8) and therefore the sequence $z^{(s)}$ generated by the method 7, converges to the real solution for any arbitrary initial approximation $z^{(0)}$ (Audu et al., 2020).

In this section, our focus is mainly to study the convergence of the TAOR method for linear systems with $M-$matrices.

Now, by applying the lemmas in section 2, the following theorems are proposed in establishing the convergence of the TAOR iterative method for M - matrices.

Theorem 3.4: If matrix A is an $M-$matrix and $A = D - L_T - U_T$ is a splitting defined in (3), then D is an $M-$matrix and $\rho(D^{-1}L_T) < 1$.

Proof:

Suppose $A = [a_{ij}]$ is a $-M$matrix with the decomposition $A = D - L_T - U_T$. Now, if we let $D - L_T$ to be a splitting of a matrix say Y, such that we have

$$Y = D - L_T \tag{9}$$

Then, obviously $A \leq Y$ and applying lemma 2.2 indicates that matrix Y is equally an M matrix since A is an M matrix. Likewise, it is observed that D is equally an M matrix too and this means that $D^{-1} \geq 0$. Also, the matrix L_T is nonnegative, that is to say $L_T \geq 0$. Having in mind that the two sub matrices D and L_T of Y are both nonnegative;

$$D^{-1} \geq 0 \quad \text{and} \quad L_T \geq 0 \tag{10}$$

This further explains that $Y = D - L_T$ is a regular splitting of Y and by application of lemma 2.3, we deduces that $\rho(D^{-1}L_T) < 1$ which completes the proof.

Theorem 3.5: If matrix A is an $M-$matrix for the range of values $0 \leq r + v \leq \omega \leq 1, (\omega \neq 0)$, then the Triple Accelerated Over-relaxation (TAOR) iterative method converges to the real solution or simply $\rho\left(T_{v,r,\omega}\right) < 1$.

Using similar idea of Salkuyeh (2011), theorem 3.2 will be established by the following proof;

Proof:

Given that $A = [a_{ij}]$ is an $M-$matrix with the usual decomposition $A = D - L_T - U_T$ and the regular splitting $A = M - N$. With respect to the TAOR method, we have the choices;

$M_T = \frac{1}{\omega}(D - rL_T - vL_T), \quad N_T = \frac{1}{\omega}\left((1-\omega)D + (\omega - r - v)L_T + \omega U_T\right) \quad (11)$

Which gives the TAOR splitting as;

$$\omega A = M_T - N_T \qquad (12)$$

With matrices $M_T = D - (r+v)L_T$ and $N_T = (1-\omega)D + \big(\omega - (r+v)\big)L_T + \omega U_T$. For the matrix $N_T = (1-\omega)D + \big(\omega - (r+v)\big)L_T + \omega U_T$, given that $L_T \geq 0$ and $U_T \geq 0$, the inequalities $\omega \geq 0$, $\big(\omega - (r+v)\big) \geq 0$ and $(1-\omega) \geq 0$ are considered which gives the condition of $0 \leq r + v \leq \omega \leq 1$, $(\omega \neq 0)$ for ensuring that N_T is nonnegative. Hence

$$N_T = \big((1-\omega)D + (\omega - r - v)L_T + \omega U_T\big) \geq 0 \qquad (13)$$

Apparently, we have $A \leq M_T$ and by implication of lemma 2.2, it follows that M_T is an M matrix and as a result $M_T{}^{-1} \geq 0$ since $A^{-1} \geq 0$. Meanwhile, from theorem 2.1, we obtain $\rho(D^{-1}L_T) < 1$. Now, in view of the fact that $0 \leq r + v \leq 1$, we get $\rho\big((v+r)D^{-1}L_T\big) < 1$. Considering the fact that $\rho((r+v)D^{-1}L_T) < 1$, $M_T = D - (r+v)L_T$ is an $M-$ splitting and as such M_T is an $M-$matrix it follows that

$$M_T{}^{-1} = (D - (r+v)L_T)^{-1} \geq 0 \qquad (14)$$

Also, matrix $M_T{}^{-1}N_T$ is examined as follows;

$$\begin{aligned} M_T{}^{-1}N_T &= [D - (v+r)L_T]^{-1} \times [(1-\omega)D + [\omega - (v+r)]L + \omega U\,] \\ &= \big(D + (v+r)L_T + (v+r)^2 L_T{}^2 + (v+r)^3 L_T{}^3 + \cdots + (v+r)^{N-1} L_T{}^{N-1}\big) \\ &\quad \times [(1-\omega)D + [\omega - (v+r)]L_T + \omega U_T\,] \end{aligned} \qquad (15)$$

Which gives

$$\begin{aligned} &(1-\omega)D + (1-\omega)(v+r)L_T + (1-\omega)(v+r)^2 L_T{}^2 + \cdots + (1-\omega)(v+r)^{N-1}L_T{}^{N-1} \\ &+ (\omega - v - r)L_T + (v+r)(\omega - v - r)L_T{}^2 + (v+r)^2(\omega - v - r)L_T{}^3 + (v+r)^3(\omega - v - r)L_T{}^4 \\ &+ \cdots + (v+r)^{N-1}(\omega - v - r)L_T{}^N + \omega U_T \\ &+ \omega(v+r)L_T U_T + \omega(v+r)^2 L_T{}^2 U_T + \omega(v+r)^3 L_T{}^3 U_T + \cdots + \omega(v+r)^{N-1}L_T{}^{N-1} \\ &\geq 0 \end{aligned} \qquad (16)$$

Multiply through 16 by D^{-1} and after letting $D^{-1}L_T = L$, $I = D^{-1}D$ and $D^{-1}U_T = U$, it becomes

$$\begin{aligned} &(1-\omega)I + (1-\omega)(v+r)L + (1-\omega)(v+r)^2L^2 + \cdots + (1-\omega)(v+r)^{N-1}L^{N-1} \\ &+ (\omega - v - r)L + (v+r)(\omega - v - r)L^2 + (v+r)^2(\omega - v - r)L^3 + (v+r)^3(\omega - v - r)L^4 \\ &+ \cdots + (v+r)^{N-1}(\omega - v - r)L^N + \omega U \\ &\geq 0 \end{aligned} \qquad (17)$$

Therefore the iteration matrix becomes

$$M_T{}^{-1}N_T = \sum_{k=0}^{\infty}(I - (r+v)L)^{-1}\left[(1-\omega)I + \big(\omega - (r+v)\big)L + \omega U\,\right] \geq 0 \qquad (18)$$

The matrix $M_T{}^{-1}N_T$ is nonnegative, therefore $\omega A = M_T - N_T$ is obviously a weak regular splitting of matrix ωA. And in view of lemma 2.3, then it means $\rho\big(M_T{}^{-1}N_T\big) < 1$ or equivalently

$$\rho((I - (r+v)L)^{-1}[(1-\omega)\mathrm{I} + (\omega - r - v)L + \omega U]) < 1 \qquad (21)$$

This implies that the iteration matrix of the TAOR method for an $M-$matrix is

$\rho\left({M_T}^{-1}N_T\right) < 1$, thus, the theorem is proved and completed.

The algorithm of the TAOR method for solving $M-$matrix linear system is described as follows

***Algorithm* 3.1:** Algorithm of TAOR Method

To solve the linear system $Az = b \quad or \quad (I - L - U)z = \bar{b}$

Step 0: Input the entries a_{ij} and b_i; $1 \le i, j \le m$ of the matrix A and b respectively

Input $v,\ r,\ \omega,\ L,\ U, I$ and $\bar{b}$

Step 1: Choose an initial guess $z_i^{(0)} = 0$ for $\mathrm{s} = 0, 1, 2, \dots, \mathrm{s}_{max}$ and for $\mathrm{i} = 1, 2, 3, \dots, m$,

where $i = 1,2,3, \dots, m$ refers to the number of iterations

Step 2: Set $T = (I - vL - rL)^{-1}\left((1-\omega)I + (\omega - v - r)L + \omega U\right)$

Set $C = (I - vL - rL)^{-1}(\omega\bar{b})$

Step 3: for $\mathrm{s} = 0,1,2, \dots, m$ then, set $z_i^{(\mathrm{s}+1)} = Tz_i^{(\mathrm{s})} + C$

If $\left\|z - z_i^{(0)}\right\| < \mathrm{TOL}$, output $(z_1, z_2, \dots, z_m)$

Step 4: update $\mathrm{s} \coloneqq \mathrm{s} + 1$

Step 5: For $s = 0, 1, 2, \dots, m$, output ("maximum number of iterations exceeded")

STOP

NUMERICAL RESULTS

In order to verify the theoretical results obtained in section 3, the convergence and efficiency of the TAOR method is tested on three M- matrices. The spectral radii of iteration matrices for the TAOR and AOR methods are computed. The results are presented in Tables 4.1, 4.2 and 4.3 and notations employed for the methods in the tables are the following;

$T_{v,r,\omega} = (I - rL - vL)^{-1}[(1-\omega)I + (\omega - r - v)L + \omega U]$: *TAOR iteration matrix*

$T_{0,r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U\,]$: *AOR iteration matrix*

Sample 4.1: We consider a 4×4 matrix of 1 from Mayaki & Ndanusa (2019) in the form

$$A_1 = \begin{pmatrix} 1 & -\frac{12}{43} & -\frac{10}{43} & 0 \\ -\frac{15}{49} & 1 & 0 & -\frac{10}{49} \\ -\frac{13}{49} & 0 & 1 & -\frac{12}{49} \\ 0 & -\frac{13}{55} & -\frac{13}{11} & 1 \end{pmatrix}$$

Sample 4.2: Considering a coefficient matrix of the linear system 1 given by

$$A_2 = \begin{pmatrix} 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \end{pmatrix}$$

Sample 4.3: In this sample, we consider a 10×10 coefficient matrix of 1 in the form

$$A_3 = \begin{pmatrix} 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & 0 \\ -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 \end{pmatrix}$$

We computed the determinants and inverses of the coefficient matrices and observed that determinant A_1, A_2 and A_3 are not equal to zero and their inverses are nonnegative $({A_1}^{-1} \geq 0,\ {A_2}^{-1} \geq 0$ and ${A_3}^{-1} \geq 0)$. They have positive diagonal and negative off-diagonal elements. Hence, the three coefficient matrices are M- matrices. All computations of the numerical illustrations are done using Maple 2017 software, on a computer 2.80Hz CPU processor and 6GB RAM memory. The range of values picked for parameter v, ω and r in tables 4.1- 4.3 are based on the restriction $0 \leq r + v \leq \omega \leq 1$, $(v \neq 0$ and $\omega \neq 0)$ placed on the TAOR scheme for convergence.

Table 1 : Comparison of Results for Sample

ω	r	v	$\rho(T_{0,r,\omega})$	$\rho(T_{v,r,\omega})$
0.1	0.04	0.05	0.9505283950	0.9498713348
0.2	0.08	0.10	0.9000097009	0.8972371369
0.3	0.12	0.15	0.8483925987	0.8417887521
0.4	0.16	0.20	0.7956211154	0.7831416062
0.5	0.20	0.25	0.7416339693	0.7208036878
0.6	0.24	0.30	0.6863637872	0.6541239335
0.7	0.28	0.35	0.6297361605	0.5822016349
0.8	0.32	0.40	0.5716684977	0.5037120192
0.9	0.36	0.45	0.5120686140	0.4165235847

Table 2: Comparison of Results for Sample 2

ω	r	v	$\rho(T_{0,r,\omega})$	$\rho(T_{v,r,\omega})$
0.1	0.04	0.05	0.9702892488	0.9697412031
0.2	0.08	0.10	0.9397055636	0.9373783471
0.3	0.12	0.15	0.9082009036	0.9026234150
0.4	0.16	0.20	0.8757229293	0.8651200559
0.5	0.20	0.25	0.8422144385	0.8244179744
0.6	0.24	0.30	0.8076127016	0.7799331029
0.7	0.28	0.35	0.7718486727	0.7308822855
0.8	0.32	0.40	0.7348460470	0.6761686319
0.9	0.36	0.45	0.6965201257	0.6141612268

Table 3: Comparison of Results for Sample 3

ω	r	v	$\rho(T_{0,r,\omega})$	$\rho(T_{v,r,\omega})$
0.1	0.04	0.05	0.9695659299	0.9690220033
0.2	0.08	0.10	0.9382651880	0.9359663473
0.3	0.12	0.15	0.9060538059	0.9005738634
0.4	0.16	0.20	0.8728842724	0.8625309526
0.5	0.20	0.25	0.8387051160	0.8214516534
0.6	0.24	0.30	0.8034604202	0.7768509636
0.7	0.28	0.35	0.7670892570	0.7281033573
0.8	0.32	0.40	0.7295250213	0.6743746009
0.9	0.36	0.45	0.6906946434	0.6145013535

RESULTS AND DISCUSSION

Tables 1 and 3 shows the performance of various spectral radii of AOR and TAOR iteration matrices for samples 1, 2 and 3 respectively. Obviously, it is observed that $\rho(T_{v,r,\omega})$ spectral radius of the TAOR method, displayed better result than $\rho(T_{0,r,\omega})$ spectral radius of the AOR method in comparison for the varied values of v, ω and r in Tables 1 and 3. Evidently, the performance confirms the effectiveness of the TAOR method for M −matrices. Hence, the TAOR method performs better than method of AOR since the spectral radii of its iteration matrices are lesser than that of AOR method.

CONCLUSION

In this paper, we studied the convergence of Triple Accelerated Over-relaxation iterative method called TAOR, for M −matrix linear systems. Results of the numerical experiments performed confirm the theoretical results of the proposed theorems. Based on the numerical results, we conclude that the TOAR method is convergent for M −matrix linear systems. It is also important to note that TAOR method converges faster than the classical AOR method.

REFERENCES

Audu, K. J., Yahaya, Y. A., Adeboye, K. R., Abubakar, U. Y., & Ndanusa, A., (2020). Triple Accelerated Over-relaxation Method for System of Linear Equations. *Journal of Science, Technology, Mathematics and Education,* 16 (2), 137-146.

Axelsson, O., (1996). *Iteration Solution Methods*. Cambridge University Press, USA.

Berman, A., & Plemmon, R. J., (1994). *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, USA.

Hadjidimos, A., (1978). Accelerated Over-relaxation Method. *Mathematics of Computation*, 32, 149-57.

Hadjidimos, A., (1980). Some basic results on M-matrices in connection with the Accelerated Over-relaxation method. *Computing*, 24, 259-268.

Liu, Q., & Chen, G., (2010). Convergence Analysis of Preconditioned AOR iterative method for Linear System. *Mathematical Problems in Engineering,* 341982, 1-14.

Mayaki, Z. & Ndanusa, A., (2019). Modified Successive Over-Relaxation (SOR) Type method for M- matrices. *Science World Journal*, 14 (4), 1-5.

Saad, Y. (1995). *Iterative Method for Sparse Linear Systems.* PWS Press, New York.

Salkuyeh, D. K., (2011). Generalised AOR Method for solving System of Linear Equations. *Australian Journal of Basic and Applied Sciences.* 5(3), 351-358.

Shi-Guang, Z., & Ting, Z. (2017). Comparison Theorems between the Preconditioned Gauss-Seidel method and the AOR method for M-matrices. *Advance in Engineering Research*, 136, 446-449.

Varga, R. S., (2000). *Matrix Iterative Analysis*. Springer Berlin, Germany.

Vatti, V. B. K., Rao, G. C., & Srinesh, P. S., (2019). Parametric Accelerated Over-Relaxation (PAOR) Method. *Numerical Optimization in Engineering and Sciences*, 979, 283-288.

Wang, L., & Song, Y., (2009). Preconditioned AOR Iterative methods for M-matrices. *Journal of Computational and Applied Numerical Mathematics*, 226(1) 114 -124.

Wu, M., Wang, L., & Song, Y., (2007). Preconditioned AOR Iterative method for Linear System. *Applied Numerical Mathematics*, 57(5-7), 672-685.

Wu, S., & Liu, Y., (2014). A New Version of Accelerated Over-relaxation iterative method. *Journal of Applied Mathematics*, 725360, 1-6.

Xue, Q., Gao, X., & Liu, X. (2013, December). Preconditioned AOR Iterative method for M-matrix. In *2013 Ninth International Conference on Computational Intelligence and Security* (pp. 372-376). IEEE.

Young, D. M., (2014). *Iterative Solutions of Large Linear Systems*, Elsevier.

Youssef, I. K., & Farid, M. M., (2015). On the Accelerated Over-relaxation method. *Pure and Applied Mathematics Journal*, 4, 26-31.