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Numerical Solution of Fokker-Planck-Kolmogorov Time Fractional Differential Equations Using Haar Wavelet Method and convergence and error analysis

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Abstract

The purpose of this paper is to present an efficient numerical method for finding numerical solutions Fokker-Planck-Kolmogorov time-fractional differential equations. The Haar Wave was the first to be introduced. The Fokker-Planck-Kolmogorov time-fractional differential equation is converted to the linear equation using the Haar wavelet operation matrix in this technique. This method has the advantage of being simple to solve. The simulation was carried out using MATLAB software. Finally, the proposed strategy was used to solve certain problems. The results revealed that the suggested numerical method is highly accurate and effective when used to Fokker-Planck-Kolmogorov time fraction differential equations. The results for some numerical examples are documented in table and graph form to elaborate on the efficiency and precision of the suggested method. Moreover, for the convergence of the proposed technique, inequality is derived in the context of error analysis.

Keywords:
Time fractional differential equations
Fokker-Planck-Kolmogorov differential equations
Haar wavelet

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INTRODUCTION

In recent decades, the use of fractional differential equations in physical systems has received a lot of attention. This type of equation can be used to simulate a variety of physical phenomena, such as damping laws and diffusion processes. Electromagnets, electrochemistry, arterial science, and fragment theory are some of the other uses. As a result, there has recently been a greater focus on the development of more effective and better solution methods for determining an approximate or exact, analytical or numerical answer to these types of equations (Momani and Odibat, 2006; Jafari and daftardar-Gejji, 2006). Some strategies for solving fractional order partial differential equations have been proposed to attain this goal. The Adomian decomposition technique, the analysis method, the hematopoietic disorder, the Wardian repetition method, and the fractional partial differential conversion method are the most often used approaches (Yulita Molliq etal, 2009; Fadime Dal, 2009; Odibat and Momani, 2009; Kurulay etal, 2010) However, only a few approaches for numerically solving fractional partial differential equations have been proposed. These are as follows: By developing Riemann-Liouville derivatives, Podlebani (1999) employed the Laplace transform method to solve fractional differential equations numerically. He also established a generalized definition of the Green function for partial differential equations of fractions with constant coefficients. Surprisingly, the wavelet method has gotten so little attention among the different methods of solution. The wavelet approach has only been used to solve fractional equations in a few articles. The wavelet approach has only been used to solve fractional order differential equations in a few studies(Yulian etal, 2011; Lepik, 2009 ; Wu, 2009; Li and Zhao, 2010).Wave, Chebyshev, Haar, and sine wavelet fractional were employed for this purpose(Chen etal, 1997; Shih etal,1986; Paraskevopoulos etal,1985). To convert differential equations into a collection of algebraic equations, a variety of orthogonal fractional can be used. Its benefits include simple, computer-based calculations and a wide range of applications, as well as the ability to solve differential equations with integer and whole orders. The utilization of wavelet theory is the most recent advancement in applied mathematics(Meerschaert and Tadjeran, 2006; Jumarie, 2006; Jumarie, 2007) For solving various practical problems in seismology, signal processing in systems, telecommunications, computer image and vision processing, elementary particles and quantum mechanics, approximation theory and locating, criminologists, genetics, and medicine, the new wavelet theory and wave approximation models have replaced classical theories, including the classical Fourier theory method. In reality, wavelet analysis as a numerical tool may substantially reduce the difficulty of large-scale computations, such as the Fourier transform, by compressing dense matrices into thin ones that can be calculated rapidly by gently altering the coefficient. Fractional differential equations are numerically solved using the Wavelet family. The orthogonally of the scale and wavelet fractional, the orthogonally of the subspaces formed by the scale and wavelet fractional, the compactness of the scale and wavelet fractional, and the symmetry of the scale function could all be grounds for selecting this wavelet. In statistical mechanics, the Fokker-Planck equation is a partial derivative differential equation that represents the temporal evolution of the density probability velocity function for a particle subjected to drag force and random forces. Brownian motion is described by this equation, which can be generalized to include observations other than velocity (Spencer etal, 1993; Floris, 2013). Previous research has presented a method for solving two-dimensional Fokker-Planck equations for non-hybrid continuous systems using the finite difference approach, and the stability and accuracy of the proposed method have been investigated(Zorzano etal, 1999; Bect etal,2006) Ismail et al. (2020) in a study showed that obtained results by Green-Haar method is better than the conventional other method. The fractional derivative of Lagrange polynomial is a big hurdle in classical differential quadrature method. To overcome this problem, Saeed and Umair (2019) represent the Lagrange polynomial in terms of shifted Legendre polynomial. They construct a transformation matrix which transforms the Lagrange polynomial into shifted Legendre polynomial of arbitrary order. Then, they obtain the new weighting coefficients matrices for space fractional derivatives by shifted Legendre polynomials and use these in conversion of a non-linear fractional partial differential equation into a system of fractional ordinary differential equations. The Haar wavelet method is utilized in this study to solve the Fokker-Planck-Kolmogorov time-fractional differential equations in the following way(Hejazi etal, 2020):

$$D_t^{\alpha}u-\frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}+(\beta-2\sigma^2)x\frac{\partial u}{\partial x}+(\beta-\sigma^2)u=R(x,t) \quad (1)$$

Initial conditions:

$$u(0,x)=f_0(x),\ u_t(0,x)=f_1(x),\quad 0\le x\le 1\ ,\ 1<\alpha<2$$

Boundary conditions:

$$u(t,0)=g_0(t),\quad u_t(t,1)=g_0(t),\quad 0\le t\le 1$$

$R(x,t)$ is the right side function of the equation, which is given for each equation.

PRELIMINARIES

Riemann-Liouville Integral and fractional derivative

Suppose that $n>0$ and fare continuous segments on the interval (α,∞) and are integrable on any finite sub-interval(α,∞). Then, the fractional Riemann-Liouville Integral f for $t>a$ of order n is defined as

$${}_aD_t^{-n}f(t)=\frac{1}{\Gamma(n)}\int_\alpha^t(t-T)^{n-1}f(T)dT, \quad (2)$$

Which can also be displayed with the symbols I_a^n orJ_a^n . In addition, if f is continuous on $[a,t]$, then $\lim\limits_{n\to\alpha}\mathrm{D}_t^{-n}f(t)=f(t)$. Furthermore, the following equation can be true:

$${}_aD_t^{\cdot}f(t)=f(t) \quad (3)$$

When$n-m\in N$, the definition of (1-1) is compatible$-m$ with -fold integral as follows:

$${}_aD_t^{-m}f(t)=\int_\alpha^t dT_1\int_\alpha^{T_1}dT_2\ldots\int_\alpha^{T_{m-1}}f(T_m)dT_m=\frac{1}{(m-1)!}\int_\alpha^t(t-T)^{m-1}f(T)dT\quad m\in N \quad (4)$$

Regarding $m\ge 0$ and $v>-1$, the integral from the defined real order in Eq. 2 has the following properties:

$$I._\alpha D_t^{-n}(t-\alpha)^v=\frac{\Gamma(v+1)}{\Gamma(n+v+1)}(t-\alpha)^{n+v}$$

$$II._\alpha D_t^{-n}k=\frac{k}{\Gamma(n+1)}(t-\alpha)^n,$$

If $f(t)$ for $t\ge a$ is continuous, then:

$$III._\alpha D_t^{-n}({}_\alpha D_t^{-m}f(t))={}_\alpha D_t^{-m}({}_\alpha D_t^{-n}f(t))={}_\alpha D_t^{-n-m}f(t).$$

Caputo fractional derivative

Caputo defined a derivative operator in 1976 that differs from previous derivatives in terms of characteristics. The symbol of this operator is as ${}_aD_*^n$ and is defined as:

$${}_aD_*^nf(t)=\frac{1}{\Gamma(m-n)}\int_\alpha^t(t-T)^{m-n-1}f^{(m)}(T)dT\qquad(m-1<n\le m)\ (5)$$
$$=\alpha D_t^{-(m-n)}f^{(m)}(t),$$

On the conditions that $n\to m$ are exercised on the f function, then the Caputo derivative transforms to the m[th] order derivative of the f(t) function. Suppose that $0\le m-1<n<m$ and function f(t) have m+1 continuous bounded derivative in the interval$[a,t]$, then by partial integration for each $t>a$ per $m=1,2,\ldots$, we have:

$$\lim_{n\to m}{}_\alpha D_*^nf(t)=\lim_{n\to m}\left(\frac{f^{(m)}(\alpha)(t-\alpha)^{m-n}}{\Gamma(m-n+1)}\int_\alpha^t(t-T)^{m-n}f^{(m+1)}(T)dT\right)$$
$$=f^{(m)}(\alpha)+\int_\alpha^tf^{(m+1)}(T)dT=f^{(m)}(t).$$

RESEARCH METHOD

Orthogonal system of block-pulse and orthogonal system of Haar:

A set m is a member of the block-pulse fractional on the interval [0.1) is defined as follows:

$$b_i(t) = \begin{cases} 1 & \frac{i}{m} \le t \le \frac{i+1}{m} \\ 0 & otherwise \end{cases} \qquad (6)$$

A set of m members of the $h_i(t)$ Haar function on the interval [0.1) is defined as follows:

$$\mathrm{h}_0(\mathrm{t}) = \frac{1}{\sqrt{\mathrm{m}}} \qquad (7)$$

$$h_i(t) = \frac{1}{\sqrt{m}} = \begin{cases} 2^{\frac{j}{2}} & x\epsilon[\xi_1, \xi_2) \\ -2^{\frac{j}{2}} & x\epsilon[\xi_2, \xi_3] \\ 0 & otherwise \end{cases} \qquad (8)$$

where $\xi_1 = \frac{k-1}{m}$ ، $\xi_2 = \frac{k-0.5}{m}$ ، $\xi_3 = \frac{k}{m}$ ، $m = 2^j$ and j=0,1,2,3 are the wavelet surface index, $1 \le k \le 2^j$ represents the transfer parameters and i index is obtained from the $i = 2^j + k$ formula. Then, the matrices of Haar wavelets coefficients for different J are defined.

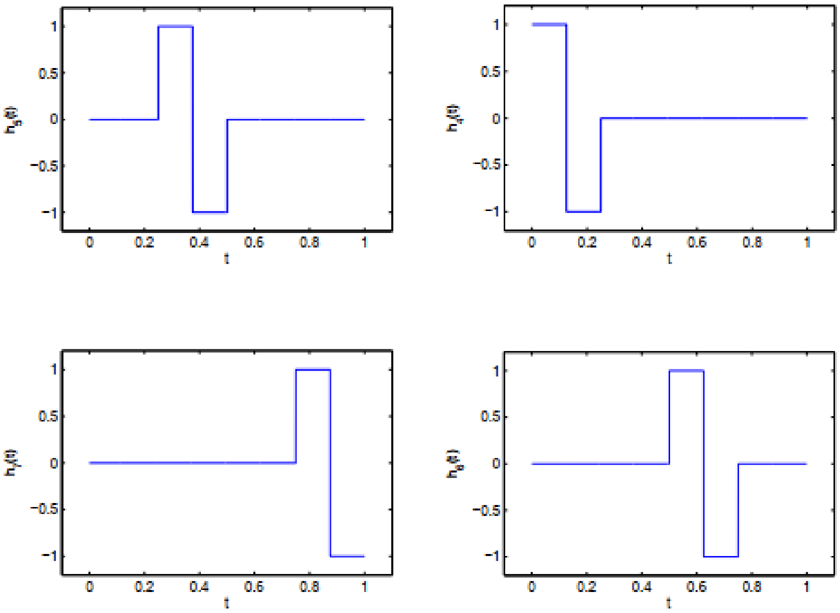

Fig. 1. Haar wavelet diagrams for j=2 in the interval [0,1]

Approximation of fractional using Haar wavelet:

The function $f(x) \in L^2[0,1]$ in the interval [0,1] using the Haar wavelet can be as follows(Razzaghi etal,2000):

$$f(t) = \sum_{i=0}^{\infty} c_i h_i(t) \quad , \quad c_i = \int_0^1 f(t)\, h_i(t) dt \qquad (9)$$

$$f(t) = \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T h_m(t \qquad (10)$$

$$C_m = [c_0, c_1, \dots, c_{m-1}]^T \qquad (11)$$

$$h_m(t) = [h_0(t), h_1(t), \dots, h_{m-1}]^T \qquad (12)$$

Where the vector C_m is the coefficient and $h_m(t)$s the vector of the Haar function. In the following, the form of the matrix of wavelet coefficients for different J is introduced. For this purpose, the interval [0,1] is divided as follows:

$$\mathrm{t}_l = \frac{2l-1}{2M} \quad , \quad l = 1,2,3, \dots, M \qquad (13)$$

Now we write equation (10) in the following form:

$$\vec{f} = c_0\overrightarrow{h_0} + c_1\overrightarrow{h_1} + \cdots + c_{m-1}\overrightarrow{h_{m-1}} \qquad (14)$$

And the discrete form of the continuous function f(t) is as follows:

$$\vec{f}^T = [\mathrm{f}_0, \mathrm{f}_1, \dots, \mathrm{f}_{\mathrm{m}-1}] \qquad (15)$$

Discrete values of f_i are obtained by means of continuous curves f(t) at intervals of $\frac{1}{m}$.

Similarly, the discrete form of Haar wave bases is as follows:

$$\overrightarrow{h_0}^T = [h_{0,0}, h_{0,1}, \dots, h_{0,m-1}] \quad (16)$$

$$\overrightarrow{h_1}^T = [h_{1,0}, h_{1,1}, \dots, h_{1,m-1}] \quad (17)$$

$$\vdots$$

$$\overrightarrow{h_{m-1}}^T = [h_{m-1,0}, h_{m-1,1}, \dots, h_{m-1,m-1}] \quad (18)$$

Therefore, the Haar wavelet matrix with dimension m is defined as follows:

$$H_{m\times m} = \begin{bmatrix} h_{0,0} & \cdots & h_{0,m-1} \\ \vdots & \ddots & \vdots \\ h_{m-1,0} & \cdots & h_{m-1,m-1} \end{bmatrix} \quad (19)$$

$$H_{m\times m} = \left[h_m\left(\frac{1}{2m}\right), h_m\left(\frac{3}{2m}\right), \dots, h_m\left(\frac{2m-1}{2m}\right)\right] \quad (20)$$

The function f (x,t) on the interval [0,1] × [0,1] can be written using the Haar wavelet as follows (wu,2009):

$$f(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{i,j}\, h_i(x)\, h_j(t) \quad (21)$$

$$c_{i,j} = \langle h_i(x), \langle f(x,t), h_{(t)j} \rangle\rangle = \int_0^1 f(x,t)\, h_i(x) dx . \int_0^1 f(x,t)\, h_j(t) \mathrm{dt} \quad (22)$$

$$F(x,t) = H_{m\times m}^T C_{m\times m} H_{m\times m}(t) \quad (23)$$

$$C_{m\times m} = \begin{bmatrix} c_{0,0} & \cdots & c_{0,m-1} \\ \vdots & \ddots & \vdots \\ c_{m-1,0} & \cdots & c_{m-1,m-1} \end{bmatrix} \quad (24)$$

Operational matrix of integral fraction of Haar wavelet:

The Haar fractional are fragmentary, fixed, they can be extended into m sentences of block-pulse fractional (Li and Zhao, 2010) and so we have:

$$h_m(t) = H_{m\times m} B_m(t) \quad (25)$$

Where:

$$B_m(t) = [b_0(t), b_1(t), \dots, b_{m-1}(t)]^T \quad (26)$$

$$H_{m\times m} = \left[h_m\left(\frac{1}{2m}\right), h_m\left(\frac{3}{2m}\right), \dots, h_m\left(\frac{2m-1}{2m}\right)\right] \quad (27)$$

The fractional integral operating matrix of a block box is as follows(Razzaghi etal, 2000):

$$(I^\alpha B_m)(t) = F^\alpha{}_{m\times m} B_m(t) \quad (28)$$

$$F^\alpha = \frac{1}{m^\alpha \Gamma(\alpha+2)} = \begin{vmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}, \; k = 1, .2, \dots, m-1$$

ow suppose that: $(I^\alpha h_m)(t) = P^\alpha{}_{m\times m} h_m(t)$, then the matrix $P^\alpha{}_{m\times m}$is the The fractional integral operating matrix of a Haar wavelet. From relations (25) and (28) we have:

$$(I^\alpha h_m)(t) \approx (I^\alpha H_{m\times m} B_m)(t) = H_{m\times m}(I^\alpha B_{m\times m})(t) = H_{m\times m} F^\alpha B_m(t) \quad (29)$$

$$P_{m\times m}^\alpha h_m(t) = P_{m\times m}^\alpha H_{m\times m} B_m(t) = H_{m\times m} F^\alpha B_m(t)$$

Finally, the fractional integral operating matrix of a Haar wavelet will be as follows:

$$P_{m\times m}^\alpha = H_{m\times m} F^\alpha H_{m\times m}{}^{-1} \quad (30)$$

THE WAVELETS METHOD FOR SOLVING DIFFERENTIONAL EQUATIONS OF FPKKER-PLANCK-KOLMOGROV FRACTIONAL ORDER

For the approximate solution of the Fokker-Planck-Kolmogorov fractional differential equation, the Haar wavelet method is explained as follows:

$$D_t^\alpha u - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + (\beta - 2\sigma^2) x \frac{\partial u}{\partial x} + (\beta - \sigma^2) u = R(x,t) \quad (31)$$

Initial conditions:

$$u(0,x) = f_0(x), \qquad u_t(0,x) = f_1(x), \quad 0 \le x \le 1$$

Boundary conditions:

$u(t,0) = g_0(t), \quad u_t(t,1) = g_0(t), \quad 0 \le t \le 1$

$R(x,t)$ Is the right-side function of the equation given for each equation.

Consider:

$$\frac{\partial^4 u(x,t)}{\partial x^2\, \partial t^2} \approx H^T_{m\times m}(x) C_{m\times m} H_{m\times m}(t) \quad (32)$$

By twice integrating with t from both sides of Eq. 32 we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx f_0''(x) + t f_1''(x) + H^T_{m\times m}(x) C_{m\times m}(I^2 H_{m\times m}(t)) \quad (33)$$

By twice integrating with x from both sides of equation (33) we have:

$$\frac{\partial\, u(t,x)}{\partial x} = \frac{\partial\, u(t,x)}{\partial x}|_{x=0} + f_0'(x) + f_0'(0) + t\big(f_1'(x) - f_1'(0)\big) + (IH^T_{m\times m}(x))^T C_{m\times m}(I^2 H_{m\times m}(t)) \quad (34)$$

$$u(t,x) \approx u(t,0) + x\frac{\partial\, u(t,x)}{\partial x}|_{x=0} + \big(f_0(x) - f_0(0) - x f_0'(0)\big) + t\big(f_1(x) - f_1(0) - x f_1'(0)\big) + \big(I^2 H_{m\times m}(x)\big)^T C_{m\times m}(I^2 H_{m\times m}(t)) \quad (35)$$

Now by applying the boundary conditions and putting x = 1, we will have:

$$u(t,1) \approx u(t,0) + x\frac{\partial\, u(t,x)}{\partial x}|_{x=0} + \big(f_0(1) - f_0(0) - x f_0'(0)\big) + t\big(f_1(1) - f_1(0) - f_1'(0)\big) + \big(I^2 H_{m\times m}(1)\big)^T C_{m\times m}(I^2 H_{m\times m}(t)) \quad (36)$$

Therefor:

$$\frac{\partial\, u(t,x)}{\partial x}|_{x=0} \approx g_1(t) - g_0(t) - \big(f_0(1) - f_0(0) - f_0'(0)\big) - t\big(f_1(1) - f_1(0) - f_1'(0)\big) - \big(I^2 H_{m\times m}(1)\big)^T C_{m\times m}(I^2 H_{m\times m}(t)) = K(t) \quad (37)$$

Now by placing K(t) in Eq. 35 we have:

$$u(t,x) \approx g_0(t) + xK(t) + \big(f_0(x) - f_0(0) - x f_0'(0)\big) + t\big(f_1(x) - f_1(0) - x f_1'(0)\big) + \big(I^2 H_{m\times m}(x)\big)^T C_{m\times m}(I^2 H_{m\times m}(t)) \quad (38)$$

Now we need the fraction derivative u (t, x) according to Equation (31). From Equation (37) we derive the order fraction α with respect to t:

$$D_t^\alpha u(t,x) \approx D_t^\alpha g_0(t) + x D_t^\alpha K(t) + \big(I^2 H_{m\times m}(x)\big)^T C_{m\times m}(I^{2-\alpha} H_{m\times m}(t)) \quad (39)$$

And we will have:

$$D_t^\alpha K(t) = D_t^\alpha g_1(t) - D_t^\alpha g_0(t) - \big(I^2 H_{m\times m}(1)\big)^T C_{m\times m}(I^{2-\alpha} H_{m\times m}(t)) \quad (40)$$

Now convert all approximations (≈) to equals (=), and place Eq. (33), (34), (38) and (39) in Eq. 31, the following linear equation is obtained:

$$D_t^\alpha g_0(t_j) + x_i D_t^\alpha K(t_j) + \big(I^2 H_{m\times m}(x_i)\big)^T C_{m\times m}(I^{2-\alpha} H_{m\times m}(t)) - \frac{1}{2}\sigma^2 {x_i}^2 (f_0''(x_i) + t_j f_1''(x_i) + H^T{}_{m\times m}(x_i) C_{m\times m}(H^2{}_{m\times m}(t_j))) + (\beta - 2\sigma^2) x_i (K(t_j) + f_0'(x_i) - f_0'(0) + t_j\big(f_1'(x_i) - f_1'(0)\big) + (IH\,{}_{m\times m}(x_i))^T C_{m\times m}(H^2{}_{m\times m}(t_j))) + (\beta - \sigma^2)(g_0(t_j) + x_i K(t_j) + \big(f_0(x_i) - f_0(0) - x_i f_0'(0)\big) + t_j\big(f_1(x_i) - f_1(0) - x_i f_1'(0)\big) + \big(I^2 H_{m\times m}(x_i)\big)^T C_{m\times m}(I^2 H_{m\times m}(t_j)) = R(x_i, t_j) \quad (41)$$

CONVERGENCE ANALYSIS

In this part, we derive inequality in the context of upper bound, which shows the convergence of Haar Wavelet for Fokker-Planck-Kolmogorov Time Fractional Differential Equations.

Theorem5.1: Suppose that the function $\frac{\partial\, u(x,t)}{\partial x}$ is continuous and bounded on (0,1) × (0,1) then:

$$\exists M > 0, \;\; \forall x, t \epsilon (0,1) \times (0,1), \left|\frac{\partial\, u(x,t)}{\partial x}\right| \le M \quad (42)$$

And also assume that $u_m(x,t)$ is an approximation of u(x,t), then we have(Wang,2014):

$$u(x,t) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} c_{nl}\, h_n(x)\, h_l(t) \quad (43)$$

Therefore, we have:

$$u(x,t) - u_m(x,t) = \sum_{n=m}^{\infty}\sum_{l=m}^{\infty} c_{nl}\, h_n(x)\, h_l(t) = \sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty} c_{nl}\, h_n(x)\, h_l(t) \quad (44)$$

Theorem5.2: Assume that $u_m(x,t)$ is an approximation of u(x,t) Then we have:

$$\|u(x,t) - u_m(x,t)\|_E \le \frac{M}{\sqrt{3}m^3} \quad (45)$$

Where:

$$\|u(x,t)\|_E = \left(\int_0^1\int_0^1 u^2(x,t)dxdt\right)^{\frac{1}{2}} \quad (46)$$

Proof. Consider:

$$\int_0^1 h_n(x)h_{n'}(x) = \begin{cases} \frac{1}{m}, & n = n' \\ 0, & n \ne n' \end{cases} \quad (47)$$

Therefore:

$$\|u(x,t) - u_m(x,t)\|_E = \left(\int_0^1\int_0^1 (u(x,t) - u_m(x,t))^2 dxdt = \sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty} c_{nl} \sum_{n'=2^{p+1}}^{\infty}\sum_{l'=2^{p+1}}^{\infty} c_{nl}\, c_{n'l'} \left(\int_0^1 h_n(x)h_{n'}(x)dx\right)\left(\int_0^1 h_l(t)h_{l'}(t)dt\right) = \frac{1}{m^2}\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty} {c_{nl}}^2 \quad (48)$$

Where:

$$c_{nl} =< h_n(x), < u(x,t), h_l(t) >> \quad (49)$$

$$u(x,t), h_l(t) >= \int_0^1 u(x,t), h_l(t)dt = \frac{2^{\frac{j}{2}}}{\sqrt{m}}\left(\int_{(k-1)2^{-j}}^{\left(k-\frac{1}{2}\right)2^{-j}} u(x,t)dt - \int_{\left(k-\frac{1}{2}\right)2^{-j}}^{k2^{-j}} u(x,t)dt\right) \quad (50)$$

Using the mean value theorem:

$$\exists t_1, t_2 : \ (k-1)2^{-j} \le t_1 \le \left(k - \frac{1}{2}\right)2^{-j} \quad (51)$$

$$\left(k - \frac{1}{2}\right)2^{-j} \le t_2 \le k2^{-j} \quad (52)$$

$$< u(x,t), h_l(t) >= \frac{2^{\frac{j}{2}}}{\sqrt{m}}\left\{\left[(k-1)2^{-j} - \left(k - \frac{1}{2}\right)2^{-j}\right]u(x,t_1) - \left[k2^{-j} - \left(k - \frac{1}{2}\right)2^{-j}\right]u(x,t_2)\right\} = \frac{2^{\frac{-j}{2}-1}}{\sqrt{m}}(u(x,t_1) - u(x,t_2)) \quad (53)$$

Therefore:

$$c_{nl} =< h_n(x), \frac{2^{\frac{-j}{2}-1}}{\sqrt{m}}\left(u(x,t_1) - u(x,t_2)\right) >= \frac{2^{\frac{-j}{2}-1}}{\sqrt{m}}\int_0^1 h_n(t)(u(x,t_1) - u(x,t_2))dt$$

$$= \frac{2^{\frac{-j}{2}-1}}{\sqrt{m}}\left(\int_0^1 h_n(t)(u(x,t_1) - \int_0^1 h_n(t)u(x,t_2))dt\right.$$

$$= \frac{1}{2m}\left(\int_{(k-1)2^{-j}}^{\left(k-\frac{1}{2}\right)2^{-j}} u(x,t_1)dt + \int_{\left(k-\frac{1}{2}\right)2^{-j}}^{k2^{-j}} u(x,t_1) - \int_{(k-1)2^{-j}}^{\left(k-\frac{1}{2}\right)2^{-j}} u(x,t_2)dt - \int_{\left(k-\frac{1}{2}\right)2^{-j}}^{k2^{-j}} u(x,t_2)\,dt\right)$$

Using the mean value theorem:

$$\exists x_1, x_2, x_3, x_4 : (k-1)2^{-j} \le x_1, x_3 \le \left(k - \frac{1}{2}\right)2^{-j}$$

$$\left(k - \frac{1}{2}\right)2^{-j} \le x_2, x_4 \le k2^{-j}$$

Where:

$$c_{nl} = \frac{1}{2m}\left\{\left[\left(k-\frac{1}{2}\right)2^{-j} - (k-1)2^{-j}\right]u(x_1,t_1) - \left[k2^{-j} - \left(k-\frac{1}{2}\right)2^{-j}\right]u(x_2,t_1) - \left[\left(k-\frac{1}{2}\right)2^{-j} - (k-1)2^{-j}\right]u(x_3,t_2) + \left[k2^{-j} - \left(k-\frac{1}{2}\right)2^{-j}\right]u(x_4,t_2)\right\} =$$

$$= \frac{1}{2^{2j+4}m^2}\left[(u(x_1,t_1) - u(x_2,t_1)) - \left(u(x_3,t_2) - u(x_4,t_2)\right)\right] \quad (54)$$

Therefore**:**

$${c_{nl}}^2 = \frac{1}{2m}\left[(u(x_1,t_1) - u(x_2,t_1)) - \left(u(x_3,t_2) - u(x_4,t_2)\right)\right]^2 \quad (55)$$

herefore, using the mean value theorem:

$$\exists \xi_1, \xi_2 \; : \quad x_1 \le \xi_1 \le x_2 \, , x_3 \le \xi_2 \le x_3$$

Where:

$$c_{nl}{}^2 = \frac{1}{2^{2j+4}m^2}\left[(x_1 - x_2)\frac{\partial\, u(\xi_1,t_1)}{\partial x} - (x_4 - x_3)\frac{\partial\, u(\xi_2,t_2)}{\partial x}\right]^2 \le$$

$$\frac{1}{2^{2j+4}m^2}\left[(x_1 - x_2)^2\left[\frac{\partial\, u(\xi_1,t_1)}{\partial x}\right]^2 + (x_4 - x_3)^2\left[\frac{\partial\, u(\xi_2,t_2)}{\partial x}\right]^2\right]^2 +$$

$$2(x_1 - x_2)(x_4 - x_3)\left|\frac{\partial\, u(\xi_1,t_1)}{\partial x}\right|\left|\frac{\partial\, u(\xi_2,t_2)}{\partial x}\right| \quad (56)$$

Therefore:

$$u_{nl}{}^2 = \frac{4M^2}{2^{4j+4}m^2} = \frac{M^2}{2^{4j+4}m^2}$$

Therefore:

$$\|u(x,t) - u_m(x,t)\|_E = \frac{1}{m^2}\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty} c_{nl}{}^2 = \frac{1}{m^2}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1} c_{nl}{}^2\right)$$

$$\le \frac{1}{m^2}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1}\frac{M^2}{2^{4j+2}m^2}\right) = \frac{M^2}{m^4}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1}\frac{1}{2^{4j+2}m^2}\right)$$

$$\frac{M^2}{3m^4}\frac{1}{2^{2(p+1)}} = \frac{M^2}{3m^6} \quad (57)$$

Therefor:

$$\|u(x,t) - u_m(x,t)\|_E \le \frac{M}{\sqrt{3}m^3}$$

SOLVING NUMERICAL EXAMPLES

Numerical solutions and errors are calculated, evaluated, and provided in tables after evaluating certain numerical instances with conditions of varying initial values. The MATLAB software is used to solve all of the examples.

Example 1: In equation 1, by placing, $\alpha = 1.1 \;, \beta = 1 \,, \sigma = 0.2, m = 3 \,, k = 2$,

Initial conditions:

$$\mathrm{u}(0,\mathrm{x}) = 0, \qquad u_t(0,x) = 0, \qquad 0 \le x \le 1$$

Boundary conditions:

$$\mathrm{u}(t,0) = 0, \quad u_t(t,1) = 0, \quad 0 \le t \le 1$$

The right-side fractional of the equation:

$$R(t,x) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{2}(\sigma x t \pi)^2 + (\beta - \sigma^2)t^2\right)\sin(\pi x) + (\beta - 2(\sigma)^2)t^2 x\pi\cos(\pi x)$$

The accurate answer of this equation in example (1) is $u(t,x) = t^2\sin(\pi x)$. Example (1) is solved by the Haar wavelet method for , $\alpha = 1.2 \;, \beta = 1 \,, \sigma = 0.2, m = 3 \,, k = 2$and its error is presented in Table 1.

Table 1: example 1 error,by placing, $\alpha = 1.2 \;, \beta = 1 \,, \sigma = 0.2, m = 3 \,, k = 2$

(x,t)	α=1.2	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.09*10^{-10}$	$2.14*10^{-10}$	$5.01*10^{-11}$	$6.10*10^{-11}$	$2.01*10^{-11}$
(3.13,3.13)	$4.12*10^{-8}$	$1.77*10^{-8}$	$2.19*10^{-8}$	$5.33*10^{-8}$	$4.19*10^{-8}$
(5.13,5.13)	$3.29*10^{-6}$	$1.34*10^{-6}$	$1.88*10^{-6}$	$4.02*10^{-6}$	$5.22*10^{-6}$
(7.13,7.13)	$2.64*10^{-5}$	$2.02*10^{-5}$	$3.11*10^{-5}$	$2.78*10^{-5}$	$3.11*10^{-5}$
(9.13,9.13)	$3.14*10^{-4}$	$3.25*10^{-4}$	$4.34*10^{-4}$	$6.28*10^{-4}$	$4.19*10^{-4}$
(11.13,11.13)	$1.43*10^{-7}$	$1.00*10^{-7}$	$2.19*10^{-7}$	$3.44*10^{-7}$	$2.01*10^{-7}$

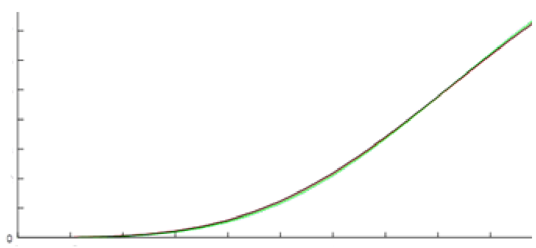

Fig.1. Relation of β and error for example 1 for $\alpha = 1.2$, $\beta = 1$, $\sigma = 0.2, m = 3$,

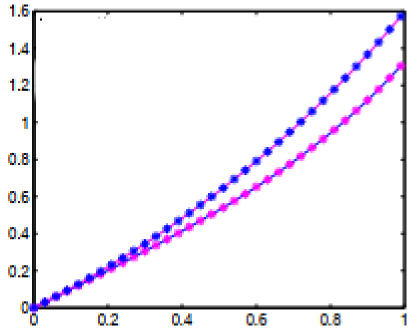

Fig.2. Approximate and exact solution, respectively for example 1 for $\alpha = 1.2$, $\beta = 1$, $\sigma = 0.2, m = 3$, $k = 2$

The method of numerical solution for $\alpha = 1.3$, $\beta = 1$, $\sigma = 0.2, m = 3$, $k = 2$is presented in Table 2.

Table 2: the numerical solution of example 1 py placing , $\alpha = 1.3$, $\beta = 1$, $\sigma = 0.2, m = 3$, $k = 2$

(x,t)	α=1.3	α=1.4	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$2.19*10^{-4}$	$2.24*10^{-4}$	$2.37*10^{-4}$	$2.34*10^{-4}$	$2.49*10^{-4}$
(3.13,3.13)	$3.22*10^{-3}$	$3.26*10^{-3}$	$3.58*10^{-3}$	$3.47*10^{-3}$	$3.55*10^{-3}$
(5.13,5.13)	$1.34*10^{-2}$	$1.24*10^{-2}$	$1.41*10^{-2}$	$1.55*10^{-2}$	$1.60*10^{-2}$
(7.13,7.13)	$1.47*10^{-2}$	$1.63*10^{-2}$	$1.62*10^{-2}$	$1.64*10^{-2}$	$1.24*10^{-2}$
(9.13,9.13)	$2.11*10^{-2}$	$2.19*10^{-2}$	$2.01*10^{-2}$	$2.01*10^{-2}$	$2.01*10^{-2}$
(11.13,11.13)	$3.10*10^{-2}$	$3.02*10^{-2}$	$3.02*10^{-2}$	$3.11*10^{-2}$	$3.25*10^{-2}$

Example 2: the numerical solution of the following equation:

In equation (1), by placing $\boldsymbol{\alpha = 1.3}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2, m = 3}$, $\boldsymbol{k = 2,}$

Initial conditions:

$$\mathrm{u}(0, x) = 0, \quad u_t(0, x) = 0, \quad 0 \leq x \leq 1$$

Boundary conditions:

$$\mathrm{u}(t, 0) = \mathrm{t}^3, \quad u_t(t, 1) = et^3, \quad 0 \leq t \leq 1$$

The right-side fractional of the equation:

$$R(t, x) = \left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} - \frac{1}{2}\sigma^2 x^2 t^3 + (\beta - 2\sigma^2) x t^3 + (\beta - \sigma^2) t^3 \right) e^x$$

The accurate response to this equation in example (2) is$u(t, x) = t^3 e^x$. Example (2) is solved by the Haar wavelet method for $\boldsymbol{\alpha = 1.3}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2, m = 3}$, $\boldsymbol{k = 2}$ and its error has been shown in Table 3.

Table 3: The error of example 2, by placing $\alpha = 1.3$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$

(x,t)	α=1.3	α=1.5	α=1.6	α=1.7	α=1.9
(1.13,1.13)	$4.21*10^{-8}$	$5.34*10^{-8}$	$5.24*10^{-8}$	$4.85*10^{-8}$	$5.45*10^{-8}$
(3.13,3.13)	$3.65*10^{-6}$	$4.01*10^{-6}$	$4.74*10^{-7}$	$4.14*10^{-6}$	$4.23*10^{-6}$
(5.13,5.13)	$2.85*10^{-9}$	$3.19*10^{-9}$	$5.22*10^{-8}$	$3.24*10^{-8}$	$3.01*10^{-8}$
(7.13,7.13)	$2.64*10^{-4}$	$2.24*10^{-4}$	$3.34*10^{-4}$	$2.31*10^{-4}$	$2.22*10^{-4}$
(9.13,9.13)	$3.10*10^{-4}$	$3.00*10^{-4}$	$4.02*10^{-4}$	$4.10*10^{-4}$	$4.10*10^{-4}$
(11.13,11.13)	$1.01*10^{-3}$	$1.14*10^{-3}$	$2.12*10^{-3}$	$1.51*10^{-3}$	$1.21*10^{-3}$

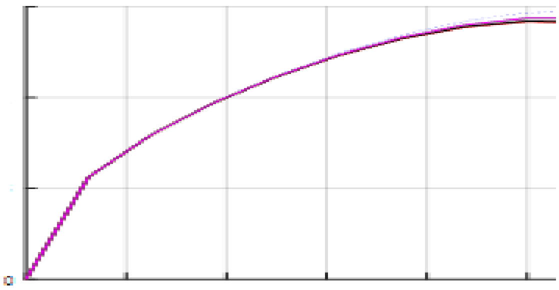

Fig.3. Relation of β and error for example2 for, $\alpha = 1.3$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

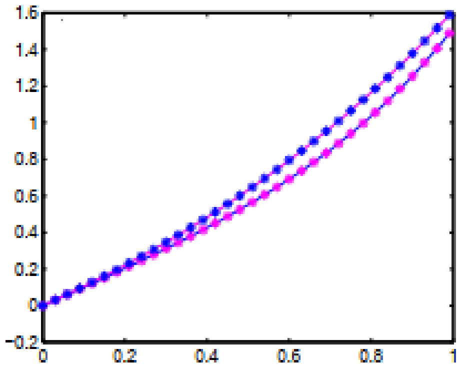

Fig.4. Approximate and exact solution, respectively example2 for, $\alpha = 1.3$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

In Table (4), the numerical solution method for $\boldsymbol{\alpha = 1.3}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$ has been shown.

Table 4: the numerical solution of example 2 by placing $\alpha = 1.3$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

(x,t)	α=1.3	α=1.4	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.19*10^{-4}$	$5.29*10^{-4}$	$5.34*10^{-4}$	$5.37*10^{-4}$	$5.41*10^{-4}$
(3.13,3.13)	$2.74*10^{-3}$	$2.80*10^{-3}$	$2.21*10^{-3}$	$2.67*10^{-3}$	$2.77*10^{-3}$
(5.13,5.13)	$1.45*10^{-2}$	$1.61*10^{-2}$	$1.39*10^{-2}$	$1.42*10^{-2}$	$1.45*10^{-2}$
(7.13,7.13)	$3.21*10^{-2}$	$3.12*10^{-2}$	$3.18*10^{-2}$	$3.14*10^{-2}$	$3.19*10^{-2}$
(9.13,9.13)	$1.11*10^{-2}$	$1.10*10^{-2}$	$1.25*10^{-2}$	$1.24*10^{-2}$	$1.11*10^{-2}$
(11.13,11.13)	$7.10*10^{-2}$	$7.02*10^{-2}$	$7.07*10^{-2}$	$3.14*10^{-2}$	$3.19*10^{-2}$

Example 3: the numerical solution of the following equation:

In equation (1), by placing , $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

Initial conditions:

$$u(0,x) = 0, \quad u_t(0,x) = 0, \quad 0 \le x \le 1$$

Boundary conditions:

$$u(t,0) = 0, \quad u_t(t,1) = t^3 sin^2 x , \quad 0 \le t \le 1$$

The right-side fractional of the equation:

$$R(t,x) = (\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (\beta - \sigma^2) t^3) sin^2 x - \sigma^2 x^2 t^3 cos(2x) + (\beta - \sigma^2) x t^3 sin(2x)$$

The accurate response to this equation in example (3) is $u(t,x) = t^3 sin^2 x$. Example (3) is solved by the Haar wavelet method for $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$ and its error has been shown in Table 5.

Table5: The error of example 3, by placing $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$3.41*10^{-8}$	$5.77*10^{-8}$	$5.22*10^{-8}$	$4.74*10^{-8}$	$5.07*10^{-8}$
(3.13,3.13)	$3.24*10^{-6}$	$4.31*10^{-6}$	$4.44*10^{-7}$	$4.24*10^{-6}$	$4.32*10^{-6}$
(5.13,5.13)	$2.88*10^{-9}$	$3.22*10^{-9}$	$5.14*10^{-8}$	$3.34*10^{-8}$	$3.17*10^{-8}$
(7.13,7.13)	$2.84*10^{-4}$	$2.31*10^{-4}$	$3.37*10^{-4}$	$2.33*10^{-4}$	$2.45*10^{-4}$
(9.13,9.13)	$3.03*10^{-4}$	$3.18*10^{-4}$	$4.21*10^{-4}$	$4.52*10^{-4}$	$4.37*10^{-4}$
(11.13,11.13)	$1.64*10^{-3}$	$1.64*10^{-3}$	$2.41*10^{-3}$	$1.67*10^{-3}$	$1.22*10^{-3}$

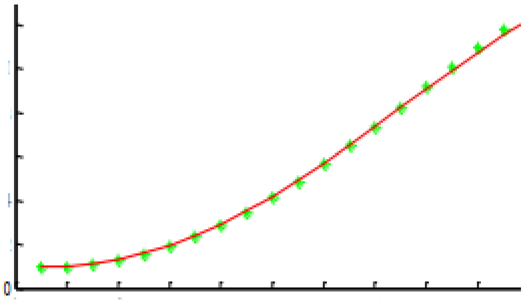

Fig.5. Relation of β and error for example3 for, $\alpha = 1.1\ ,\beta = 0.5\,,\sigma = 0.2, m = 3\,,k = 2$

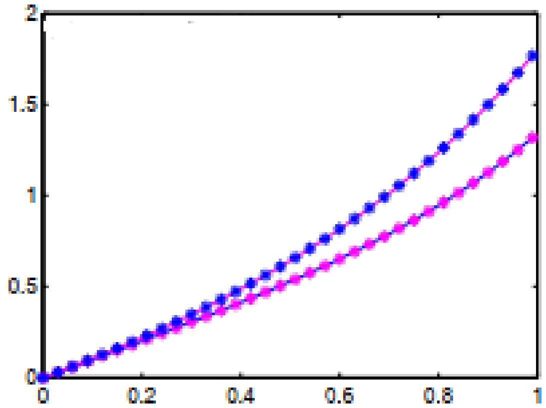

Fig.6. Approximate and exact solution, respectively for example3 for, $\alpha = 1.1\ ,\beta = 0.5\,,\sigma = 0.2, m = 3\,,k = 2$

In Table 6, the numerical solution method for $\alpha = 1.1\ ,\beta = 0.5\,,\sigma = 0.2, m = 3\,,k = 2$has been shown.

Table 6: the numerical solution of example 3by placing $\alpha = 1.1\ ,\beta = 0.5\,,\sigma = 0.2, m = 3\,,k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.75*10^{-4}$	$5.27*10^{-4}$	$5.55*10^{-4}$	$5.65*10^{-4}$	$5.31*10^{-4}$
(3.13,3.13)	$2.27*10^{-3}$	$2.33*10^{-3}$	$2.63*10^{-3}$	$2.14*10^{-3}$	$2.82*10^{-3}$
(5.13,5.13)	$1.55*10^{-2}$	$1.64*10^{-2}$	$1.37*10^{-2}$	$1.66*10^{-2}$	$1.34*10^{-2}$
(7.13,7.13)	$3.22*10^{-2}$	$3.22*10^{-2}$	$3.41*10^{-2}$	$3.23*10^{-2}$	$3.23*10^{-2}$
(9.13,9.13)	$1.21*10^{-2}$	$1.14*10^{-2}$	$1.12*10^{-2}$	$1.31*10^{-2}$	$1.21*10^{-2}$
(11.13,11.13)	$7.54*10^{-2}$	$7.19*10^{-2}$	$7.34*10^{-2}$	$3.30*10^{-2}$	$3.12*10^{-2}$

DISCUSSION AND CONCLUSION

These equations are difficult to solve but can be solved using a variety of numerical methods. The Fokker-Planck-Kolmogorov time-fractional differential equations are the focus of this research. The Haar wavelet technique is one of these methods. The highest order of the derivative in the equation is estimated in terms of Haar fractional for each of the variables using this procedure. Finally, the proposed method was used to solve certain numerical cases. The results revealed that the proposed approach for Fokker-Planck-Kolmogorov time-fractional differential equations is numerically and practically very efficient. As a result, because many differential equations in various fields of research do not have approximate solutions, the wavelet approach is recommended as a valid and trustworthy method for solving such equations. Therefore, the present article can be used as a starting point for other wavelet-based applications.

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