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Legendre Wavelet Method for a Class of Fourth-Order Boundary Value Problems

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Abstract

In this paper we apply an approximate method based on Galerkin approach with Legendre wavelets basis, on a class of fourth order boundary value problems. The approach reduces the main equation to a system of linear algebraic equations that could be solved numerically. The operational matrix of the method is obtained, and the convergence of the method is proved. we approximate the solution and its higher order derivatives, for some special examples and compare the results with some other numerical methods. The results show the effectiveness of the proposed method.

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INTRODUCTION

Fourth order boundary value problems have many applications in boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology. From the theoretical point of view, the necessary and sufficient conditions for existence and uniqueness of these equations are contained in a comprehensive survey in a book by Agarwal (Agarwal, 1997). Moreover his work on fourth order boundary value problems in special cases are in his previous work (Agarwal,1989). Before his work Aftabizadeh present the proof of sufficient conditions for the existence and uniqueness of the solution for some special cases in his work (Aftabizade, 1980). Because of its applications, only positive solutions of the proposed equations are useful. Y.Li has considered the existence of positive solution for fourth order boundary value problems (Li, 2003). In recent years these problems have extensively studied by using diverse methods, including fixed point theorems on cones (Jankowski, 2010), the method of lower and upper solutions (Bai, 2007; pao, 1999) the iterative method (Agarwal, 1984), critical point theory(Han, 2007), and the shooting method(Amster, 2008). Numerical methods for obtaining the approximate solution for fourth order boundary value problems have been applied, among them are Adomian decomposition method (Kelesoglu, 2014), variational iteration method (Aslamnoor &Mohyedin, 2007), sinc - Galerkin and Laplace Adomian method which makes the Adomian series convergent when it diverges in its original space(Hajji & Khaled, 2008).

In this paper we consider the fourth order boundary value problem to be the problem of bending rectangular clamped beam of length l resting on an elastic foundation. The vertical deflection w of the beam satisfies the system(Siddiqi& Akram,2008).

$$\left[L+\left(\frac{K}{D}\right)\right]w=D^{-1}q(x), \quad (1)$$

Where

$$L=\frac{d^4}{dx^4}, w(0)=w(l)=w'(0)=w'(l)=0,$$

where D is the flexural rigidity of the beam, and k is the spring constant of the elastic foundation and the load q(x) acts vertically downwards per unit length of the beam. Mathematically, the system (1) belong to a general class of boundary value problems of the form,

$$\left(\frac{d^4}{dx^4}+f(x)\right)y(x)=g(x), x\in[a,b],$$

$$y(a)=\alpha_0, y(b)=\alpha_1,$$
$$y'(a)=\gamma_0,\ \ y'(b)=\gamma_1 \quad (2)$$

Where α_i;γ_i;i=0,1 are finite real constants and the functions f(x)and g(x) are continuous on [a,b]. The analytic solution of (2) for special choices of f(x)and g(x) are easily obtained, but for arbitrary choices, the analytic solution cannot be determined.

some of the previous work on this equation are (Usmani, 1980&1987). In the mentioned papers spline methods have been applied to the proposed equation. Here we apply Legendre wavelet method to this equation (Venkatesh& Ayyaswamy, 2012). The rest of the paper is organized as follows: In section 2 Legendre wavelets, it's convergence and its applications in function approximation is considered. In section 3 the proposed method is applied to some special cases of the main equation, and the results are compared to quantic spline solutions.

LEGENDRE WAVELET

Wavelet theory is an improvement of Fourier analysis. It was first introduced by (Haar, 1910); The most important step that has led to the prosperity of the wavelets was the invention of multiresolution analysis (MRA) by Mallat and Meyer (Meyer, 1989; Mallat, 1989&1999). Daubechies (Daubechies, 1992&1988) created her own wavelets based on MRA, that are orthogonal and smooth as much as needed with compact supports. Morlet developed and implemented the technique of scaling and shifting (Mackenzie, 2001). Wavelet basis has been applied in vast range of engineering sciences (Biazar & Ebrahimi, 2012; Biazar & Goldoust, 2013; Biazar & Goldoust, 2019; Biazar & Ebrahimi, 2010; Azizi et al., 2016); particularly, they are used very successfully for wave form representation and segmentations in signal analysis and time-frequency analysis and in the mathematical sciences. It is used in thriving manner for solving variety of linear and nonlinear differential and partial differential equations and fast algorithms for easy implementation (Balaji, 2014). Moreover, wavelets build a connection with fast numerical algorithm. Wavelet basis, specially multi wavelets, have been applied for solving different kinds of functional equations (Razzaghi &Yousefi, 2001). Some of wavelet basis are constructed based on orthogonal polynomials, among them are Legendre wavelets and Chebyshev wavelets. These wavelet bases are orthogonal and have compact supports that makes them more appropriate for solving boundary value problems. To construct a wavelet basis one uses dilations and translations of a signal function (with some properties) called the mother wavelet. When the dilation parameter and translation parameter are continuously changed we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\psi\left(\frac{x-b}{a}\right), \quad a,b \in \mathbb{R}, a \neq 0. \tag{3}$$

If we restrict the parameters a and b to take discrete values $a=a_0^{-k}$ and $b=nb_0\, a_0^{-k}$)where $a_0>1$,$b_0>0$, and n,k positive integers, then one has the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a_0|^{\frac{k}{2}}\psi(a_0{}^k x - nb_0), \tag{4}$$

here $\psi_{k,n}$ (x) forms a wavelet basis forL^2 (R). In particular, when $a_0=2$ and$b_0=1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. In the following Legendre wavelets are presented.

LEGENDRE WAVELET IN ONE DIMENSION

Legendre wavelets $\psi_{m,n}$ (x)= $\psi(k,\bar{n},m,x)$ have four arguments:

$\bar{n}=2n-1$, $\quad n=1,2,\ldots,2^{k-1}$,

where k that can assume any positive integer is the order of Legendre polynomials and x∈[0,1]. They are defined as:

$$\psi_{n,m}(x) = \begin{cases} (m+\frac{1}{2})^{\frac{1}{2}} 2^{\frac{k}{2}} p_m(2^k x - \bar{n}), & \frac{\bar{n}-1}{2^k} \le x < \frac{\bar{n}+1}{2^k}, \\ 0, & otherwise. \end{cases} \tag{5}$$

Where $m=0,1,2,\ldots,M-1, n=1,2,3,\ldots,2^{k-1}$. In equation (5) the coefficient $(m+1)^{1/2}$ is for orthonormality, the dilation parameter is $a= 2^{-k}$ and translation parameter is $b=\bar{n}2^{-k}$. Here P_m (x) for m=0,1,… are the well-known Legendre polynomials of order m, which are orthogonal with respect to the weight function w(x)=1 on interval [-1,1] and satisfy the following recursive formula:

$$p_0(x) = 1, \quad p_1(x) = x,$$

$$p_{m+1}(x) = \left(\frac{2m+1}{m+1}\right) x p_m(x) - \left(\frac{m}{m+1}\right) p_{m-1}(x). \tag{6}$$

Any function f(x) defined over [0,1) may be expanded as:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x) \tag{7}$$

Because of orthonormality we have $C_{n,m}=\langle f(x),\psi_{n,m}\rangle$, where $\langle .,. \rangle$ denotes the inner product. If the infinite series in equation (7) is truncated, then it can be written as:

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{8}$$

Where C and Ψ(x) are 2^{k-1} M×1 matrices given by:

$$C=[c_{10},\ldots,c_{1(M-1)},c_{20},\ldots,c_{2(M-1)},\ldots,c_{2^{k-1}0},\ldots,c_{2^{k-1}(m-1)}]^T,$$

$$\psi=[\psi_{10},\ldots,\psi_{1(M-1)},\psi_{20},\ldots,\psi_{2(M-1)},\ldots,\psi_{2^{k-1}0},\ldots,\psi_{2^{k-1}(m-1)}]^T.$$

CONVERGENCE ANALYSIS

Theorem 1: (convergence theorem)

The series solution

$$u(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(x) \qquad (9)$$

Using Legendre wavelet method converges to u(x).

Proof (Sahu & Ray, 2015).

Theorem 2: If $y(x) \in L^2(R)$ be a continuous function defined on [0,1] and $\|y(x)\| \le M_y$, then the Legendre wavelet expansion defined in the above equation converges uniformly and also (Sahu &Ray, 2016).

$$\|c_{n,m}\| \le 2\sqrt{m+\frac{1}{2}}\sqrt{\frac{2}{2n+1}}\,2^{\frac{-k}{2}} M_y.$$

OPERATIONAL MATRIX OF THE PROPOSED METHOD

Based on (Razzaghi &Yousefi, 2001), the operational matrix of the method could be obtained as follows:

In general, we have

$$\int_0^t \Psi(t)\,dt = P\Psi(t),$$

Where $\Psi(t)$ is defined before, and P is a $(2^{k-1}M)\times(2^{k-1}M)$ matrix given by

$$P = \frac{1}{2^K}\begin{bmatrix} L & F & F & . & F \\ 0 & L & F & . & F \\ . & 0 & \ddots & \ddots & . \\ . & . & . & . & F \\ 0 & 0 & . & 0 & L \end{bmatrix},$$

Where F and L are M×M matrices given by

$$F = \begin{bmatrix} 2 & 0\cdots & 0 \\ 0 & 0\cdots & 0 \\ \vdots & \vdots\ \vdots & \vdots \\ 0 & 0\cdots & 0 \end{bmatrix},$$

And

$$L = \begin{bmatrix} 1 & \frac{1}{3^{\frac{1}{2}}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}}{3\times 5^{\frac{1}{2}}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{5^{\frac{1}{2}}}{5\times 3^{\frac{1}{2}}} & 0 & -\frac{5^{\frac{1}{2}}}{5\times 7^{\frac{1}{2}}} & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\frac{7^{\frac{1}{2}}}{7\times 5^{\frac{1}{2}}} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{(2M-3)^{\frac{1}{2}}}{(2M-3)\times(2M-5)^{\frac{1}{2}}} & 0 & \frac{(2M-3)^{\frac{1}{2}}}{(2M-3)\times(2M-1)^{\frac{1}{2}}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{(2M-1)^{\frac{1}{2}}}{(2M-1)\times(2M-3)^{\frac{1}{2}}} & 0 \end{bmatrix}$$

Let $Y^{(4)}(x)=C^T\Psi(x)$, so we have $Y(x)=C^T P^4 \Psi(x)+P(x)$.

Where P(t) is a polynomial of degree 3. The coefficients of P could be obtained by the boundary conditions.

Let $P(x)=Q^T\Psi(x)$, $f(x)=F^T\Psi(x)$, and $g(x)=G^T\Psi(x)$.

Then the discretization of the equation will be as follows:

$C^T\Psi(x)+F^T\Psi(x)[\Psi^T(x)P^4C+\Psi^T(x)Q]=G^T\Psi(x)$

$C^T\Psi(x)+F^T\Psi(x)\Psi^T(x)P^4C+F^T\Psi(x)\Psi^T(x)Q=G^T\Psi(x)$

The following property of the product of two Legendre wavelet function vectors will also be used:

$C^T\Psi(x)\Psi^T(x)\approx\Psi^T(x)\tilde{C}$,

$\tilde{C}$ is a (2^(k-1) M)×(2^(k-1) M) matrix.

SOME SPECIAL CASES OF THE MAIN EQUATION

Case1: Consider the following equation:

$$\begin{cases} y^{(iv)}+4y=1, & -1\le x\le 1 \\ y(-1)=y(1)=0, y'(-1)=-y'(1)=\dfrac{\sinh(2)-\sin(2)}{4(\cosh(2)+\cos(2)} \end{cases} \qquad (10)$$

The analytic solution of the given problem is

$$y(x) = 0.25\left(1 - \frac{2(\sin(1)\sinh(1).\sin(x)\sinh(x) + \cos(1)\cosh(1).\cos(x)\cosh(x))}{\cos(2) + \cosh(2)}\right).$$

The coefficients for M=16 and k=1 are as follows:

$$y \approx c_1\Psi_{1,0} + c_2\Psi_{1,1} + c_3\Psi_{1,2} + c_4\Psi_{1,3} + c_5\Psi_{1,4} + c_6\Psi_{1,5} + c_7\Psi_{1,6} + c_8\Psi_{1,7} + c_9\Psi_{1,8} + c_{10}\Psi_{1,9}$$

$$+c_{11}\Psi_{1,10} + c_{12}\Psi_{1,11} + c_{13}\Psi_{1,12} + c_{14}\Psi_{1,13} + c_{15}\Psi_{1,14} + c_{16}\Psi_{1,15}$$

$$c_1 = 0.0083223150, \qquad c_2 = -0.0765219904,$$

$$c_3 = 0.0106300249, \qquad c_4 = 0.0181292960,$$

$$c_5 = 0.0031635090, \qquad c_6 = 0.0002252174,$$

$$c_7 = -0.0000035548, \qquad c_8 = -0.0000053546,$$

$$c_9 = -0.0000004778, \qquad c_{10} = -0.0000000198,$$

$$c_{11} = 0.0000000001, \qquad c_{12} = 0.0000000002,$$

$$c_{13} = c_{14} = c_{15} = c_{16} = 0$$

The results are presented in the following table.

Table1: Maximum absolute errors for problem by LWM method

	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
M =8	1.8×10^{-5}	4×10^{-5}	3×10^{-4}	3×10^{-3}	2×10^{-2}
M =16	2×10^{-13}	3×10^{-12}	3×10^{-11}	6×10^{-11}	10×10^{-9}

Table2: Maximum absolute errors forh=1/8,h=1/16,h=1/32 by spline method

h	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
1/8	6.35×10^{-5}	9.42×10^{-5}	4.06×10^{-4}	1.00×10^{-3}	2.54×10^{-4}
1/16	1.33×10^{-5}	2.01×10^{-5}	9.61×10^{-5}	2.41×10^{-4}	5.33×10^{-5}
1/32	3.17×10^{-6}	4.85×10^{-6}	2.35×10^{-5}	5.96×10^{-5}	1.27×10^{-5}

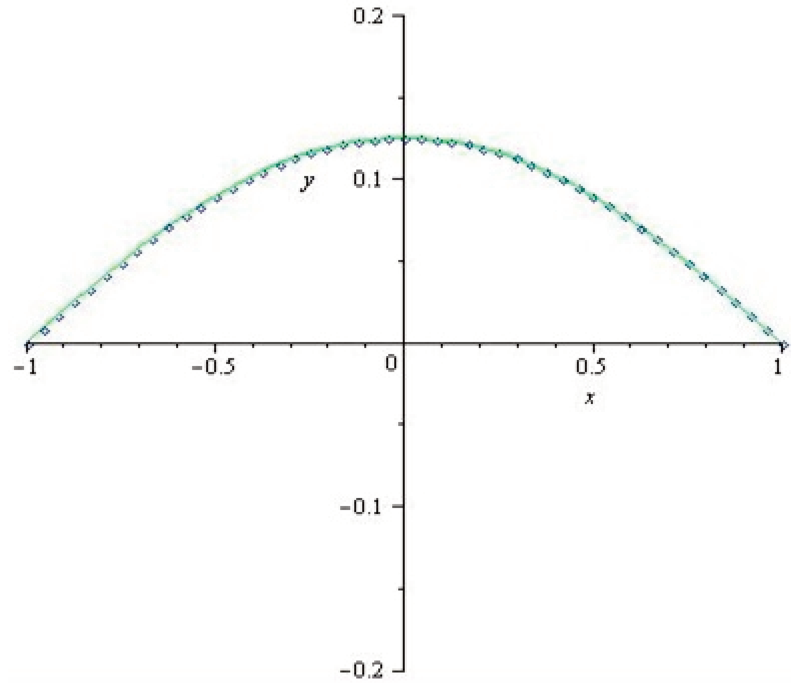

Fig.1.Comparison between exact solution and approximate solution by LWM method for M=8

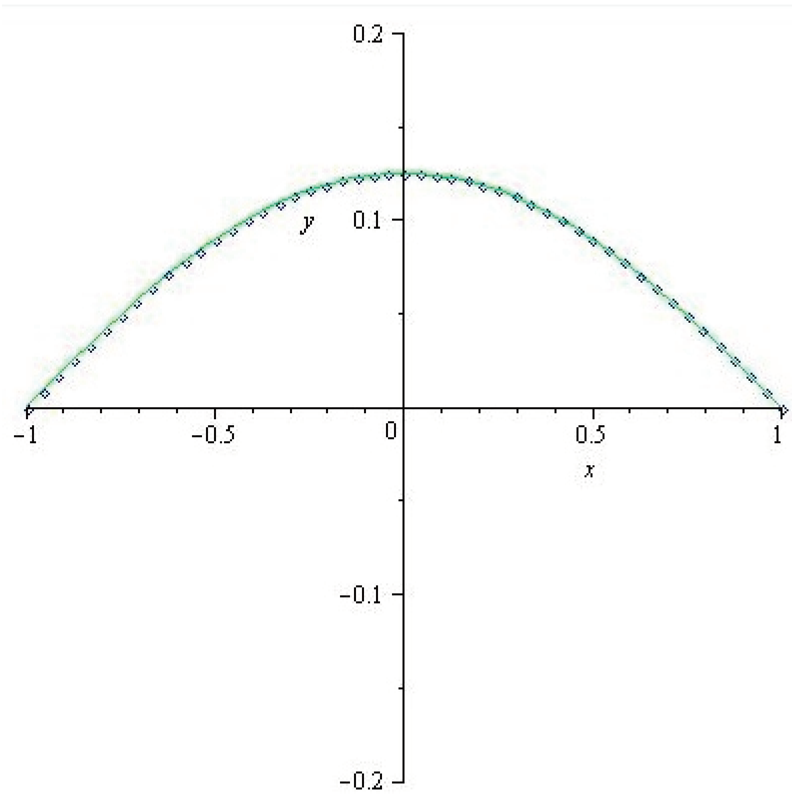

Fig.2.Comparison between exact solution and approximate solution by LWM method for M=16

Case2: Consider the special case of main equation as follows:

$$\begin{cases} y^{(iv)} + xy = -(8 + 7x + x^3)e^x, 0 \leq t \leq 1 \\ y(0) = y(1) = 0, y'(0) = 1, y'(1) = -e \end{cases} \quad (11)$$

The analytic solution of the above differential system is

$$y(x)=x(1-x)\, e^x.$$

The results are presented in the following table:

Table3: Maximum absolute errors for problem by LWM method

	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
M =8	2×10^{-6}	1×10^{-5}	4×10^{-5}	2×10^{-4}	5×10^{-13}
M =16	2×10^{-17}	7×10^{-17}	2×10^{-16}	2×10^{-15}	2×10^{-11}

Table4: Maximum absolute errors forh=1/8,h=1/16,h=1/32 by spline method

h	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
1/8	3.18×10^{-4}	1×10^{-3}	4.9×10^{-3}	4.12×10^{-2}	1.91×10^{-4}
1/16	4.17×10^{-5}	1.49×10^{-4}	1.5×10^{-3}	1.09×10^{-2}	2.17×10^{-5}
1/32	1.08×10^{-5}	3.46×10^{-5}	3.94×10^{-4}	2.7×10^{-3}	6.15×10^{-6}

Case3: Consider the special case of main equation as follows:

$$\begin{cases} y^{(iv)} - y = -4(2x\cos(x) + 3\sin(x), \quad -1 \leq x \leq 1 \\ \qquad y(-1) = y(1) = 0, \qquad\qquad y'(-1) = y'(1) = 2\sin(1) \end{cases} \quad (12)$$

The analytic solution of the above differential system is

$y(x)=(x^2-1)\sin(x)$

The results are presented in the following table

Table5: Maximum absolute errors for problem by LWM method

	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
M =8	1×10^{-5}	1×10^{-4}	5×10^{-4}	1×10^{-3}	2×10^{-3}
M =16	4×10^{-12}	1×10^{-11}	1.5×10^{-11}	7×10^{-10}	2×10^{-8}

Table6: Maximum absolute errors forh=1/8,h=1/16,h=1/32 by spline method

h	y_i	y_i'	y_i''	y_i'''	$y_i^{(4)}$
1/8	1.93×10^{-4}	6.01×10^{-4}	2.3×10^{-3}	1.8×10^{-2}	1.93×10^{-4}
1/16	3.40×10^{-5}	1.14×10^{-4}	7.90×10^{-4}	4.7×10^{-3}	3.40×10^{-5}
1/32	7.83×10^{-6}	2.72×10^{-5}	2.05×10^{-4}	1.2×10^{-3}	7.83×10^{-6}

CONCLUSION

In this paper we proposed LWM for a class of fourth order boundary value problems. The results obtained has more efficiency compare to other numerical methods. Moreover, the rate of convergence is higher than spline method. For Large amount of M, the size of the linear system is Large and because of discretization the system is ill-conditioned and we need to apply some regularization on it.

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