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Numerical solution of Fredholm and Volterra integral equations using the normalized Müntz−Legendre polynomials

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Abstract

The current research approximates the unknown function based on the normalized Müntz−Legendre polynomials (NMLPs) in conjunction with a spectral method for the solution of nonlinear Fredholm and Volterra integral equations. In this method, by using operational matrices, a system of algebraic equations is derived that can be readily handled through the use of the Newton scheme. The stability, error bound, and convergence analysis of the method are discussed in detail by preparing some theorems. Several illustrative examples are provided formally to show the efficiency of the proposed method.

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INTRODUCTION

Integral equations particularly Fredholm and Volterra integral equations play a key role in a vast area of biology, economics and engineering [1–5]. For this reason, integral equations have gained a special interest in the last few decades. General schemes to solve integral equations are classified into two categories of analytical and numerical methods. Exact analytical methods are usually not available for solving integral equations. Consequently, numerical methods have been adopted to find approximate solutions of integral equations. One of the numerical schemes for handling integral equations is to apply orthogonal polynomials such as Müntz polynomials. In this respect, many researchers have utilized the Müntz polynomials to solve many equations. For example, In [6], Mokhtari and his colleagues used the Tau method on the basis of the Müntz-Legendre polynomials to solve the fractional differential equations. Esmaeili and his colleagues [7] have solved the fractional differential equations by using collocation method based on the Müntz polynomials. Mokhtari [8] solved the second-order Abel integral equations in 2016 using Galerkin's method based on the Müntz Legendre polynomials. Yüzbaşı et al. found the solution of the linear Fredholm differential-integral equation by exerting a collocation method in terms of the Müntz-Legendre polynomials [9]. A fractional differential system has been solved by Aghashahi and Rasouli [10] by using the Müntz-Legendre polynomials. In 2018, Rahim Khani and Ordokhany [11] applied the Müntz−Legendre polynomials to solve the Bagley-Torvik equation in a large interval. Rahimkhani et al. [12] have solved the fractional Pantograph differential equations using Müntz−Legendre wavelet operational matrix [12]. Our focus in the present work to solve the Fredholm and Volterra integral equations as

$$y(x) = f(x) + \lambda_1 \int_0^1 k_1(x,t,y(t))\,dt + \lambda_2 \int_0^x k_2(x,t,y(t))\,dt, \qquad 0 \le x \le 1, \tag{1}$$

by proposing a new scheme based on the normalized Müntz−Legendre polynomial in conjunction with a spectral method. It is worthy of note that $y(x)$ is an unknown, $f(x)$ is a known function, k_1 and k_2 are linear or nonlinear functions and λ_1 and λ_2 are arbitrary constants.

The rest of paper is organized as follows: In Section 2, the Müntz−Legendre polynomials are explained; in the third section, the solution method is introduced in detail; several theorems regarding the stability, error bound, and convergence analysis of the method are presented in Section 4; in Section 5, several illustrative examples are used to examine the proposed methodology and its results and at the end, concluding remarks are provided in the last section.

BASIC FORMULATION OF THE MUNTZ AND MUNTZ-LEGENDER POLYNOMIALS

In this section, the Müntz polynomials and their basic properties are reviewed systematically. Further information can be found in [13, 14].

Müntz−Legendre polynomials

In this subsection, first the Müntz Theorem [15-17] which a generalization of Weierstrass' theorem is defined. After that, the relevance of Müntz−Legendre polynomials and Jacobi polynomials is given in another Theorem.

Theorem 1 (Müntz's Theorem). Suppose that $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\inf_k \lambda_k > -\frac{1}{2}$. Then span $\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $L^2[0,1]$ if and only if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k + \frac{1}{2}} = \infty$ [18].

It should be mentioned that the Müntz polynomial $\sum_{k=0}^{n} c_k x^{\lambda_k}$ can be orthogonalized on $L^2[0,1]$. In this regard, the orthogonal Müntz polynomials that were first introduced by Armenian mathematician can be defined as

$$L_n(\Lambda,x)=\sum_{k=0}^{n} c_{n,k} x^{\lambda_k}, \quad c_{n,k}=\frac{\prod_{j=0}^{n-1}(\lambda_k+\lambda_j+1)}{\prod_{j=0, j\neq k}^{n}(\lambda_k-\lambda_j)}, \quad k=0,1,\ldots,n, \quad n=0,1,2,\ldots.$$

in which $L_n(\Lambda,x)$ is the orthogonal Müntz polynomial of order n associated with Λ.

MLPs are orthogonal in $L^2[0,1]$ with respect to the weight function of Legendre polynomials, namely

$$\int_0^1 L_n(x)L_m(x)\,dx=\frac{\delta_{n,m}}{(2\lambda_n+1)}, \quad n\geq m, \quad n,m=0,1,2,\ldots,$$

where $\delta_{n,m}$ indicates the Kronecker Delta. The recursive relation for the MLPs can be written as

$$L_n(x)=L_{n-1}(x)-(\lambda_n+\lambda_{n-1}+1)x^{\lambda_n}\int_x^1 t^{-\lambda_n-1}L_{n-1}(t)\,dt, \qquad x\in(0,1].$$

By assuming $\lambda_k=\alpha k$, $k=0,1,\ldots,n$ which α is a real constant, the MLPs can be defined on $[0,T]$ as

$$L_n(x,\alpha)=\sum_{k=0}^{n} c_{n,k}\left(\frac{x}{T}\right)^{\alpha k}, \qquad c_{n,k}=\frac{(-1)^{n-k}}{\alpha^k k!(n-k)!}\prod_{i=0}^{n-1}((k+i)\alpha+1).$$

Theorem 2. If $\alpha>0$ and $x\in[0,T]$, then

$$L_n(x,\alpha)=J_n^{0,\frac{1}{\alpha}-1}\left(2\left(\frac{x}{T}\right)^{\alpha}-1\right), \quad n=0,1,2,\ldots,$$

where $J_n^{0,\frac{1}{\alpha}-1}(x)$ is the Jacobi polynomial with parameters 0 and $\frac{1}{\alpha}-1$.

Proof: Please see [19].

Furthermore, the NMLPs can be defined as below:

$$\bar{L}_n(x)=\sqrt{2\varepsilon_n+1}\,L_n(x).$$

Operational matrices of the normalized Müntz-Legendre polynomials

In this subsection, the operational matrices for NMLPs including the operational matrixes of integral and product are introduced.

The operational matrix of integral

By a simple calculation, the integration of the vector $\bar{L}(x)$ can be derived as

$$\int_0^x \bar{L}(t)\,dt=P\,\bar{L}(x),$$

where P is the $(N+1)\times(N+1)$ operational matrix of integration. For instance, if $N=3$ and $\alpha=\frac{1}{2}$, then P can be extracted

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{5}\sqrt{2} & \frac{1}{30}\sqrt{3} & 0 \\ -\frac{1}{5}\sqrt{2} & 0 & \frac{1}{21}\sqrt{2}\sqrt{3} & \frac{1}{35}\sqrt{2} \\ -\frac{1}{30}\sqrt{3} & -\frac{1}{21}\sqrt{2}\sqrt{3} & 0 & \frac{2}{45}\sqrt{3} \\ 0 & -\frac{1}{35}\sqrt{2} & -\frac{2}{45}\sqrt{3} & 0 \end{bmatrix}$$

The operational matrix of product

Similarly, simple calculations show that he product of two vector bases of the NMLPS can be shown as

$$\bar{L}(x)\bar{L}^T(x)Y\approx\tilde{Y}\bar{L}(x),$$

where Y is the $(N+1)\times 1$ vector and $\tilde{Y}$ is a $(N+1)\times(N+1)$ matrix that is named the operational matrix of product. Again if $N=3$ and $\alpha=\frac{1}{2}$, then $\tilde{Y}$ can be expressed as

$$\tilde{Y}=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & \frac{3\sqrt{3}}{5}c_3+c_1-\frac{2\sqrt{2}}{5}c_2 & \frac{-16\sqrt{2}}{35}c_3+\frac{3\sqrt{3}}{5}c_2+\frac{3\sqrt{6}}{7}c_4 & \frac{3\sqrt{6}}{7}c_3-\frac{10\sqrt{2}}{21}c_4 \\ c_3 & \frac{-16\sqrt{2}}{35}c_3+\frac{3\sqrt{3}}{5}c_2+\frac{3\sqrt{6}}{7}c_4 & \frac{-16\sqrt{2}}{35}c_2+c_1-\frac{16}{21}c_4+\frac{24\sqrt{3}}{35}c_3 & \frac{3\sqrt{6}}{7}c_2+\frac{5\sqrt{3}}{7}c_4+\frac{16}{21}c_3 \\ c_4 & \frac{3\sqrt{6}}{7}c_3-\frac{10\sqrt{2}}{21}c_4 & \frac{3\sqrt{6}}{7}c_2+\frac{5\sqrt{3}}{7}c_4+\frac{16}{21}c_3 & \frac{-10\sqrt{2}}{21}c_2+\frac{5\sqrt{3}}{7}c_3+c_1-\frac{72}{77}c_4 \end{bmatrix}.$$

EXPLAINING OF THE SOLUTION METHOD

In order to solve the equation (1) by using the normalized Müntz-Legendre polynomials, without loss of generality, consider the following case

$$y(x)=f(x)+\lambda_1\int_0^1 k_1(x,t)g_1(y(t))\,dt+\lambda_2\int_0^x k_2(x,t)g_2(y(t))\,dt, \qquad 0\le x\le 1,$$

where g_1 and g_2 are linear or nonlinear functions. Now, the approximate solution of the above integral equation can be considered as a linear combination of the NMLPs as

$$y(x)=\sum_{n=0}^{N} c_n\bar{L}_n(x)=\bar{L}^T(x)C=C^T\bar{L}(x),$$

where $\bar{L}_n(x)$, $n=0,1,\dots,N$ are the NMLPs and $\bar{L}$ and C are the following vectors

$$\bar{L}=[\bar{L}_0(x),\bar{L}_1(x),\dots,\bar{L}_N(x)]^T,$$
$$C=[c_0,c_1,\dots,c_N]^T.$$

Also, other expressions of the above equation can be approximated as

$$f(x)=\bar{L}^T(x)F=F^T\bar{L}(x),$$
$$k_i(s,t)=\bar{L}^T(x)K_i\bar{L}(t), \quad i=1,2,$$
$$g_i(y(x))=\bar{L}^T(x)Y_i=Y_i^T\bar{L}(x) \quad i=1,2,$$

where F, K_1 and K_2 are known and Y_1 and Y_2 are vectors of elements of the vector C. Inserting the above approximations into equation (1) yields

$$\bar{L}^T(x)C=\bar{L}^T(x)F+\lambda_1\int_0^1(\bar{L}^T(x)K_1\bar{L}(t))(\bar{L}^T(t)Y_1)dt+\lambda_2\int_0^x(\bar{L}^T(x)K_2\bar{L}(t))(\bar{L}^T(t)Y_2)dt,$$
$$=\bar{L}^T(x)F+\lambda_1\bar{L}^T(x)K_1Y_1+\lambda_2\bar{L}^T(x)K_2\tilde{Y}_2P\bar{L}(x),$$

where P and $\tilde{Y}_2$ are operational matrices given in the previous section.

The obtained equations can be solved by a spectral method such as the Galerkin method. Therefore, we consider the following approximation for solving the above system

$$\bar{L}^T(x)K_2\tilde{Y}_2P\bar{L}(x)=X^T\bar{L}(x)=\bar{L}^T(x)X.$$

Thus, one has

$$\bar{L}^T(x)C-\bar{L}^T(x)F-\lambda_1\bar{L}^T(x)K_1Y_1-\lambda_2\bar{L}^T(x)X \approx 0.$$

Now by using the Galerkin method in the interval $[0,1]$, the following nonlinear algebraic system is yielded

$$C-F-\lambda_1K_1Y_1-\lambda_2X \approx 0.$$

Finally, through the use of the Newton's iterative method for the above system, the elements of vector C are determined.

STUDY OF STABILITY AND ERROR ANALYSIS OF NMLPs METHOD

In this section, the stability, error bound and convergence of the method based on the NMLPs are studied.

Theorem 3 (Stability). Suppose that $y_N(x)$ and $y(x)$ are the approximate and exact solutions of the equation (1). Also, assume that $e_N(x)$ is the error of the approximated solution and $r_N(x)$ is the residual function corresponding to the approximated solution. If $\left(|\lambda_1|\eta_1+|\lambda_2|\eta_2\right)<1$ then one has,

$$\|E\| \le \eta\,\|R\|,$$

where $\eta>0$ and $\|E\|=\underset{x\in[0,1]}{Max}\|e_N(x)\|$ and $\|R\|=\underset{x\in[0,1]}{Max}\|r_N(x)\|$.

Proof: It is clear that $y_N(x)$ and $y_N(x)+e_N(x)$ satisfy the equation (1), accordingly

$$y_N(x)=f(x)+\lambda_1\int_0^1 k_1(x,t,y_N(t))\,dt+\lambda_2\int_0^x k_2(x,t,y_N(t))\,dt+r_N(x), \tag{2}$$

and

$$y_N(x)+e_N(x)=f(x)+\lambda_1\int_0^1 k_1(x,t,y_N(t)+e_N(t))\,dt+\lambda_2\int_0^x k_2(x,t,y_N(t)+e_N(t))\,dt, \tag{3}$$

Subtracting (2) from (3) results in

$$e_N(x)=-r_N(x)+\lambda_1\int_0^1\left(k_1(x,t,y_N(t)+e_N(t))-k_1(x,t,y_N(t))\right)dt$$
$$+\lambda_2\int_0^x\left(k_2(x,t,y_N(t)+e_N(t))-k_2(x,t,y_N(t))\right)dt.$$

Due to the following Lipchitz's conditions,

$$\|k_1(x,t,y_N(t)+e_N(t))-k_1(x,t,y_N(t))\|\le\eta_1\|e_N(t)\|,$$
$$\|k_2(x,t,y_N(t)+e_N(t))-k_2(x,t,y_N(t))\|\le\eta_2\|e_N(t)\|,$$

where η_1 and η_2 are positive constants, we will get

$$\|e_N(x)\|\le\|r_N(x)\|+|\lambda_1|\left|\int_0^1\|k_1(x,t,y_N(t)+e_N(t))-k_1(x,t,y_N(t))\|dt\right.$$
$$+|\lambda_2|\left|\int_0^x\|k_2(x,t,y_N(t)+e_N(t))-k_2(x,t,y_N(t))\|dt\right.$$
$$\le\|r_N(x)\|+|\lambda_1|\left|\int_0^1\eta_1\|e_N(t)\|dt+|\lambda_2|\right|\int_0^x\eta_2\|e_N(t)\|dt.$$

If $\|E\|=\underset{x\in[0,1]}{Max}\|e_N(x)\|$ and $\|R\|=\underset{x\in[0,1]}{Max}\|r_N(x)\|$, then one has,

$$\|E\|\le\|R\|+\left(|\lambda_1|\eta_1+|\lambda_2|\eta_2\right)\|E\|.$$

Thus $\|E\|\le\eta\|R\|$, where

$$\eta=\frac{1}{1-\left(|\lambda_1|\eta_1+|\lambda_2|\eta_2\right)}.$$

Theorem 4 (The error bound). If $y(x)\in C^{N+1}[0,1]$, then error bound for its approximation is obtainable as follows:

$$\|e_N(x)\|=\|y(x)-y_N(x)\|\le\frac{CM}{(N+1)!},$$

where $M=Max\left|y^{(N+1)}(x)\right|$ for $x\in[0,1]$ and C is a constant that to come later.

Proof: Suppose $y_N(x)$ is the best approximation of $y(x)$ based on NMLPs and also, the Taylor expansion of $y(x)$ around $a\in[0,1]$ as follows

$$P_N(x)=y(a)+y'(a)(x-a)+y''(a)\frac{(x-a)^2}{2!}+\ldots+y^{(N)}(a)\frac{(x-a)^N}{n!},$$

Therefore,

$$\left|y(x)-P_N(x)\right|\le M\frac{|x-a|^{N+1}}{(N+1)!}.$$

where $M=\underset{\xi\in(a,x)}{Max}\left|y^{(N+1)}(\xi)\right|$. According to the property of least square orthogonal polynomials [20], then

$$\|y(x)-y_N(x)\|^2=\int_0^1\left(y(x)-y_N(x)\right)^2dx$$
$$\le\int_0^1\left(y(x)-P_N(x)\right)^2dx$$
$$\le\int_0^1\left(\frac{M(x-a)^{N+1}}{(N+1)!}\right)^2dx=\frac{M^2}{((N+1)!)^2}\int_0^1(x-a)^{2N+2}dx.$$

Thus $\|y(x)-y_N(x)\|\le\frac{CM}{(N+1)!}$, where

$$C=\sqrt{\frac{(1-a)^{2N+3}+a^{2N+3}}{2N+3}}.$$

Theorem 5 (Convergence). Suppose functions k_1 and k_2 satisfy in the following conditions

$$i)\ \|k_1(x,t,y(t))-k_1(x,t,y_N(t))\|\le\beta_1\|y(t)-y_N(t)\|,$$

$$ii)\ \|k_2(x,t,y(t))-k_2(x,t,y_N(t))\|\le\beta_2\|y(t)-y_N(t)\|.$$

If N tend to infinity, then

$$\|e_N(x)\|=\|y(x)-y_N(x)\|\to 0.$$

Proof: Because $y_N(x)$ satisfy in equation (1), one has

$$y_N(x) = f(x) + \lambda_1 \int_0^1 k_1(x,t,y_N(t))\,dt + \lambda_2 \int_0^x k_2(x,t,y_N(t))\,dt + r_N(x),$$

By subtracting equations (1) and (4), we will have:

$$y(x) - y_N(x) = -r_N(x) + \lambda_1 \int_0^1 \left(k_1(x,t,y(t)) - k_1(x,t,y_N(t))\right) dt$$
$$+\lambda_2 \int_0^x \left(k_2(x,t,y(t)) - k_2(x,t,y_N(t))\right) dt\ .$$

Therefore,

$$\left|y(x) - y_N(x)\right| \le \left|r_N(x)\right| + \left|\lambda_1\right| \int_0^1 \left|k_1(x,t,y(t)) - k_1(x,t,y_N(t))\right| dt$$
$$+ \left|\lambda_2\right| \int_0^x \left|k_2(x,t,y(t)) - k_2(x,t,y_N(t))\right| dt$$
$$\le \left|r_N(x)\right| + \left|\lambda_1\right| \int_0^1 \beta_1 \left|y(t) - y_N(t)\right| dt + \left|\lambda_2\right| \int_0^x \beta_2 \left|y(t) - y_N(t)\right| dt\ .$$

By using the previous theorem

$$\left|e_N(x)\right| \le \left|r_N(x)\right| + \frac{\left|\lambda_1\right| \beta_1 C M}{(N+1)!} + \frac{\left|\lambda_2\right| \beta_2 C M}{(N+1)!},$$

Because $\left|r_N(x)\right| \to 0$, when $N \to \infty$, then the prove is complete.

IMPROVED SOLUTION IN LINEAR CASE

Now, we can improve the approximate solution using error estimation, provided that the g_1 and g_2 are linear functions and suppose $e_N(x)$ as Müntz's approximation error function. So that $y_N(x)$ applies in equation (1) with residual function, $r_N(x)$ .Then,

$$e_N(x) - \lambda_1 \int_0^1 k_1(x,t)(g_1(y(t)) - g_1(y_N(t)))dt - \lambda_2 \int_0^x k_2(x,t)(g_2(y(t)) - g_2(y_N(t)))dt = r_N(x). \quad (4)$$

Therefore

$$e_N(x) - \lambda_1 \int_0^1 k_1(x,t) g_1(e_N(t)) dt - \lambda_2 \int_0^x k_2(x,t) g_2(e_N(t)) dt = r_N(x). \quad (5)$$

By solving Equation (12) by NMLP method, we obtain gain an approximate solution $(e_{N,M}(x))$ for the $e_N(x)$, that can be used to improve the approximate solution equation (1). Therefore $y_{N,M}(x)$ is the better than $y_N(x)$, such that,

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x).$$

NUMERICAL APPLICATIONS

In this section, to illustrate the numerical application of this method, some examples are given and comparison of the absolute error of the proposed method with some methods is given. All results are computed with the mathematical software ***MAPLE 18*** with ten significant digits. Also, some of numerical results discuss the performance of our method, we have obtained the absolute error and E_2 error defined as follows

$$error_{NMLPM}(x) = \left| y(x) - y_{NMLPM}(x) \right|,$$

$$E_2 = \left(\int_0^1 (y(x) - y_{NMLPM}(x))^2 dx \right)^{\frac{1}{2}}.$$

Example 1. consider the following Vollterra-Fredholm integral equation,

$$y(x) = f(x) + \int_0^x (x-t) y^2(t) dt + \int_0^1 (x+t) y(t) dt, \quad (6)$$

where $f(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4}$ and the exact solution is $y(x) = x^2 - 2$.

According to the method presented, we have

$$y(x) \simeq \bar{L}^T(x) C, \qquad f(x) \simeq \bar{L}^T(x) F\ ,$$
$$y^2(t) \simeq Y_1^T \bar{L}(x),$$
$$x - t \simeq \bar{L}^T(x) K_1 \bar{L}(t),$$
$$x + t \simeq \bar{L}^T(x) K_2 \bar{L}(t).$$

By replacing the above relations in equation (6), for $\alpha = 1$, $N = 8$,the following linear algebraic system is obtained,

$$\bar{L}^T(x) C - \bar{L}^T(x) F - \bar{L}^T(x) K_1 \tilde{Y}_1 P \bar{L}(x) - \bar{L}^T(x) K_2 C \simeq 0,$$

By using the newton method, the unknown vector C is obtain as following

$$C \simeq [-1.666666666, 0.2886751348, 0.07453559916, -1.979927054 \times 10^{-11}, 6.842772931 \times 10^{-12}, -3.399236740 \times 10^{-13}, -1.737992881 \times 10^{-13}, -3.233116770 \times 10^{-15}, 1.820681636 \times 10^{-15}].$$

As a result, the approximation solutions will be as follows

$y(x) \simeq 9.661332292\times10^{-11}x^{8} - 3.434784486\times10^{-10}x^{7} - 9.82223417\times10^{-10}x^{6} + 1.120175776\times10^{-9}x^{5} + 2.088806711\times10^{-10}x^{4} - 3.516999999\times10^{-9}x^{3} + 1.0000000002x^{2} + 1.289433624\times10^{-9}x - 2.0000000000.$

In Tables 1, the NMLPs approximate and exact solutions of $y(x)$ for $\alpha = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, $N = 8$, for some values of x are presented and absolute error of the suggested method have been reported in Tables 2 and also in Tables 3, the error E_2 of NMLPs method have been compared with the hat functions method (HFM) [21], modification of hat function method (MHFM) [22] and Triangular functions method (TFM) [23]. It is clear that the NMLPs method is better the other method for $n = 8$. In Figure 1, the plot of exact and present solutions and also absolute error function of approximate solution has been shown.

Table 1: Numerical results of $y_{NMLPM}(x)$ for $N = 8$ and various values of x in Example 1.

x	$y_{NMLPM}(x)$, $\alpha = 1/2$	$y_{NMLPM}(x)$, $\alpha = 1/3$	$y_{NMLPM}(x)$, $\alpha = 1/4$	$y_{NMLPM}(x)$, $\alpha = 1$	$y_{exact}(x)$
0.0	–1.996109585	–2.011381672	–2.239843426	–1.999999999	–2
0.2	–1.959623863	–1.959858038	–1.957842584	–1.959999999	–1.96
0.4	–1.840393993	–1.840015706	–1.840385935	–1.839999998	–1.84
0.6	–1.640062945	–1.640076207	–1.641842162	–1.6399999998	–1.64
0.8	–1.359485799	–1.359931226	–1.360332139	–1.359999998	–1.36
1	–1.001161497	–1.000304068	–0.9984295801	–0.999999999	–1

Table 2: The absolute error of $y_{NMLPM}(x)$ for $N = 8$ and various values of x in Example 1.

x	$error_{NMLPM}(y(x))$ $\alpha = \frac{1}{2}$	$error_{NMLPM}(y(x))$ $\alpha = \frac{1}{3}$	$error_{NMLPM}(y(x))$ $\alpha = \frac{1}{4}$	$error_{NMLPM}(y(x))$, $\alpha = 1$
0	3.8904155×10^{-3}	1.13816726×10^{-2}	0.239843672	1×10^{-9}
0.2	3.761376×10^{-4}	1.419624×10^{-4}	2.1569069×10^{-3}	1.451851106×10^{-9}
0.4	3.9398971×10^{-4}	1.57058×10^{-4}	3.865289×10^{-4}	1.800481328×10^{-9}
0.6	6.29449×10^{-5}	7.620733×10^{-5}	1.84287083×10^{-3}	1.909342474×10^{-9}
0.8	5.142012×10^{-4}	6.87739×10^{-5}	3.330277×10^{-4}	1.692821891×10^{-9}
1	1.1614974×10^{-3}	3.040675×10^{-4}	1.5693407×10^{-3}	1.088144573×10^{-9}

Table 3: Assessment of the E_2 error for Example 1 with other methods.

	HF method, [21]	*TF method*, [23]	*MHF method*, [22]	*present method*, $N=8$
E_2 ,	$m=8,\ 1.2\times10^{-2}$ $m=16,\ 3\times10^{-3}$ $m=32,\ 7.5\times10^{-4}$	$m=8,\ 2.7\times10^{-3}$ $m=16,\ 7.1\times10^{-4}$ $m=32,\ 3.8\times10^{-3}$	$m=8,\ 6\times10^{-4}$ $m=16,\ 7.6\times10^{-5}$ $m=32,\ 1.0\times10^{-5}$	$\alpha=1,\ 6.947814781\times10^{-10}$ $\alpha=\frac{1}{2},\ 3.863388797\times10^{-4}$ $\alpha=\frac{1}{3},\ 1.613652090\times10^{-4}$ $\alpha=\frac{1}{4},\ 1.842321063\times10^{-4}$

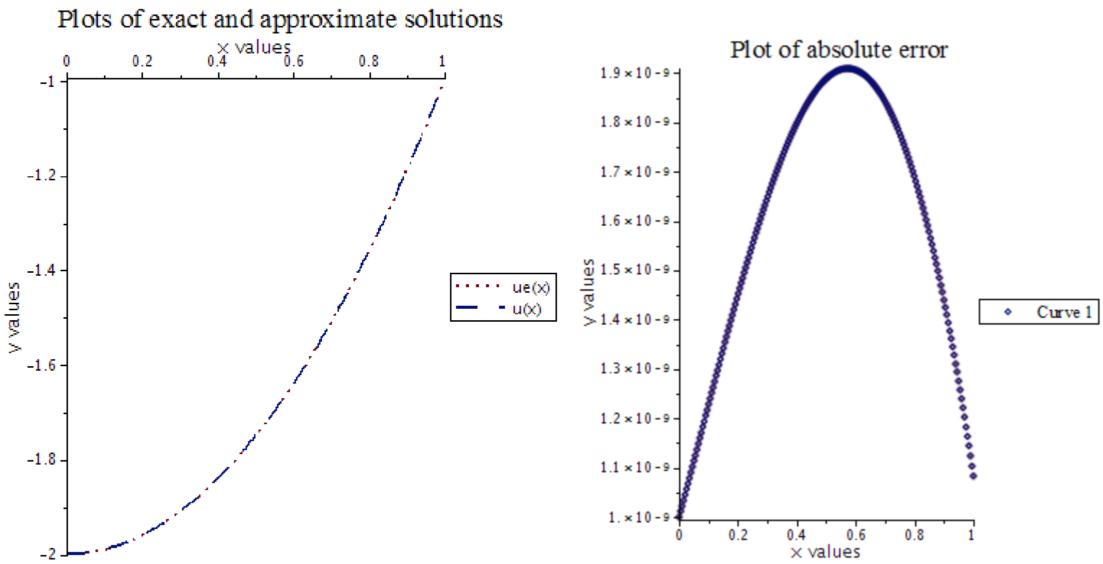

Fig.. 1: Plots of exact and approximate solutions and absolute error for $\alpha=1$ in Example 1.

Example 2. In this example, study the following nonlinear Fredholm- Volterra integral equations

$$y(x)=e^{-x}-e^{x}(h(x)-1)+\int_0^{h(x)} e^{x+t}\, y(t)dt-\int_0^1 e^{x+t}$$

with $h(x)=\ln(x+1)$ and exact solution $y(x)=e^{-x}$. Solving above equation by NMLPs method for $N=5$ and $\alpha=1$, the unknown vectors C can be derived as

$C \simeq [0.6321315468, -0.1794982362, 0.02301041819,$
$-0.001947002095,$
$0.0001283429039, -0.00007488634086].$

Thus

$y_{NMLPs}(x)\simeq -0.062589213435x^5+0.1834250433x^4-$
$0.2960168274x^3$
$+0.5500659176x^2-1.007484357x+$
$1.000269153.$

In Tables 4 and 5, the obtained approximation solutions for some values of $\alpha=1,\frac{1}{2},\frac{1}{3},\frac{1}{4}$ have been compared with exact solution, and also, the absolute errors of NMLP method have been compared with the TFM method [24]. Plots of exact and approximate solutions and absolute error for $\alpha=1$ will be shown in Fig. 2.

Table 4 : Numerical results of $y(x)$ for various values of x in Example 2.

x	$y_{NMLPM}(x)$, $\alpha=1/2$	$y_{NMLPM}(x)$, $\alpha=1/3$	$y_{NMLPM}(x)$, $\alpha=1/4$	$y_{NMLPM}(x)$, $\alpha=1$	$y_{exact}(x)$
0.0	1.000246877	0.011787150	1.033032457	1.000253867	1
0.2	0.8187307531	0.8185871686	0.8183380787	0.8186821113	0.8187307531
0.4	0.6703083206	0.6702346418	0.6701956364	0.6703905575	0.6703200460
0.6	0.5488085704	0.5489285954	0.5490568918	0.5487712487	0.5488116361
0.8	0.4493785786	0.4492446600	0.4490879803	0.4493792537	0.4493289641
1	0.3677367931	0.3675004027	0.366625778	0.3676879637	0.3678796612

Table 5: Comparison of absolute error of $y(x)$ with method in [24] for various values of x in Example 2.

x	$error_{NMLPM}(y(x))$ $\alpha=\frac{1}{2}$	$error_{NMLPM}(y(x))$ $\alpha=\frac{1}{3}$	$error_{NMLPM}(y(x))$ $\alpha=\frac{1}{4}$	$error_{NMLPM}(y(x))$, $\alpha=1$	$error_{TFM}(y(x))$,
0	2.46877×10^{-4}	1.1787150×10^{-2}	3.3032457×10^{-2}	2.56867×10^{-4}	2.270623×10^{-4}
0.2	2.31046×10^{-5}	1.435845×10^{-4}	3.926744×10^{-4}	4.86418×10^{-5}	2.981265×10^{-4}
0.4	1.17254×10^{-5}	8.54042×10^{-5}	1.244096×10^{-4}	7.05115×10^{-5}	4.427352×10^{-4}
0.6	3.0657×10^{-6}	1.169593×10^{-4}	2.452557×10^{-4}	4.03874×10^{-5}	3.760377×10^{-4}
0.8	4.95545×10^{-5}	8.43041×10^{-5}	2.409838×10^{-4}	5.02896×10^{-5}	2.095614×10^{-4}
1	1.426481×10^{-4}	3.790385×10^{-4}	1.2536632×10^{-3}	1.914775×10^{-4}	2.159911×10^{-4}

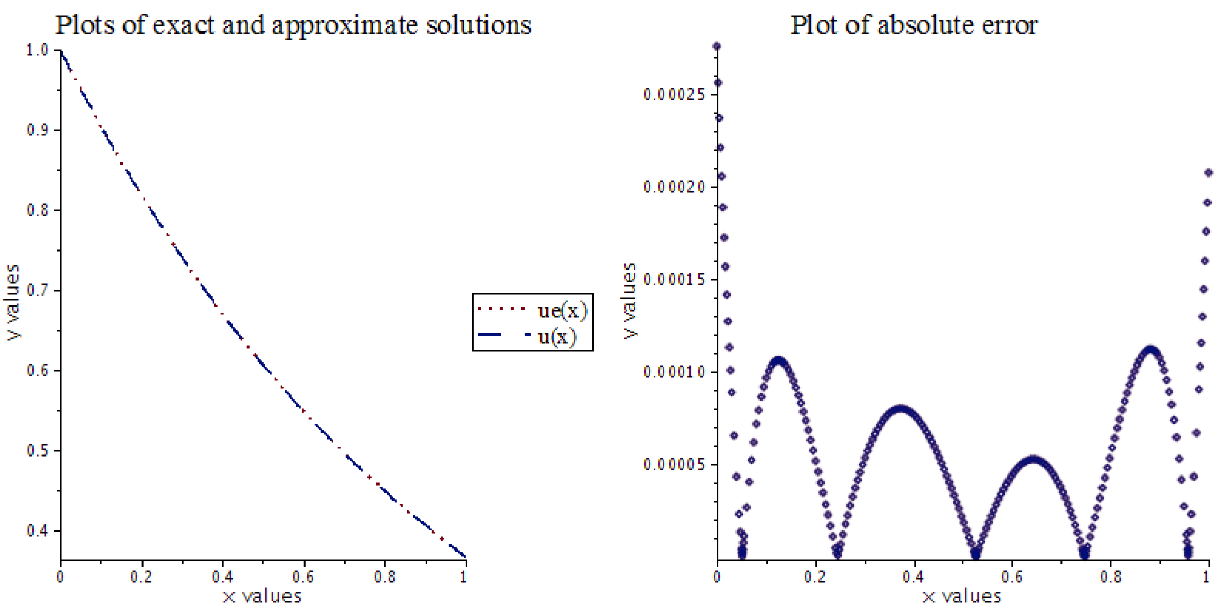

Fig. 2: Plots of exact and approximate solutions and absolute error for $\alpha=1$ in Example 2

Example 3. Pay attention to the nonlinear Fredholm-Volterra integral equation

$$y(x)=\frac{1}{6}x+\frac{1}{2}xe^{-x^2}+\int_0^x xt\,e^{-y^2(t)}dt-\int_0^1 xy^2(t)dt$$

where exact solution is $y(x)=x$. Using this method, the unknown vector C is obtained as follows for $N=5$ and $\alpha=1$.

$C=[0.4999836222, 0.2886612833, -$
$0.000007431701592, -0.0000006384896657,$
$-0.000005304690537, -0.000002649341547].$

Also the following approximate solution would be achieved.

$y_{NMLPs}(x)\simeq -0.002214291707x^5+$
$0.004421744255x^4-0.003030535185x^3$
$+0.0001200553650x^2+0.9999036855x+$
7.61030612×10^{-7}.

In Table 6, the computed solutions for some values of $\alpha=1,\frac{1}{2},\frac{1}{3}$ have been compared with exact solution and also, in Table 7, the absolute errors of NMLP method have been compared with method in [25]. Plots of exact and approximate solutions and absolute error will be showed Fig. 3.

Table 6 : Numerical results of $y(x)$ for various values of x in Example 4.

x	$y_{NMLPM}(x)$, $\alpha=1/2$	$y_{NMLPM}(x)$, $\alpha=1/3$	$y_{NMLPM}(x)$, $\alpha=1$	$y_{exact}(x)$
0.0	0.0048225456	0.0036075686	7.61030612×10^{-7}	0
0.2	0.2007172671	0.0.2002835014	0.1999964222	0.2
0.4	0.3997371331	0.3999064981	0.3999900122	0.4
0.6	0.5994398261	0.5996192782	0.5999844714	0.6
0.8	0.8001797156	0.7994538177	0.7999824781	0.8
1	1.002008398	0.9993982454	0.9999014193	1

Table 7: Comparison of absolute error of $y(x)$ with method in [25] for various values of x in Example 4.

x	$error_{NMLPM}(y(x))$ $\alpha=\frac{1}{2}$	$error_{NMLPM}(y(x))$ $\alpha=\frac{1}{3}$	$error_{NMLPM}(y(x))$, $\alpha=1$	$error(y(x))$, [25]
0	4.8225456×10^{-3}	3.6075686×10^{-3}	7.61030612×10^{-7}	0
0.2	7.17267042×10^{-4}	2.8350140×10^{-4}	3.577718808×10^{-6}	6.66696141×10^{-2}
0.4	2.6286691×10^{-4}	9.350194×10^{-5}	9.98785698×10^{-6}	1.20842155×10^{-1}
0.6	5.6017398×10^{-4}	3.8072182×10^{-4}	1.55286057×10^{-5}	1.43213404×10^{-1}
0.8	1.7971560×10^{-4}	5.4618227×10^{-4}	1.75218105×10^{-5}	1.14769599×10^{-1}
1	2.008398×10^{-3}	6.017546×10^{-4}	9.8580741×10^{-4}	2.59596401×10^{-2}

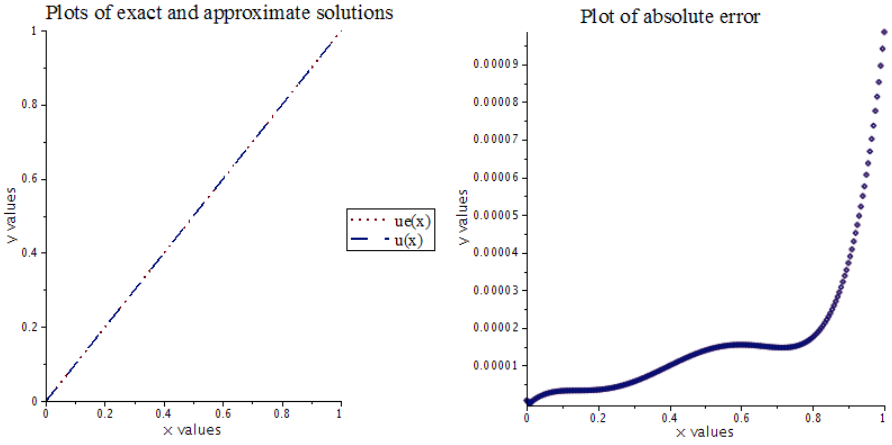

Fig. 3. Plots of exact and approximate solutions and absolute error for $\alpha = 1$ in Example 3.

Example 4. One of the most important equations in physics is the Love equation, which is solved in this work by using the NMLPs method.

$$y(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{y(t)}{1+(x-t)^2}\,dt = 1, \qquad -1 \le x \le 1.$$

To change the integral interval, using the following variable change,

$$x' = \frac{1}{2}(x+1) \quad , \quad t' = \frac{1}{2}(t+1).$$

We have the following equivalent equation,

$$\bar{y}(x') - \frac{2}{\pi}\int_{0}^{1} \frac{\bar{y}(t')}{1+4(x'-t')^2}\,dt' = 1, \qquad 0 \le x' \le 1.$$

By choosing $\alpha = 1$ and $N = 10$, the following approximate solution will be obtained.

$$y_{NMLPM}(x) \simeq -0.001493476576\,x^{10} - 0.00195268387\,x^8 + 1\times 10^{-10} x^7 + 0.0201293310\,x^6 - 8.0\times 10^{-10} x^5 + 0.0157152220\,x^4 + 3.9\times 10^{-9} x^3 - 0.3117398584\,x^2 + 1.34\times 10^{-9} x + 1.919032343.$$

The comparison between Chebyshev approximation method (CHAM) [24] and our approximation for $y(x)$ have been reported in Table 8 and also value of residual function for $\alpha = 1, 2$ and some values of x are presented in this table. The plots of these functions are shown in the Fig. 4.

Table 8: The approximate solutions and value of residual function for $y(x)$ in Example 4.

x	$y_{CHAM}(x)$	$y_{NMLPM}(x)$ $\alpha = 1$	$y_{NMLPM}(x)$ $\alpha = 2$	$Res(y_{NMLPM}(x))$ $\alpha = 1$	$Res(y_{NMLPM}(x))$ $\alpha = 2$
0	1.91903	1.916156219	1.92784695	4.808×10^{-7}	2.77584×10^{-3}
0.2	1.90659	1.906589107	1.904352046	1.71×10^{-7}	2.21696×10^{-3}
0.4	1.86964	1.869637235	1.871303721	4.6659×10^{-7}	1.6834×10^{-3}
0.6	1.80974	1.809740139	1.807932625	9.48×10^{-7}	1.7924×10^{-3}
0.8	1.73075	1.730744439	1.732697079	1.3480×10^{-5}	1.9656×10^{-3}
1	1.63969	1.639690334	1.632140508	4.8864×10^{-5}	7.5338×10^{-3}

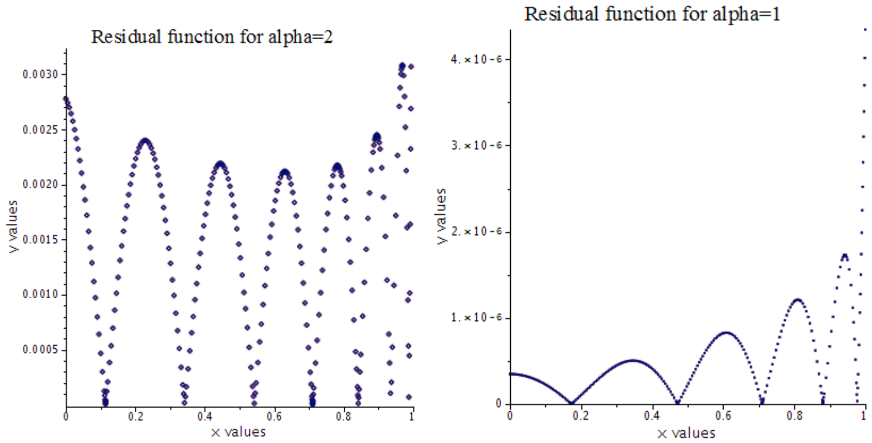

Fig.4. Plots of Residual function for $\alpha = 1$ and $\alpha = 2$ in Example 4.

CONCLUSION

In this work, we present a new method for solving Fredholm-Volterra integral equations based on the Müntz−Legendre polynomials. This method has two benefits: first, Solving the equation turns into an algebraic system by using operational matrices and Galerkin method, which is easier to solve, Secondly, due to normalized polynomials, the calculations will be simpler and faster. These advantages are confirmed by same illustrative examples. Also some example is compared with other method therefore it can be seen that the NMLPs method is better.

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