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A novel study on nonlinear fractional differential equations: general solution

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Abstract

In the present article, the Abel's technique has been developed to finding a general solution of the modified linear first-order ordinary differential equations in the sense of the truncated M-fractional derivative. By using proposed approach, a general solution of two well-recognized nonlinear first-order ordinary differential equations, Bernoulli and Riccati, in agreement with truncated M-fractional derivative have been obtained. For each equation, some examples are presented for satisfactory and efficiency of the proposed method.

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INTRODUCTION

Over the past few decades, researchers' results are showed that fractional calculus is used to achieve more accurate results in studies and applications of differential equations. Although it has only been highlighted since 1974, after the first international conference on fractional accounts, it has been shown that in modeling problems at natural phenomena are more accurate [1-3]. Its development, with helping of the older famous mathematicians such as Leibniz, L'Hopital, Fourier, Laplace, Lagrange, Abel, Euler, Liouville, Riemann, Grunwald, Letnikov, Hadamard, Riesz, Mittag-Leffler, Hardy, Weyl, Feller, Levy, Littlewood and as well as recently Caputo and others contemporary mathematicians was becoming possible. It is possible to define various derivatives and integrals fractional qua each definition has its own strengths and weaknesses and its own properties and thus any of them have a valuable fractional calculus in theory and applications. At present, there are countless and important definitions of types of fractional derivatives, each with its own characteristics and applications [4]. In 2017, Sousa and et al., introduced an M-fractional derivative involving a Mittag-Leffler function with one parameter that also satisfies the properties of integer-order calculus [5,6]. In this sense, Sousa and Oliveira introduced a truncated M-fractional derivative type that unifies four existing fractional derivative types mentioned above and which also satisfied the classical properties of integer-order calculus [7].

Definition 1. Given a function $f:[0,\infty)\to R$, then the truncated M-fractional derivative of f of order α is defined by

$${}_i\mathcal{D}_M^{\alpha,\beta} f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t\ {}_i\mathbb{E}_\beta(\varepsilon t^{-\alpha})\right)-f(t)}{\varepsilon},\ t>0,\ \alpha\in(0,1),\ \beta>0, \qquad (1)$$

where ${}_i\mathbb{E}_\beta(.)$, is the Mittag-Leffler function with one parameter. Note that if f is α-differentiable in some $(0,a), a>0$, and $\lim_{x\to 0^+}{}_iD_M^{\alpha,\beta} f(t)$ exists, then one can define [5-7]

$${}_iD_M^{\alpha,\beta} f(0)=\lim_{x\to 0^+}{}_iD_M^{\alpha,\beta} f(t)$$

$${}_iD_M^{\alpha,\beta} f(0)=\lim_{x\to 0^+}{}_iD_M^{\alpha,\beta} f(t)$$

Definition 2. Given a function $f:[a,\infty)\to R, a\geq 0$, the local M-fractional integral of f order α is defined by

$${}_M\mathcal{T}_a^{\alpha,\beta} f(t)=\Gamma(\beta+1)\int_a^t \frac{f(x)}{x^{1-\alpha}}dx,\quad \alpha\in(0,1),\beta>0, \qquad (2)$$

where the integral is the usual Riemann improper integral. For simplicity we show, ${}_M\mathcal{T}_0^{\alpha,\beta} f(t)={}_M\mathcal{T}^{\alpha,\beta} f(t)$. One of the well results is the following [6,7].

A. (Inverse theorem) ${}_iD_M^{\alpha,\beta}\left({}_M\mathcal{T}_a^{\alpha,\beta} f(t)\right)=f(t)$,

B. (Fundamental theorem of calculus) ${}_M\mathcal{T}_a^{\alpha,\beta}\left({}_iD_M^{\alpha,\beta} f(t)\right)=f(t)-f(0)$.

The study of fractional differential equations has demonstrated very valuable over time. Solving fractional differential equations is very important but there are many fractional differential equations which can't be solved analytically. The authors of this article suggest dear researchers to refer to the articles cited to see some useful methods for solving fractional differential equations [9-22].

ABEL'S TECHNIQUE FOR M-FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, the Abel's technique of the solution of first order linear M-fractional differential equations is pronounced. We well know that by choosing an appropriate dependent variable change, Bernoulli and Riccati M-fractional differential equations will be converted in to a linear first-order M-fractional differential equation. Then a general solution of the M-fractional Bernoulli and Riccati differential equations are presented.

Linear First-order M-fractional differential equations

The most important type of M-fractional derivative differential equations is the linear M-fractional differential equations, in which the M-fractional derivative fractional of highest order is a linear function of the lower M-derivative fractional. Thus, the general linear first-order fractional differential equation in accordance with truncated M-fractional derivative is presented as

$${}_iD_M^{\alpha,\beta} f(t)+g(t)f(t)=h(t),\alpha\in(0,1),\beta>0, \qquad (3)$$

where g(t) and h(t) are α-differentiable functions and f(t) is an unknown function.

With multiplied equation (3) by $e^{\left({}_M\mathcal{T}^{\alpha,\beta} g(t)\right)}$, will be obtained e^((_M^)T^(α,β) g(t)) (〖(_i^)D〗_M^(α,β) f(t)+g(t)f(t)=h(t)) (4)

〖(_i^)D〗_M^(α,β) f(t) e^((_M^)T^(α,β) g(t))+g(t) e^((_M^)T^(α,β) g(t)) f(t)=h(t) e^((_M^)T^(α,β) g(t)). (5)

According to product rule derivative in equation (5), will be gotten

〖(_i^)D〗_M^(α,β) (f(t) e^((_M^)T^(α,β) g(t)))=h(t) e^((_M^)T^(α,β) g(t)), (6)

Now by local M-fractional integration of Equation (6), will be had

f(t) e^((_M^)T^(α,β) g(t))=(_M^)T^(α,β) (h(t).e^((_M^)T^(α,β) g(t)))+C, (7)

wherever C is constant and arbitrary [8,9]. By simplifying the above equation (7), we will get

f(t)=e^(-(_M^)T^(α,β) g(t)) [(_M^)T^(α,β) (h(t).e^((_M^)T^(α,β) g(t)))+C], (8)

as a general solution of the M-fractional differential equation (3).

Example 1. The M-fractional relaxation-oscillation differential equation with initial value [10]

〖(_i^)D〗_M^(0.5,β) f(t)=-f(t), 0≤t≤1, u(0)=1, (9)

according to equation (8), has a particular solution such as

f(t)=exp⁡(-2Γ(β+1) t^0.5).

An exact solution of the M-fractional relaxation-oscillation equation (9) with different values β, are plotted in Fig. 1.

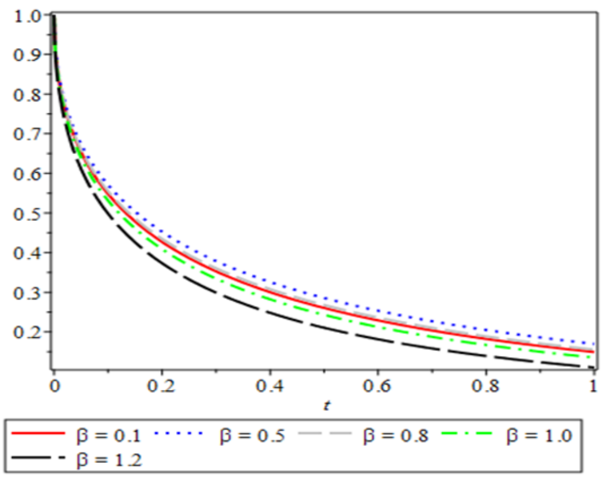

Fig. 1. An analytical solution of relaxation-oscillation equation (9), for β=0.1,0.5,0.8,1.0,1.2.

Example 2. Linear M-fractional relaxation-oscillation differential equation with initial value [10]

〖(_i^)D〗_M^(0.5,β) f(t)=1+f(t), 0≤t, f(0)=0, (10)

according to modify Abel's approach proposed in section (2.1), a particular solution is as the following

f(t)=exp⁡〖(2Γ(β+1)t^0.5)-1〗.

An exact solution of the M-fractional relaxation-oscillation equation (9) with different values β, are plotted in fig.2.

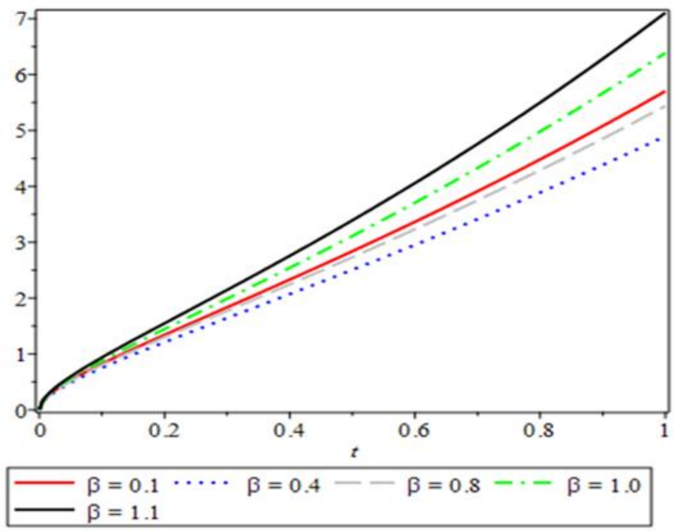

Fig. 2. An analytical solution of relaxation-oscillation equation (10), for β=0.1,0.4,0.8,1,1.1.

M-fractional Bernoulli differential equation

The Bernoulli differential equation is a controversial subject that is widely used in mathematics and the engineering and physical sciences, such as the logistic equation plays an important role in population dynamics, a field that models the evolution of populations of plants, animals or humans over time. A general form of the M-fractional Bernoulli differential equations is as follows,

〖(_i^)D〗_M^(α,β) f(t)+g(t)f(t)=h(t) (f(t))^n,n≠0,1,n∈R,α∈(0,1), β>0, (11)

where g(t) and h(t) are α-differentiable functions and f(t) is an unknown function. By multiplying two sides of the above equation in (f(t))^n, we get

(f(t))^(-n) 〖(_i^)D〗_M^(α,β) f(t)+g(t) (f(t))^(1-n)=h(t). (12)

Using of the change of dependent variable z(t)=(f(t))^(1-n), can be reduced equation (12), to a linear first-order M-fractional differential equation as the following form

$$(1-n)\,{}_{i}D_{M}^{\alpha,\beta} z(t)+g(t)z(t)=h(t). \quad (13)$$

According to the modify Abel's technique in section (2.1) a general solution of equation (13), is as follows,

$$z(t)=e^{-{}_{M}T^{\alpha,\beta}\left((1-n)g(t)\right)}\left[-{}_{M}T^{\alpha,\beta}\left((1-n)h(t).e^{{}_{M}T^{\alpha,\beta}\left((1-n)g(t)\right)}\right)+C\right]. \quad (14)$$

By replacing $z(t)=(f(t))^{1-n}$, in equation (14), a general solution of equation (11), can be expressed as follows form

$$f(t)=\left(e^{-{}_{M}T^{\alpha,\beta}\left((1-n)g(t)\right)}\left[-{}_{M}T^{\alpha,\beta}\left((1-n)h(t).e^{{}_{M}T^{\alpha,\beta}\left((1-n)g(t)\right)}\right)+C\right]\right)^{\frac{1}{1-n}}. \quad (15)$$

that C is constant and arbitrary.

Example 3. A general solution of the M-fractional Bernoulli differential equations,

$${}_{i}D_{M}^{0.5,\beta} f(t)+f(t)=(f(t))^2, \quad (16)$$

in accordance with equation (15), can be presented as the following

$$u(t)=\left(1+Ce^{2\Gamma(\beta+1)\sqrt{t}}\right)^{-1},$$
$$u(t)=\frac{1}{1+CE_1\left(2\Gamma(\beta+1)\sqrt{t}\right)}.$$

In figure 3. an exact solution of the M-fractional Bernoulli equation (14) for C=0.5, with different values β, are planned.

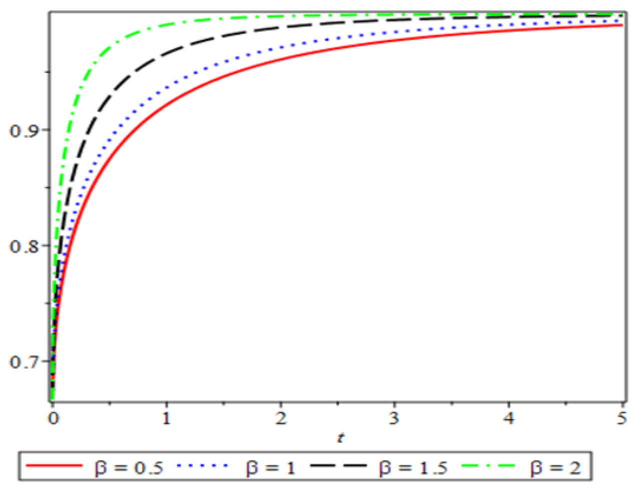

Fig. 3. An analytical solution of Bernoulli equation (16), for β=0.5,1,1.5,2,C=0.5.

Example 4. The M-fractional Bernoulli differential equation

$${}_{i}D_{M}^{0.5,\beta} u(t)+\sqrt{t}\,u(t)=\left(\sqrt{t}\,e^{-2t}\right)(u(t))^{-1}, \quad (17)$$

in accordance with modify Abel's technique for Bernoulli equation proposed in this section, has a general solution as the following form

$$u(t)=\sqrt{\frac{\Gamma(\beta+1)}{\Gamma(\beta+1)-1}e^{-2t}+Ce^{-2\Gamma(\beta+1)t}}, \quad u(t)=\sqrt{\frac{\Gamma(\beta+1)}{\Gamma(\beta+1)-1}E_1(-2t)+CE_1(-2\Gamma(\beta+1)t)},$$

In fig. 4. an exact solution of the M-fractional Bernoulli equation (17) for C=5, with different values β, are planned.

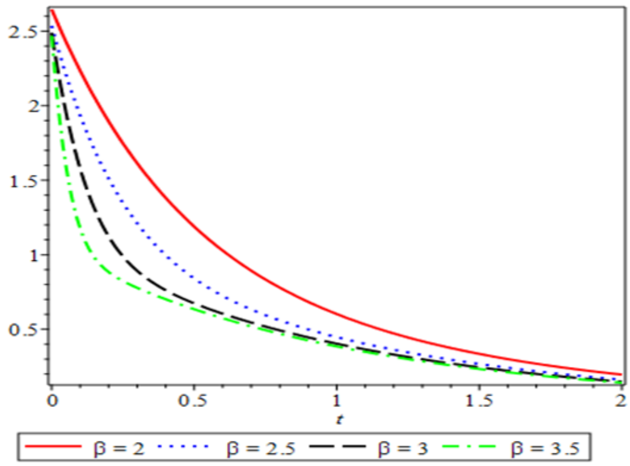

Fig. 4. An analytical solution of Bernoulli equation (17), for β=2,2.5,3,3.5,C=5.

M-fractional Riccati differential equation

The Riccati differential equation, is named in honor to the Italian Nobleman Count Jacopo Francesco Riccati. It is a typical first-order nonlinear ordinary differential equation that plays significant role in mathematics and physics. we well known that a one-dimensional static Schrodinger equation is closely related to the Riccati differential equation [11,12]. A natural extension of a first order M-fractional differential equation is the M-fractional Riccati differential equations,

$${}_{i}D_{M}^{\alpha,\beta} f(t)=g(t)+h(t)\,f(t)+r(t)(f(t))^2, \quad \alpha\in(0,1), \quad \beta>0, \quad (18)$$

where g(t), h(t), and r(t) are α-differentiable functions, and f(t) is an unknown function.

If a particular solution $f_1(t)$ is known, then general solution has the form $f(t)=f_1(t)+u(t)$ where u(t) is a general solution of M-fractional Bernoulli differential equation as follows form

$${}_{i}D_{M}^{\alpha,\beta} u(t)+\left(-h(t)-2r(t)\,f_1(t)\right)u(t)=r(t)(u(t))^2. \quad (19)$$

So that according to the modify Abel's method for Bernoulli equation presented in section (2.2) solution function $u(t)$, of equation (19), can be calculated.

Example 5. Determine a general solution of the M-fractional Riccati differential equations,

$${}_{i}D_M^{0.5,\beta} f(t)=-t\sqrt{t}+\frac{1}{2\Gamma(\beta+1)\sqrt{t}} f(t)+\sqrt{t}\,(f(t))^2, \quad (20)$$

since $f_1(t)=\sqrt{t}$ is an obvious particular solution. To obtain a general solution of Riccati differential equation, suppose $f(t)=f_1(t)+u(t)$, where the function $u(t)$ is denoted by

$$u(t)=\frac{2\sqrt{t}\, e^{\frac{4}{3}\Gamma(\beta+1)t\sqrt{t}}}{C-e^{\frac{4}{3}\Gamma(\beta+1)t\sqrt{t}}}.$$

So, a general solution of M-fractional Riccati equation (20), is as the following

$$f(t)=\sqrt{t}+\frac{2\sqrt{t}\, e^{\frac{4}{3}\Gamma(\beta+1)t\sqrt{t}}}{C-e^{\frac{4}{3}\Gamma(\beta+1)t\sqrt{t}}},$$

$$f(t)=\frac{\sqrt{t}\,[C+E_1(\frac{4}{3}\Gamma(\beta+1)t\sqrt{t})]}{C-E_1(\frac{4}{3}\Gamma(\beta+1)t\sqrt{t})},$$

whereas C is constant and arbitrary. An exact solution of the M-fractional Riccati equation (20) for $C=-1$, with different values β, are shown in figure 5.

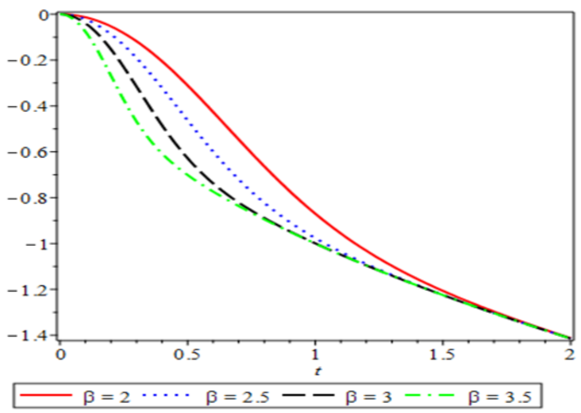

Fig. 5. An analytical solution of Riccati equation (20), for $\beta=0.5,0.8,1,1.2, C=-1$.

Example 6. Find the general solution of the following M-fractional Riccati differential equations,

$${}_{i}D_M^{0.5,\beta} f(t)+2t^2\sqrt{t}=\frac{1}{\Gamma(\beta+1)\sqrt{t}} f(t)+2\sqrt{t}\,(f(t))^2, \quad (21)$$

which has $f_1(t)=t$, as an obvious particular solution.

Suppose $f(t)=f_1(t)+u(t)$, where the function $u(t)$ is denoted by

$$u(t)=\frac{2t e^{2\Gamma(\beta+1)t^2}}{C-e^{2\Gamma(\beta+1)t^2}}.$$

Consequently, a general solution of the M-fractional Riccati equations (21), is as follows form

$$f(t)=t+\frac{2t e^{2\Gamma(\beta+1)t^2}}{C-e^{2\Gamma(\beta+1)t^2}},$$

$$f(t)=\frac{t[C+E_1(2\Gamma(\beta+1)t^2)]}{C-E_1(2\Gamma(\beta+1)t^2)}.$$

An exact solution of the M-fractional Riccati equation (21) for $C=-4$, with different values β, are shown in fig.6.

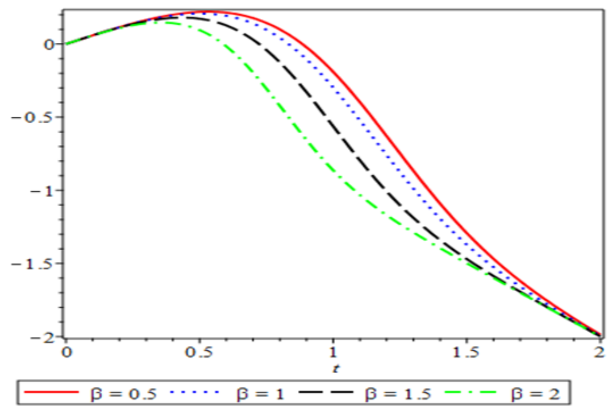

Fig. 6. An analytical solution of Riccati equation (21), for $\beta=0.5,1,1.5,2, C=-4$.

CONCLUSION

In the proposed of present investigation, the Abel's method has been developed for solving linear first-order M-fractional differential equations. By using of the Abel's approach, two well-known nonlinear Bernoulli and Riccati equations in the sense of truncated M-fractional derivative have been solved. The relaxation-oscillation M-fractional differential equations has been resolved regarding the general solution of linear first-order M-fractional differential equations. This method leads to the exact solution, thus some illustrative examples have been presented, to corroborate the satisfactory implementation of the proposed technique in solving the local M-fractional differential equations.

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