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Application of Adomian Decomposition Method and Variational Iteration Method to Dynamical System Problems

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Abstract

Adomian decomposition method and He's variational Iteration method are applied to nonlinear oscillator problems that involve conservative type of oscillators. The methods proved to be effective for the general and specific cases due to their algorithms that admit nonlinear terms in the problems. The two methods are tested on some specific problems in the literature, and the results obtained compared favourably with those obtained via the use of Energy balance method.

Keywords:
Nonlinear oscillator
Dynamic system
Energy balance method
Adomian decomposition method
Variational iteration method

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INTRODUCTION

Nonlinear oscillator problems originated from dynamical systems that involve both conservative and non-conservative oscillations. Problems of this nature prominent in engineering and physics. These problems are modelled into nonlinear second order initial value problems that are mostly difficult or impossible to handle using any of the existing analytical methods. Methods of solutions abound in the literature especially after the introduction of energy balanced method by (He, 1999). Other methods proposed include: analytical approximation technique (Wu, Sun & Lim, 2006), rational harmonic balance method (Belendez, Gimeno, Belendez & Hernandaz, 2009), high-order energy balanced method (Durmaz & Kaya, 2012), nonlinear oscillator with discontinuity was equally solved using He's energy balance method (Zhang, Xu, & Chang, 2009), just to mention a few.

More accurate approximate analytic solution of pendulum with rotating support was proposed by (Molla & Alam, 2017). Energy balance method that admits both collocation and Ritz-Galerkin methods was implemented in (Hermann & Saravi, 2006) to solve nonlinear oscillator problems and the results they obtained for different order of approximations are presented. In the present work, Adomian decomposition method and variational iteration methods are applied to the same family of problems, and the results obtained are in consonance with those obtained in literature. The limitation in applications of energy balance method (EBM) calls for more general approach, thus the need for the present work. The said limitation is that EBM cannot be applied whenever there is a product of the independent and dependent variables in the nonlinear model equation (Hermann & Saravi, 2016).

NONLINEAR OSCILLATOR PROBLEM

The motion of a general nonlinear oscillator is modelled by the second order nonlinear initial value problem.

$$\ddot{x}+\alpha x^k=0, \qquad x(0)=A, \quad \dot{x}=0 \qquad (1)$$

where α is a non-zero constant and k is a positive integer. In addition, x is the displacement, $\dot{x}$ is the velocity, while $\ddot{x}$ is the acceleration of the oscillator.

APPLICATION OF ADOMIAN DECOMPOSITION METHOD

Before discussing the application of ADM to IVP in (1), a brief review of the method is presented as follows.

Consider the general non-linear problem

$$Lx(t)+N(x(t))+Rx(t)=g(t), \qquad (2)$$

where L represents the highest order linear differential operator, *N(x)* is the non-linear term, *Rx* is the remaining linear term and *g(t)* represents the inhomogeneous term.

Comparing (1) and (2), we have

$$Lx= d^2x/dt^2, \quad N(x)=\alpha x^k, \quad R(x)=0 \quad \text{and } g(t)=0 \qquad (3)$$

The solution of (3) is given by

$$x(t) = x_0(t) - \alpha \int_0^t \int_0^T N(x(\xi))d\xi d\tau \qquad (4)$$

The initial approximation x_0 (t) is derived through

$$x_0(t)= \psi_0(t) + h(t) \qquad (5)$$

ψ_0 (t) is derived generally from the Taylor's series

$$\psi_0(t) = A + t\dot{x}(0) + \frac{t^2}{2!}\ddot{x}(0) + \cdots \qquad (6)$$

Thus for problem (1)

$$\psi_0(t)=A+t\dot{x}(0)=A, \qquad (7)$$

while *h(t)* on the other hand is given by

$$h(t) = \int_0^t \int_0^T g(\xi)d\xi d\tau \qquad (8)$$

That is

$$h(t) = \int_0^t \int_0^T 0.\, d\xi d\tau = 0. \qquad (9)$$

Therefore

$$x_0(t)=A.$$

To get the other terms in the ADM, we use the recurrence relation

$$x_n(t) = -\alpha \int_0^t \int_0^T A_{n-1}(\xi) d\xi d\tau \ , \quad n \geq 1. \qquad (11)$$

where A_{n-1} (t) represents the Adomian polynomial corresponding to the non-linearity $N(x)=x^k$. Meanwhile, the Adomian polynomials are generated using the general formula reported by (Adomian, 1994) as

$$A_n(t) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N\left(\sum_{i=0}^{n} \lambda^i x_i \right) \right]_{\lambda=0}. \qquad (12)$$

The final solution is given by

$$x(t) = \sum_{n=0}^{\infty} x_n(t). \qquad (13)$$

APPLICATION OF VARIATION ITERATION METHOD

He's variational iteration method is applied on problem (1) as follows.

The correction functional corresponding to (1) is given as

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(t)(\ddot{x}_n(\tau) + \alpha x_n(\tau)^k) d\tau,$$

$$n \geq 1 \qquad (14)$$

The initial approximation x_0 (t) is obtained through

$$x_0(t)=x(0)+t\dot{x}(0) \qquad (15)$$

$$x_0(t)=A+t(0)=A. \qquad (16)$$

The Lagrange multiplier $\lambda(t)$ is optimally obtained using

$$\lambda(t) = \frac{(-1)^m}{(m-1)!} (\tau - t)^{m-1}, \qquad (17)$$

where m is the order of the differential equation. Therefore, since $m=2$ in this case,

$$\lambda(t)=\lambda\text{-}t. \qquad (18)$$

$$x_{n+1}(t) = x_n(t) + \int_0^t (\tau - t)(\ddot{x}_n(\tau) + \alpha x_n(\tau)^k d\tau,$$

$$n=1,2,\ldots \qquad (19)$$

$$x_1(t) = x_0(t) + \int_0^t (\tau - t)(\ddot{x}_0(\tau) + \alpha x_0(\tau)^k) d\tau \qquad (20)$$

$$x_1(t) = A + \int_0^t (0 + \alpha A^k) d\tau \qquad (21)$$

$$x_1(t) = A + \alpha A^k \int_0^t (\tau - t) d\tau \qquad (22)$$

$$x_1(t) = A + \alpha A^k \left[\frac{\tau^2}{2} - t\tau \right]_0^t \qquad (23)$$

$$x_1(t) = A - \frac{\alpha A^k t^2}{2}. \qquad (24)$$

The iteration continues until desired accuracy is achieved. The final result is therefore given by

$$x(t) = \lim_{n \to \infty} x_n(t).$$

NUMERICAL EXPERIMENTS USING ADM AND VIM

In this section, two tested problems are considered for the implementation of the algorithms that were described in the sequel. As a result of lengthy computations involved in some cases, few terms of the polynomial solutions are listed, while the computations are carried out using all the terms. Mathematica 12 is the tool used in writing the codes.

Problem 1 (Molla and Alam, 2017)

Consider the model equation of a pendulum attached to a rotating support expressed as

$$\ddot{x} + \sin(x)(1 - \Lambda\cos(x)) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \qquad (26)$$

where $\Lambda = \frac{\Omega^2 r}{y}, x = \omega t.$

Without loss of generality, we set $\Lambda=1$. Equation (26) can be expanded up to the seventh terms as follows:

$$\ddot{x} + \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{80} = 0, \ x(0) = A, \ \dot{x}(0) = 0. \qquad (27)$$

Solution (Adomian decomposition method)

$$x(t) = x_0(t) - \frac{1}{2}\int_0^t \int_0^T N_1(\xi) d\xi d\tau + \frac{1}{8}\int_0^t \int_0^T N_2(\xi) d\xi d\tau$$

$$-\frac{1}{80}\int_0^t \int_0^T N_3(\xi) d\xi d\tau, \qquad (28)$$

where $N_1(x)=x^3$, $N_2(x)=x^5$ and $N_3(x)=x^7$ are the non-linearities in the given problem.

The Adomian polynomials corresponding to the non-linearities x^3, x^5 and x^7 are readily available in (Yisa, 2019), and they are as shown below.

For $N_1(x)=x^3$:

$A_{1,0}=x_0^3, A_{1,1}=3x_0^2 x_1, A_{1,2}=3x_0^2 x_2+3x_0 x_1^2, A_{1,3}=3x_0^2 x_3+6x_0 x_1 x_2+x_1^3$, etc.

For $N_2(x)=x^5$:

$A_{2,0}=x_0^5, A_{2,1}=5x_0^4 x_1, A_{2,2}=5x_0^4 x_2+10x_0^3 x_1^2, A_{2,3}=5x_0^2 x_3+20x_0^3 x_1 x_2+10x_0^2 x_1^3$, etc.

For $N_3(x)=x^7$:

$A\_3,0=x_0^7, A_{3,1}=7x_0^6 x_1, A_{3,2}=7x_0^6 x_2+21x_0^5 x_1^2, A_{3,3}=7x_0^6 x_3+42x_0^5 x_1 x_2+35x_0^4 x_1^3$, etc.

$$x_{n+1}(t)=x_n(t)-\frac{1}{2}\int_0^t\int_0^\tau A_{1,n-1}(\xi)d\xi d\tau+\frac{1}{8}\int_0^t\int_0^\tau A_{2,n-1}(\xi)\quad \int_0^t\int_0^\tau A_{3,n-1}(\xi)d\xi d\tau, \qquad (29)$$

The first approximation $x_0(t)$ in (29) is derived as follows.

$$x_0(t)=\psi_0(t)+h(t). \qquad (30)$$

where

$$\psi_0(t)=x(0)+t\dot{x}(0)=A. \qquad (31)$$

and

$$h(t)=\int_0^t\int_0^\tau 0.\,d\xi d\tau. \qquad (32)$$

Therefore,

$$x_0(t)=A. \qquad (33)$$

The other members are derived using (29) as follows:

$$x_1(t)=\frac{1}{2}\int_0^t\int_0^\tau A^3 d\xi d\tau+\frac{1}{8}\int_0^t\int_0^\tau A^5 d\xi d\tau-\frac{1}{80}\int_0^t\int_0^\tau A^7 d\xi d\tau \qquad (34)$$

and (34) gives

$$x_1(t)=-\frac{1}{4}A^3t^2+\frac{A^5t^2}{16}-\frac{A^7t^2}{160}. \qquad (35)$$

Likewise

$$x_2(t)=\frac{A^5t^4}{32}-\frac{A^7t^4}{48}+\frac{3A^9t^4}{512}-\frac{A^{11}t^4}{1280}+\frac{7A^{13}t^4}{153600}, \qquad (36)$$

$$x_3(t)=-\frac{3}{640}A^7t^6+\frac{3A^9t^6}{512}-\frac{1391A^{11}t^6}{460800}+\frac{61A^{13}t^6}{73728}-\frac{809A^{15}t^6}{6144000}+\frac{431A^{17}t^6}{3686400}-\frac{7A^{19}t^6}{14747600} \qquad (37)$$

and so on.

Thus, the solution is a polynomial expression in t which is now obtained by adding (33), (34) and (36).

$$x(t)=A-\left(\frac{A^3}{4}-\frac{A^5}{16}-\frac{A^7}{160}\right)t^2+\left(\frac{A^5}{32}-\frac{A^7}{48}+\frac{3A^9}{512}-\frac{A^{11}}{1280}+\frac{7A^{13}}{153600}\right)t^4-\left(\frac{3A^7}{640}-\frac{3A^9}{512}+\frac{1391A^{11}}{460800}-\cdots\right)t^6 \qquad (38)$$

Solution (Variational Iteration Method)

The IVP is

$$\ddot{x}+\frac{x^3}{2}-\frac{x^5}{8}+\frac{x^7}{80}=0,\quad x(0)=A,\quad \dot{x}(0)=0. \qquad (39)$$

The corresponding correction functional is

$$x_{n+1}(t)=x_n(t)+\int_0^t\lambda(t)\left(\ddot{x}_n(\tau)+\frac{x_n(\tau)^3}{2}-\frac{x_n(\tau)^5}{8}+\frac{x_n(\tau)^7}{80}\right)d\tau, \qquad (40)$$

where $\lambda(t)$ is the Lagrange multiplier.

The initial approximation $x_0(t)$ is obtained just like in the ADM as $x_0(t)=A$.

The Lagrange multiplier on the other hand is derived optimally using the formula stated earlier in (17) as $c=\tau-t$.

Using $x_0(t)$ and $\lambda(t)$ in (40), we have

$$x_1(t) = x_0(t) + \int_0^t (\tau - t)\left(\ddot{x}_0(\tau) + \frac{x_0(\tau)^3}{2} - \frac{x_0(\tau)^5}{8} + \frac{x_0(\tau)^7}{80}\right) d\tau, \quad (41)$$

$$x_1(t) = A + \int_0^t (\tau - t)\left(0 + \frac{A^3}{2} - \frac{A^5}{8} + \frac{A^7}{80}\right) d\tau \quad (42)$$

$$x_1(t) = A - \frac{A^3t^2}{4} + \frac{A^5t^2}{16} - \frac{A^7t^2}{160}. \quad (43)$$

$$x_2(t) = A - \left(\frac{A^3}{4} - \frac{A^5}{16} + \frac{A^7}{160}\right)t^2 + \left(\frac{A^5}{32} - \frac{A^7}{48} + \frac{3A^9}{512} - \frac{A^{11}}{1280} + \frac{7A^{13}}{153600}\right)t^4 - \left(\frac{A^7}{320} - \frac{A^9}{240} + \frac{169A^{11}}{76800} - \frac{31A^{13}}{51200} \ldots\right)t^6 + \cdots \quad (44)$$

The final result is obtained as

$$x(t) = \lim_{n\to\infty} x_n(t). \quad (45)$$

$$x(t) = A - \left(\frac{A^3}{4} - \frac{A^5}{16} + \frac{A^7}{160}\right)t^2 + \left(\frac{A^5}{32} - \frac{A^7}{48} + \frac{3A^9}{512} - \frac{A^{11}}{1280} + \frac{7A^{13}}{153600}\right)t^4 - \left(\frac{A^7}{320} - \frac{A^9}{240} + \frac{169A^{11}}{76800} - \frac{31A^{13}}{51200} \ldots\right)t^6 + \cdots \quad (46)$$

Table 1: Numerical Results Obtained by Fixing the Values of the Constant A=1

t	Molan and Alam (2017)	Present Work (ADM)	Present Work (VIM)
0	1.000000	1.000000	1.000000
1	0.820853	0.820650	0.820682
2	0.410502	0.409427	0.410955
3	-0.046006	-0.092119	-0.050743
4	-0.501056	-0.554804	-0.513993

Problem 2 (Durmaz and Kaya, 2012)

Consider a cubic-quintic Duffing oscillator that is governed by the differential equation 4

$$u'' + \alpha u + \varepsilon u^3 + \lambda u^5 = 0, \quad \text{where } \alpha > 0 \quad (47)$$

with the initial conditions

$$u(0) = A,\ u'(0) = 0. \quad (48)$$

Solution (Adomian Decomposition Method)

$$u_{n+1}(x) = u_o(x) - \alpha \int_0^x \int_0^\tau u_n(t)\,dt\,d\tau - \varepsilon \int_0^x \int_0^\tau N_1(u)\,dt\,d\tau - \lambda \int_0^x \int_0^\tau N_2(u)\,dt\,d\tau \quad (49)$$

Equation (49) can as well be written as

$$u_{n+1}(x) = u_o - \alpha \int_0^x \int_0^\tau u_n(t)\,dt\,d\tau - \varepsilon \int_0^x \int_0^\tau A_{1,n}(t)\,dt\,d\tau - \lambda \int_0^x \int_0^\tau A_{2,n}(t)\,dt\,d\tau \quad (50)$$

Here again,

$$u_0(x) = A. \quad (51)$$

The other members are to be obtained from

$$u_{n+1}(x) = -\alpha \int_0^x \int_0^T u_n(t)\,dt\,d\tau - \varepsilon \int_0^x \int_0^T A_{1,n}(t)\,dt\,d\tau - \lambda \int_0^x \int_0^T A_{2,n}(t)\,dt\,d\tau, \quad n \geq 0 \quad (52)$$

The corresponding Adomian polynomials are as follows.

For the non-linearity $N_1(u) = u^3$:

$A_{1},0=u_0^3, A_{1},1=3u_0^2u_1, A_{1},2=3u_0^2u_2+3u_0u_1^2, A_1,3=3u_0^2u_3+6u_0u_1u_2+u_1^3,$

And for the non-linearity $N_2(u)=u^5$:

$A_2,0=u_0^5, A_2,1=5u_0^4u_1, A_2,2=5u_0^4u_2+10u_0^3u_1^2, A_2,3=5u_0^4u_3+20u_0^3u_1u_2+10u_0^2u_1^3, etc.$

Using $u_0(x)$ and the corresponding Adomian polynomials in (52), we have

$$u_1(x)=-\alpha\int_0^x\int_0^\tau u_0(t)dtd\tau-\varepsilon\int_0^x\int_0^\tau A_{1,0}(t)dtd\tau-\lambda\int_0^x\int_0^\tau A_{2,0}(t)dtd\tau, \quad (53)$$

This yield

$$u_1(x)=-\frac{1}{2}Ax^2\alpha-\frac{1}{2}A^3x^2\varepsilon-\frac{1}{2}A^5x^2\lambda. \quad (54)$$

Also,

$$u_2(x)=\frac{1}{24}Ax^4\alpha^2+\frac{1}{6}A^3x^4\alpha\varepsilon+\frac{1}{8}A^5x^4\varepsilon^2+\frac{1}{4}A^5x^4\alpha\lambda+\frac{1}{3}A^7x^4\varepsilon\lambda+\frac{5}{24}A^9x^4\lambda^2, \quad (55)$$

$$u_2(x)=\frac{Ax^8\alpha^4}{40320}-\frac{A^2x^6\alpha^2\varepsilon}{20160}-\frac{A^4x^6\alpha\varepsilon^2}{60}+\frac{A^5x^8\alpha^2\varepsilon^2}{70}\dots \quad (56)$$

Finally, the solution is obtained through

$$u(x)=u_0(x)+u_1(x)+u_2(x)+u_3(x)+\cdots \quad (57)$$

$$u(x)=A-\left(\frac{A\alpha}{2}+\frac{A^3\varepsilon}{2}+\frac{A^5\lambda}{2}\right)x^2+\left(\frac{A\alpha^2}{24}+\frac{A^3\alpha\varepsilon}{6}+\frac{A^5\alpha\lambda}{4}+\frac{A^7\varepsilon\lambda}{3}+\frac{5A^9\lambda^2}{24}\right)x^6+\cdots \quad (58)$$

Solution (Variational Iteration Method)

From (47), the corresponding correction functional is

$$u_{n+1}(x)=u_n(x)+\int_0^x(\tau-x)(u_n''(\tau)+\alpha u_n(\tau)+\varepsilon u_n(\tau)^3+\lambda u_n(\tau)^5)d\tau,\ n\geq 0,1,2,\dots \quad (59)$$

Then we have

$$u_1(x)=u_0(x)+\int_0^x(\tau-x)(u_0''(\tau)+\alpha u_0(\tau)+\varepsilon u_0(\tau)^3+\lambda u_0(\tau)^5)d\tau \quad (60)$$

$$u_1(x)=u_0(x)+\int_0^x(\tau-x)(0+\alpha A+\varepsilon A^3+\lambda A^5)d\tau \quad (61)$$

Hence

$$u_1(x)=A-\frac{x^2\alpha A}{2}-\frac{x^2\varepsilon A^3}{2}-\frac{x^2\lambda A^5}{2}. \quad (62)$$

The next iteration yields

$$u_2(x)=A-\left(\frac{\alpha A}{2}+\frac{\varepsilon A^3}{2}+\frac{\lambda A^5}{2}\right)x^2+\left(\frac{A\alpha^2}{24}+\frac{A^3\alpha\varepsilon}{6}+\frac{A^5\varepsilon^3}{8}+\frac{A^5\alpha\lambda}{4}+\frac{A^7\varepsilon\lambda}{3}+\frac{5A^9\lambda^2}{24}\right)x^4-\left(\frac{A^3\alpha^2\varepsilon}{40}+\frac{A^5\alpha\varepsilon^2}{20}+\frac{A^7\varepsilon^3}{40}+\cdots\right)x^6+\cdots \quad (63)$$

The final answer is given by

$$u(x)=\lim_{n\to\infty}u_n(x). \quad (64)$$

$$u(x)=A-\left(\frac{\alpha A}{2}+\frac{\varepsilon A^3}{2}+\frac{\lambda A^5}{2}\right)x^2+\left(\frac{A\alpha^2}{24}+\frac{A^3\alpha\varepsilon}{6}+\frac{A^5\varepsilon^3}{8}+\frac{A^5\alpha\lambda}{4}+\frac{A^7\varepsilon\lambda}{3}+\frac{5A^9\lambda^2}{24}\right)x^4-\left(\frac{A^3\alpha^2\varepsilon}{40}+\frac{A^5\alpha\varepsilon^2}{20}+\frac{A^7\varepsilon^3}{40}+\cdots\right)x^6+\cdots \quad (65)$$

Problem 3 (Abdulhafeez, 2016)

Consider the excited Duffing oscillator under damping effect modelled by the IVP:

$$\frac{d^2x}{dt^2}+\eta\frac{dx}{dt}+\omega^2x+\alpha x^3=\Lambda\sin(\Omega t) \quad (66)$$

with the initial conditions

$$x(0)=a \quad and \quad x(0)=b \quad (67)$$

Where x is the position coordinate which is a function of the time t, ω is the system natural frequency, η is a scalar parameter indicating the damping factor in Duffing equation, α is a nonlinear parameter factor, Λ and Ω are the forcing amplitude and frequency, respectively.

For the first set of solutions, we shall take $\Lambda=0$ in (66) so that the equation reduces to

$$\frac{d^2x}{dt^2}+\eta\frac{dx}{dt}+\omega^2x+\alpha x^3=0 \quad (68)$$

Solution (Adomian decomposition method)

Using (67) and (68), the initial approximation

$$x_0(t)=\psi_0(t)+h(t), \quad (69)$$

where $\psi_0(t)=x(0)+t\dot{x}(0)=a+tb$

and $h(t)=\int_0^t\int_0^\varsigma 0.\,d\xi d\varsigma=0$

Hence,

$$x_0(t)=a+bt. \quad (70)$$

recurrence relation through which other members of the series can be obtained is given as

$$x_{n+1}(t)=-\eta\int_0^t\int_0^\varsigma x_n(\xi)d\xi d\varsigma-\omega^2\int_0^t\int_0^\varsigma x_n(\xi)d\xi d\varsigma-\alpha\int_0^t\int_0^\varsigma A_n(\xi)d\xi d\varsigma, \quad n\geq 0 \quad (71)$$

Where $A_n(t)$ represents the Adomian polynomials corresponding to the nonlinearity $N(x)=x^3$.

$$x_1(t)=-\eta\int_0^t\int_0^\varsigma x_0(\xi)d\xi d\varsigma-\omega^2\int_0^t\int_0^\varsigma x_0(\xi)d\xi d\varsigma-\alpha\int_0^t\int_0^\varsigma A_0(\xi)d\xi d\varsigma \quad (72)$$

$$x_1(t)=-\eta\int_0^t\int_0^\varsigma bd\xi d\varsigma-\omega^2\int_0^t\int_0^\varsigma(a+b\xi)\,d\xi d\varsigma-\alpha\int_0^t\int_0^\varsigma(a+b\xi)^3\,d\xi d\varsigma \quad (73)$$

$$x_1(t)=-\left(\frac{a^3\alpha}{2}+\frac{b\eta}{2}+\frac{a\omega^2}{2}\right)t^2-\left(\frac{1}{2}a^2b\alpha+\frac{b\omega^2}{6}\right)t^3-\frac{1}{4}ab^2t^4\alpha-\frac{1}{20}b^3\alpha t^5 \quad (74)$$

The iteration continues and the final solution is given by

$$x(t)=\sum_{n=0}^{\infty}x_n(t) \quad (75)$$

Abdulhafeez (2016) use the following values for the constant to arrive at the polynomial he got. The values used are
a=1.0, b=0.0, ω=1.0, η=0.05 and α=0.15.
Using these values, we get

$$x(t)=1-0.575t^2+0.0095833t^3+0.0693594t^4-0.00138839t^5\ldots \quad (76)$$

which is the similar to the result obtained by the same author despite using modified differential transform method.

Solution (Variational Iteration method)

Solving (68) by VIM, we have the correction functional as

$$x_{n+1}(t)=x_n(t)+\int_0^t(\tau-t)(\ddot{x}_n(\tau)+\eta\dot{x}_n(\tau)+\omega^2x_n(\tau)+\alpha x_n(\tau)^3d\tau, \quad n\geq 0 \quad (77)$$

The initial approximation $x_0(t)$ is obtained as

$$x_0(t)=x(0)+t\dot{x}(0)=a+bt \quad (78)$$

$$x_1(t)=x_0(t)+\int_0^t(\tau-t)(\ddot{x}_0(\tau)+\eta\dot{x}_0(\tau)+\omega^2x_0(\tau)+\alpha x_0(\tau)^3d\tau. \quad (79)$$

$$x_1(t)=a+bt-\left(\frac{a^3\alpha}{2}+\frac{b\eta}{2}+\frac{a\omega^2}{2}\right)t^2-\left(\frac{1}{2}a^2b\alpha+\frac{b\omega^2}{6}\right)t^3-\frac{1}{4}ab^2\alpha t^4-\frac{1}{20}b^3\alpha t^5 \quad (80)$$

$$x(t)=1-0.575t^2+0.0095833t^3+0.694792t^4+\cdots \quad (81)$$

Problem in (66) is again restricted with Λ=0151 as against zero in the previous section. Taking Ω=0.8, (66) becomes

$$\frac{d^2x}{dt^2}+\eta\frac{dx}{dt}+\omega^2x+\alpha x^3=\frac{3}{10}\sin\frac{4}{5}t, \quad (82)$$

with the initial conditions $x(0)=a$ and $\dot{x}(0)=b$.

Solution (ADM)

The initial approximation is obtained as follows:

$$x_0(t)=\psi_0(t)+h(t) \quad (83)$$
$$\psi_0(t)=x(0)+t\dot{x}(0)=a+bt$$

and

$$h(t)=\frac{3}{10}\int_0^t\int_0^\tau \sin\left(\frac{4}{5}\xi\right)d\xi d\tau$$

$$h(t)=\frac{3t}{8}-\frac{15}{32}\sin\left[\frac{4t}{5}\right]$$

Substituting for $\psi_0(t)$ and $h(t)$ in (83) to get

$$x_0(t)=a+bt+\frac{3}{8}t-\frac{15}{32}\sin\left(\frac{4}{5}t\right) \quad (84)$$

The other members of the series are obtained using the recurrence relation

$$x_{n+1}(t) = -\eta \int_0^t \int_0^\tau x_n(\xi) d\xi d\tau - \omega^2 \int_0^t \int_0^\tau x_n(\xi) d\xi d\tau - \alpha \int_0^t \int_0^\tau A_n(\xi) d\xi d\tau \quad (85)$$

where $A_n(t)$ are the same Adomian polynomials used in (71). Thus, we have

DISCUSSION OF RESULTS

Three nonlinear conservative oscillator problems are considered. The three are drawn from the existing literature in order to be able to track the accuracy of the present work. In the case of Problem 1 that was taken from Mulla and Alam (2017), the values of the constants provided by the authors are adopted and the results are as depicted in Table 1. The results showed a reasonable level of agreement in the results. Problem 2 on the other hand could not be tabulated due to the fact that Durmaz and Kaya (2012) only tabulated values of the frequencies ω that were obtained from second order approximation obtained through the application of energy balance method. The values of some constants such as A_1 and A are not explicitly known, thus the solution of *u(x)* could not be vividly determined as presented in their work. Problem 3 has its solutions in the literature as contained in Abdulhafeez (2016) in polynomial form, and the same approach adopted here, using the values of the constant used by the author. The results obtained in the present work match the earlier results term by term.

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