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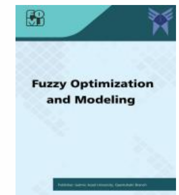

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An Efficient Approach based on Wu's Method for Solving Fully Fuzzy Polynomial Equations System

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A B S T R A C T

This article introduces a productive algebraic approach to identifying positive solutions for a system of fully fuzzy polynomial equations (FFPEs). To achieve this, the FFPEs system is transformed into a comparable system of crisp polynomial equations. The Wu's algorithm is then employed to solve the set of crisp polynomial equations as the solution method. This algorithm results in the solution of characteristic sets that are readily solvable. A key benefit of the proposed method is that all the solutions are obtained simultaneously. The article concludes by presenting some practical examples to demonstrate the efficacy of the proposed method.

1. Introduction

The fully fuzzy polynomial equations (FFPEs) system is one of the subjects in applied mathematics that plays an important role in many applications such as science, engineering, economics and so on [17, 18, 31]. Conventional methods for solving crisp linear systems are generalized to solving the FFPEs system, such as methods based on iterative methods [15], decomposition processes [14], nonlinear programming methods [27], and parametric functions methods [32, 33]. Buckley and Qu in [7, 8, 9] have studied the solution of FFPEs system with two variables and presented the necessary and sufficient conditions for the existence of a fuzzy solution. These methods usually face two major problems. The first problem is that they need to choose the

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useful beginning point. Another is that these methods are not able to find all the answers of the system at the same time. In this paper, we present two efficient algebraic approaches that, in addition to solving the problems mentioned above, seek out the solutions of fully fuzzy polynomial equations system. To achieve this goal, we find the fuzzy solutions of the following system, including s polynomial equation with n unknown:

$$\begin{cases} f_1(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \tilde{b}_1, \\ \vdots \\ f_l(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \tilde{b}_l, \\ \vdots \\ f_s(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \tilde{b}_s, \end{cases} \tag{1}$$

where all coefficients and right hand values $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_s$ and unknowns $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are positive fuzzy numbers (FNs).

Our proposed method is on the base of Wu's algorithm to solve FFPEs systems [35]. The theory of Ritt and some efficient algorithms for zero decomposition of arbitrary systems of polynomials have been considerably improved by Wu Wen-Tsun since 1980 [34, 36]. The method of Ritt-Wu was successfully implemented in many engineering and science problems [37]. In comparison with Gröbner method, this method is more effective for solving real polynomial equations systems (PESs) in some cases e.g. [11, 21, 26]. Using Wu's algorithm for solving PESs results in solving characteristic sets. Because of triangular structure of these sets we can simply find the variety of these sets by a forward substituting. The essential idea of our proposed method is based on transforming the fully fuzzy system (1) into a crisp system and achieving a system of $3s$ polynomial equations such that the solutions of the new system may be obtained by a successful scheme of solving systems. When we use Wu's algorithm, some kinds of the crisp system are found. Therefore, all positive solutions of the original system can be found.

The structure of the paper is organized as follows. Section 2 presents the related work to this proposed method. Section 3 includes some required and necessary definitions and results about FNs and the system of FFPEs. Section 4 has two subsections. Wu's algorithm and varieties are illustrated in the first subsection. To find all solutions of the system of FFPEs, an algorithm is proposed in the second subsection. Some illustrative examples are given in Section 5 to show the efficiency of the algorithms. In the end, the sum up of the paper is given in Section 6.

2. Related work

Fully fuzzy polynomial equations systems (FFPES) have gained significant attention in recent years due to their applicability in various fields, including engineering, economics, and decision-making. These systems involve FNs, which are a generalization of crisp numbers and can represent uncertainty and imprecision [38]. This literature review aims to provide an overview of the main methods and techniques used to solve FFPES, as well as their applications and limitations. FNs were first introduced by Zadeh [38] as a way to represent uncertainty in numerical values. An FN is a convex, normalized fuzzy set on the real line, usually represented by a membership function. The most common type of FNs is triangular FNs (TFNs), which are defined by three parameters: lower limit, upper limit, and modal value [23, 24]. Fuzzy arithmetic operations, such as addition, subtraction, multiplication, and division, are essential for solving FFPES. These operations are performed using the extension principle, which extends the operations from crisp numbers to FNs [16]. Several other methods have been proposed for fuzzy arithmetic, including the α-cut method [22, 23] and the vertex method [10].

2.1 Methods for Solving Fully Fuzzy Polynomial Equations System

2.1.1 Fuzzy Coefficient Method (FCM)

The FCM was first proposed by Abbasbandy and Otadi [1] to solve FFPES. This method involves transforming the fuzzy polynomial equation system into a crisp polynomial equation system by defuzzifying the fuzzy coefficients using a defuzzification method, such as the centroid method or the signed distance method. The crisp system is then solved using traditional numerical methods, and the solutions are fuzzified to obtain the fuzzy solutions. The FCM is a popular method for solving FFPES due to its simplicity and ease of implementation. The main idea behind this method is to transform the fuzzy polynomial equation system into a crisp polynomial equation system by defuzzifying the fuzzy coefficients. Defuzzification is the process of converting an FN into a crisp number, which can be done using various methods, such as the centroid method, the signed distance method, or the mean of maxima method. Once the fuzzy coefficients are defuzzified, the crisp polynomial equation system can be solved using traditional numerical methods, such as the Newton-Raphson method, the bisection method, or the secant method. After obtaining the crisp solutions, they are fuzzified to obtain the fuzzy solutions. Fuzzification is the process of converting a crisp number into an FN, which can be done using various methods, such as the extension principle or the inverse of the defuzzification method used earlier.

2.1.2 α-cut Method

The α-cut method, introduced by Klir and Yuan [22], is another approach to solving FFPES. This method involves converting the fuzzy polynomial equation system into a set of interval polynomial equation systems using α-cuts. Each interval system is then solved using interval arithmetic and numerical methods, and the solutions are combined to obtain the fuzzy solutions. The main idea behind this method is to convert the fuzzy polynomial equation system into a set of interval polynomial equation systems using α-cuts. An α-cut of an FN is an interval that contains all the real numbers whose membership degree in the FN is greater than or equal to α. By varying α from 0 to 1, a family of interval polynomial equation systems is obtained. Each interval system can be solved using interval arithmetic and numerical methods, such as the interval Newton method, the interval bisection method, or the interval Krawczyk method. The solutions of the interval systems are combined to obtain the fuzzy solutions, which can be represented as a union of intervals or as an FN with a membership function.

2.1.3 Homotopy Analysis Method (HAM)

The Homotopy Analysis Method (HAM) is a semi-analytical technique that has been applied to solve FFPES [5]. HAM constructs a homotopy between the original fuzzy polynomial equation system and a simpler auxiliary system, which can be easily solved.

2.1.4 Hybrid Methods

Hybrid methods combine the advantages of the FCM and α-cut methods to solve FFPES more efficiently and accurately. For example, Allahviranloo et al. [3] proposed a hybrid method that uses the FCM to obtain an initial guess for the solutions and then refines the solutions using the α-cut method and Newton's method.

2.2 Applications and Limitations

FFPES have been applied in various fields, such as engineering, economics, and decision-making. For example, fuzzy polynomial equation systems have been used to model and solve problems in structural engineering [25], economic equilibrium [28], and multi-objective optimization [20]. However, there are some limitations to the existing methods for solving FFPES. One limitation is the computational complexity, especially for high-dimensional systems and large α-cut levels. Another limitation is the dependence on the choice of defuzzification method in the FCM, which can affect the accuracy of the solutions. As mentioned earlier, one of the main limitations of the existing methods for solving FFPES is the computational complexity. This is particularly true for high-dimensional systems and large α-cut levels, which can lead to a combinatorial explosion in the number of interval systems that need to be solved. To overcome this limitation, researchers have proposed various techniques, such as adaptive α-cut selection, parallel computing, and approximation methods [4]. Another limitation is the dependence on the choice of defuzzification method in the FCM, which can affect the accuracy of the solutions. Different defuzzification methods may lead to different crisp polynomial equation systems, and hence different solutions. To address this issue, researchers have proposed various techniques, such as sensitivity analysis, robust optimization, and multi-objective optimization [20]. In addition to the applications mentioned earlier, FFPES have been used in various other fields, such as:

- Environmental modeling: Fuzzy polynomial equation systems have been used to model and analyze uncertain environmental processes, such as groundwater flow, pollutant transport, and air quality [6].
- Control systems: Fuzzy polynomial equation systems have been used to design and analyze fuzzy controllers for complex systems, such as robotic manipulators, aircraft, and power systems [30].
- Image processing: Fuzzy polynomial equation systems have been used to develop image processing algorithms that can handle uncertainty and imprecision in pixel values, such as edge detection, segmentation, and enhancement [29].

3. Preliminaries

In this section, some required background and notation of fuzzy set theory and the system of FFPEs are given.

Definition 1. [19] A fuzzy subset $\tilde{u}$ of $\mathbb{R}$ is defined by its membership function

$$\mu_{\tilde{u}}: \mathbb{R} \longrightarrow [0,1],$$

which assigns a real number in the interval $[0,1]$ to each element $x \in \mathbb{R}$ and the value $\mu_{\tilde{u}}(x)$ shows the grade of membership of x in $\tilde{u}$.

Definition 2. [19] An FN $\tilde{u}$ is a fuzzy set like $\mu_{\tilde{u}}: \mathbb{R} \longrightarrow [0,1]$ which satisfies:

1. $\mu_{\tilde{u}}$ is upper semi-continuous,
2. $\mu_{\tilde{u}}$ is normal, i.e., there exist an element t_0 such that $\mu(t_0) = 1$;
3. $\mu_{\tilde{u}}$ is fuzzy convex, i.e., $\mu(\lambda t_1 + (1-\lambda)t_2) \geq \min\{\mu(t_1), \mu(t_2)\}, \forall t_1, t_2 \in \mathbb{R}, \forall \lambda \in [0,1]$;
4. $\text{supp}(\tilde{u})$ is bounded, where $\text{supp}(\tilde{u}) = \text{cl}(\{t \in \mathbb{R}: \mu(t) \geq 0\})$, and cl is the closure operator.

The set of all FNs is denoted by $\mathbb{R}_F$.

Definition 3. [19] An arbitrary FN $\tilde{u}$ in parametric form is denoted by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, for all $r \in [0,1]$, which satisfy the following conditions:

- $\underline{u}(r)$ is a bounded left continuous non decreasing function on [0, 1],
- $\overline{u}(r)$ is a bounded left continuous non increasing function on [0, 1],
- $\underline{u}(r) \leq \overline{u}(r)$.

The crisp number λ is simply represented by $\underline{u}(r) = \overline{u}(r) = \lambda$, for all $r \in [0,1]$. An FN $\tilde{u}$ can be represented by its r-cuts $\tilde{u}_r = \{x \in \mathbb{R}: \mu_u(x) \geq r\}$, for $0 < r \leq 1$ and $\tilde{u}_0 = \overline{\cup_{r\in(0,1]} \tilde{u}_r}$.

It is important to observe that the r-cuts of an FN are intervals that are both closed and bounded. The fuzzy arithmetic, which is based on the Zadeh extension principle [39] can be computed using interval arithmetic [22] applied to the r-cuts. To refer to the r-cut of any arbitrary FN $\tilde{u}$, we use the notation $\tilde{u}_r = [\underline{u}(r), \overline{u}(r)]$.

If we have an FN $\tilde{u} = (\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, we can express that $\tilde{u} > 0$ if $\underline{u}(0) > 0$, and $\tilde{u} < 0$ if $\overline{u}(0) < 0$.

Definition 4. [19] The arithmetic operations for any given FNs $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ are defined as follows:

1. The FNs $\tilde{u}$ and $\tilde{v}$ are equal if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$.
2. The sum of FNs $\tilde{u}$ and $\tilde{v}$, denoted as $\tilde{u} \oplus \tilde{v}$, is as $(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$. Expressed in terms of r-cuts, the sum of FNs $\tilde{u}$ and $\tilde{v}$, denoted as $(\tilde{u} \oplus \tilde{v})_r$, is equal to $[\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$.
3. The difference between FNs $\tilde{u}$ and $\tilde{v}$, denoted as $\tilde{u} - \tilde{v}$, is equal to $(\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$. Expressed in the language of r-cuts, the difference between $\tilde{u}$ and $\tilde{v}$, denoted as $(\tilde{u} - \tilde{v})_r$, can be written as $[\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)]$.
4. If $\tilde{u}, \tilde{v} \geq 0$ then the expression $\tilde{u} \otimes \tilde{v}$ can be rewritten as $(\underline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r))$, where the dot represents a product operation. In the context of r-cuts, $(\tilde{u} \otimes \tilde{v})_r$ is equal to $[\underline{u}(r).\underline{v}(r), \overline{u}(r).\overline{v}(r))]$.

Out of the different kinds of imprecise numbers, the triangular FN is the most commonly used. It can be expressed as a set of three values, as follows:

$$\widetilde{A} = (a_1, a_2, a_3),$$

the given sequence of numbers, $a_1 \leq a_2 \leq a_3$, can be understood as membership functions and satisfies the set of conditions (1) an increasing function on $[a_1, a_2]$, and (2) a decreasing function on $[a_2, a_3]$:

$$\mu_{\tilde{u}}(x) = \begin{cases} 0 & x \leq a_1, \\ \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2} & a_2 \leq x \leq a_3, \\ 0 & x > a_3. \end{cases} \quad (2)$$

Based on Definition 2 and the aforementioned points, it can be inferred that the r-cut $\tilde{u}_r$ of $\tilde{u}$ is a closed interval in $\mathbb{R}$ every r within the range of 0 to 1. This implies that $\tilde{u}_r$ is both compact and convex, making it a subset of $\mathbb{R}$.

The r-cut of the FN (2) can be expressed as:

$$\tilde{u}_r = [\underline{\tilde{u}}_r, \overline{\tilde{u}}_r] = [a_1 + (a_2 - a_1)r, a_3 - (a_3 - a_2)r)].$$

Definition 5. An FN in the shape of a triangle, denoted as $\widetilde{A} = (a_1, a_2, a_3)$, is considered positive if its first value, a_1, is greater than zero. Conversely, $\widetilde{A}$ is as a negative triangular FN if its third value, a_3, is less than zero.

Definition 6. If the values of the three parameters in two triangular FNs, denoted as $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$, are the same, then the two FNs are considered equal. Specifically, the first parameter of $\widetilde{A}$ must be equal to the first parameter of $\widetilde{B}$, the second parameter of $\widetilde{A}$ must be equal to the second parameter of $\widetilde{B}$, and the third parameter of $\widetilde{A}$ must be equal to the third parameter of $\widetilde{B}$.

Definition 7. There are three actions that can be executed on triangular FNs. If we have $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$, then the following operations can be performed.

- $-\widetilde{B} = (-b_3, -b_2, -b_1)$,
- $\widetilde{A} - \widetilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$,
- $\widetilde{A} \oplus \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Suppose that $\widetilde{A}$ and $\widetilde{B}$ are two triangular FNs that are positive. In that case, their triangular FN product, denoted by $\widetilde{A} \otimes \widetilde{B}$, is equal to $(a_1 b_1, a_2 b_2, a_3 b_3)$.

Definition 8. A fuzzy vector is defined as a vector $\tilde{x}$ consisting of FNs, where each element $\tilde{x}_i$ is an FN.

Definition 9. A fuzzy solution of system (1) is defined as a fuzzy vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ that satisfies the equations $f_l(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = \tilde{b}_l$ for $1 \le l \le s$.

Definition 10. In the context of system (1), a positive fuzzy solution is defined as a fuzzy vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ where each component $\tilde{x}_i$ is greater than zero, for $1 \le i \le n$.

It should be noted that the notations introduced in this subsection define how FFPEs are treated in system (1), which is as follows:

$$f_l(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = \sum_{i=1}^{n} c_{li} \otimes \tilde{x}_i \oplus \sum_{i=1}^{n} \sum_{j=1}^{n} c_{lij} \otimes \tilde{x}_i \otimes \tilde{x}_j \oplus \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{lijk} \otimes \tilde{x}_i \otimes \tilde{x}_j \otimes \tilde{x}_k \oplus \ldots = \tilde{b}_l$$

for $1 \le l \le s$.

Assuming that we use triangular FNs to represent all the parameters $c_{li}, c_{lij}, \ldots$ and unknowns x_i, $i = 1, \cdots, x_n$, denoted by $(c'_{li}, {}_{li}, c''_{li})$, $(c'_{lij}, c_{lij}, c''_{lij}), \ldots$ and (x'_i, x_i, x''_i), respectively, we can express system (1) as:

$$\sum_{i=1}^{n} (c'_{li}, c_{li}, c''_{li}) \otimes (x'_i, x_i, x''_i) \oplus \sum_{i=1}^{n} \sum_{j=1}^{n} (c'_{lij}, c_{lij}, c''_{lij}) \otimes (x'_i, x_i, x''_i) \otimes (x'_j, x_j, x''_j) \oplus \ldots = (b'_l, b_l, b''_l).$$

Assuming

$$(c'_{li}, c_{li}, c''_{li}) \otimes (x'_i, x_i, x''_i) = (m'_{li}, m_{li}, m''_{li})$$

and

$$(c'_{lij}, c_{lij}, c''_{lij}) \otimes (x'_i, x_i, x''_i) \otimes (x'_j, x_j, x''_j) = (n'_{lij}, n_{lij}, n''_{lij}).$$

Then for $1 \le l \le s$, we have:

$$\sum_{i=1}^{n} (m'_{li}, m_{li}, m''_{li}) \oplus \sum_{i=1}^{n} \sum_{j=1}^{n} (n'_{lij}, n_{lij}, n''_{lij}) \oplus \ldots = (b'_l, b_l, b''_l).$$

The system represented by equation (1) can now be transformed into a clear and precise system as follows:

$$\begin{cases} \sum_{i=1}^{n} m'_{1i} + \sum_{i=1}^{n}\sum_{j=1}^{n} n'_{1ij} + \ldots = b'_{1}, \\ \sum_{i=1}^{n} m_{1i} + \sum_{i=1}^{n}\sum_{j=1}^{n} n_{1ij} + \ldots = b_{1}, \\ \sum_{i=1}^{n} m''_{1i} + \sum_{i=1}^{n}\sum_{j=1}^{n} n''_{1ij} + \ldots = b''_{1}, \\ \vdots \\ \sum_{i=1}^{n} m'_{li} + \sum_{i=1}^{n}\sum_{j=1}^{n} n'_{lij} + \ldots = b'_{l}, \\ \sum_{i=1}^{n} m_{li} + \sum_{i=1}^{n}\sum_{j=1}^{n} n_{lij} + \ldots = b_{l}, \\ \sum_{i=1}^{n} m''_{li} + \sum_{i=1}^{n}\sum_{j=1}^{n} n''_{lij} + \ldots = b''_{l}, \\ \vdots \\ \sum_{i=1}^{n} m'_{si} + \sum_{i=1}^{n}\sum_{j=1}^{n} n'_{sij} + \ldots = b'_{s}, \\ \sum_{i=1}^{n} m_{si} + \sum_{i=1}^{n}\sum_{j=1}^{n} n_{sij} + \ldots = b_{s}, \\ \sum_{i=1}^{n} m''_{si} + \sum_{i=1}^{n}\sum_{j=1}^{n} n''_{sij} + \ldots = b''_{s}. \end{cases} \tag{3}$$

By deriving a set of 3s polynomial equations, which is commonly referred to as the crisp form of system (1), we can establish a necessary and sufficient condition for the existence of a positive solution to the system. This condition is presented in the following theorem.

Theorem 1. [2] The system of fully fuzzy polynomial equations represented by (1) possesses a positive fuzzy solution if and only if the 0-cut system of the equations has a positive solution.

4. Resolution of FFPEs systems via Wu's method

This section outlines a method for solving a system of FFPEs using Wu's algorithm.

4.1 Wu's Algorithm and Varieties

This subsection begins by introducing characteristic sets, followed by an explanation of Wu's Algorithm and its relationship with varieties. Let $\Gamma = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ with characteristic zero. The variables x_1, ldots, x_n are ordered such that $x_i < x_j$ for $i < j$. If we select the variable x_m, then a polynomial $f \in \Gamma$ can be expressed as a univariate polynomial in x_m of form

$$f = I_t x_m^t + I_{t-1} x_m^{t-1} + \cdots + I_0$$

Here, t represents the degree of f with respect to x_m, denoted by $\deg_{x_m}(f)$, and

$$I_i \in \mathbb{K}[x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n]$$

for $0 \leq i \leq t$.

The leading coefficient I_t of f with respect to x_m is denoted by $lc(f, x_m)$. The class of f is defined as the greatest subscript c of x appearing in f, denoted by class(f). The class of a constant is defined to be zero. The leading variable and initial of f are denoted by lv(f) and ini(f), respectively, where x_c is the leading variable and $lc(f, x_c)$ is the initial of f. A polynomial $g \in \Gamma$ is considered reduced with respect to f if $\deg_{x_c}(g) < \deg_{x_c}(f)$, where $c = class(f) \neq 0$. The polynomial g is reduced with respect to $F \subset \Gamma$ if g is reduced with respect to any $f \in F$. A partial order on polynomials is defined as follows: let $f, g \in \Gamma$. The polynomial g has a higher rank than f and is denoted by $f < g$ if one of the conditions holds (1) $class(f) < class(g)$ and (2) $class(f) = class(g) = c$ and $\deg_{x_c}(f) < \deg_{x_c}(g)$.

If $class(f) = class(g) = c$ and $\deg_{x_c}(f) = \deg_{x_c}(g)$, or both polynomials are constant, then we consider f and g to be equivalent, denoted $f \sim g$. An ordered polynomial set $F = \{f_1, f_2, \ldots, f_r\}$ is a triangular set if either

$r = 1$ or $\mathrm{class}(f_1) < \mathrm{class}(f_2 \cdots < \mathrm{class}(f_r)$. The triangular set F is called an ascending set if f_j is reduced with respect to f_i for $i < j$. The partial order on polynomials is extended to provide a partial order for ascending sets. Let $F = \{f_1, \dots, f_r\}$ and $G = \{g_1, \dots, g_k\}$ be ascending sets. We say $F < G$ if one of the following two cases holds:

1. If $j \leqslant \min\{r, k\}$ such that $f_i \sim g_i$ for $i < j$, but $f_j < g_j$.
2. $f_i \sim g_i$ and $r > k$ for all $i \leqslant k$.

If two ascending sets are incomparable, we write $F \sim G$. When $F < G$, we say that F has a lower rank than G. An ascending set of lowest rank consisting of polynomials from F is called the basic set of F. We now introduce an interesting division for multivariable polynomials known as the pseudo division.

Proposition 1. [13] Given $f, g \in \Gamma$ with $\mathrm{class}(f) = c$, there exists an equation of the form $I_c^m g = qf + r$ where $q, r \in \Gamma$, $I_c = \mathrm{ini}(f)$, $m \geqslant 0$, and either $r = 0$ or r is reduced with respect to f.

The polynomial r mentioned in Proposition 4.1 is referred to as the pseudo remainder of g on pseudo division by f, denoted by $\mathrm{prem}(g, f)$. Given an ascending set $F = \{f_1, \dots, f_r\}$ and $g \in \Gamma$, we can obtain the following remainder formula through successive pseudo divisions:

$$I_1^{s_1} I_2^{s_2} \cdots I_r^{s_r} g = \sum q_i f_i + R \tag{4}$$

Here, $I_i = \mathrm{ini}(f_i)$, $s_i \geqslant 0$, $q_i \in \Gamma$, and R is reduced with respect to F. If we choose each s_i to be as small as possible, then R is unique and denoted by $\mathrm{prem}(g, F)$. For a finite subset G of Γ, we define $\mathrm{prem}(G, F) = \{\mathrm{prem}(g, F) | g \in G\}$. The ideal generated in Γ by F is den by $\langle F \rangle$.

Definition 11. [18] An ascending set B in Γ is considered a characteristic set of a non-empty polynomial set $F \subset \Gamma$ if $B \subset \langle F \rangle$ and $\mathrm{prem}(F, B) = \{0\}$.

Given $F \subset \Gamma$, the set $V(F) = \{(a_1, \dots, a_n) \in \mathbb{K}^n |\ \ f(a_1, \dots, a_n) = 0 \ \ \forall \text{ f in } F\}$ is referred to as the variety defined by F. For a polynomial set $G \subset \Gamma$, we define $V(F/G) = V(F) \setminus V(G)$, which is called a quasi-algebraic variety. The main properties of characteristic sets are summarized in the following theorem.

Theorem 2. [34] (Wu's Well-ordering Principle) Suppose B is a characteristic set of $F \subset \Gamma$. Then we have

$$V(F) = V(B/I_B) \cup \bigcup_{b \in B} V(F \cup B \cup \{\mathrm{ini}(b)\})$$

where $I_B = \prod_{b \in B} \mathrm{ini}(b)$.

The Wu's algorithm is presented based on Wu's Well-ordering Principle Theorem to provide all necessary characteristic sets for computing $V(F)$.

Algorithm 1. [35] (Wu Method)

Input: $F \subset \Gamma$, a non-empty set
Output: Z, a set of characteristic sets such that $V(F) = \bigcup_{B \in Z} V(B/I_B)$, where $I_B = \prod_{b \in B} \mathrm{ini}(b)$.

1. $Z := \emptyset$, $D := \{F\}$
2. While $D \neq \emptyset$ Do
   - 2.1. Pick an element F' from D
   - 2.2. $D := D \setminus \{F'\}$
   - 2.3. Choose a characteristic set B of F'
   - 2.4. If $B \neq \{1\}$ then

2.4.1. $Z := Z \cup \{B\}$
2.4.2. $D := D \bigcup \ \ \cup_{b \in B} \{F' \cup B \cup \{ini(b) | ini(b) \neq 1\}\}$
3. Return Z

Using Wu's algorithm, we can express $V(F)$ as a union of quasi-algebraic varieties of characteristic sets. As a result, we can easily find $V(F)$ since these sets are straightforward to solve.

Example 1. We can apply Wu's algorithm to $F = \{xy + x + y, xy^2 + x + y\}$ with $y < x$. We start with $F' := F$ and $D = \emptyset$. The set $B = \{y^3 - y^2, xy + x + y\}$ is a set, so we set $Z := \{B\}$. We have $ini(xy + x + y) = y + 1$ and $ini(y^2 - y^3) = 1$, so we set $D := \{F' \cup \{y + 1\}\}$. We then set $F' := \{xy + x + y, xy^2 + x + y, y + 1\}$. set $\{1\}$ is a characteristic set of F', and $D = \emptyset$. Therefore, the output is $Z = \{\{y^3 - y^2, xy + x + y\}\}$ and

$$V(F) = V(\{y^3 - y^2, xy + x + y\}) \backslash V(y + 1) = \{(x = 0, y = 0), (x = -\tfrac{1}{2}, y = 1)\}.$$

4.2 Main algorithm for solving a system of FFPEs based on Wu's method

In this subsection, we describe an algorithm for solving a system of FFPEs based on the previous discussions. To obtain the form of characteristic sets of this system, we proceed as follows. Consider the FFPE system (2). Let $F = \{f_1, f_2, \ldots, f_{3s}\}$ be the set of polynomials in its crisp form system, and let

$$\{x'_1, x_1, x''_1, x'_2, x_2, x''_2, \ldots, x'_n, x_n, x''_n\}$$

be the set of variables that appear in f_i's, ordered as $x''_n < x_n < x'_n < \ldots < x''_2 < x_2 < x'_2 < x''_1 < x_1 < x'_1$. Then, every characteristic set of F in the ring $R = \mathbb{R}[x'_1, x_1, x''_1, x'_2, x_2, x''_2, \ldots, x'_n, x_n, x''_n]$ has a structure as follows:

$$\begin{cases} g_{1,1}(x''_n) \in R & \\ g_{2,i}(x_n, x''_n) \in R, & i = 1, \cdots, p_2 \\ g_{3,i}(x'_n, x_n, x''_n) \in R, & i = 1, \cdots, p_3 \\ \vdots & \\ g_{3n-3,i}(x''_1, \ldots, x'_n, x_n, x''_n) \in R, & i = 1, \cdots, p_{3n-3} \\ g_{3n-2,i}(x_1, x''_1, \ldots, x'_n, x_n, x''_n) \in R, & i = 1, \cdots, p_{3n-2} \\ g_{3n-1,i}(x'_1, x_1, x''_1, \ldots, x'_n, x_n, x''_n) \in R, & i = 1, \cdots, p_{3n-1}. \end{cases} \quad (5)$$

Using the above discussions, we can present the following algorithm for finding positive solutions to a system of FFPEs:

Algorithm 2. *(Main Algorithm)*

Input: The system of FFPEs $\mathbb{F}$

Output: The set of positive solutions, i.e., S for $\mathbb{F}$

1. Compute the parametric form of $\mathbb{F}$
2. Compute the 0-cut system, i.e., $\mathbb{F}'$
3. $S' := WuMethod(\mathbb{F}')$
4. If $\mathbb{F}'$ has positive solution then go to 5 else go to 7
5. Compute the crisp form of system, i.e., $\mathbb{F}''$
6. $S := WuMethod(\mathbb{F}'')$
7. System $\mathbb{F}$ does not has any positive solution
8. End

5. Numerical examples

In this section, we provide several numerical examples to demonstrate the effectiveness of our method.

Example 2. Consider the following system of FFPEs:

$$\begin{cases} \tilde{x}_1 \oplus \tilde{x}_1 \otimes \tilde{x}_2 = (3,10,18), \\ (2,3,4) \otimes \tilde{x}_1 \oplus \tilde{x}_1^2 \oplus (1,2,4) \otimes \tilde{x}_2 = (5,18,41). \end{cases}$$

We will use our algorithm to solve this system. The parametric form of the system is as follows:

$$\begin{cases} [\underline{x_1}(r), \overline{x_1}(r)] \oplus [\underline{x_1}(r), \overline{x_1}(r)] \otimes [\underline{x_2}(r), \overline{x_2}(r)] = [3+7r, 18-8r], \\ [2+r, 4-r] \otimes [\underline{x_2}(r), \overline{x_2}(r)] \oplus [\underline{x_1}(r)^2, \overline{x_1}(r)^2] \oplus [1+r, 4-2r] \otimes [\underline{x_2}(r), \overline{x_2}(r)] = [5+13r, 41-23r]. \end{cases}$$

It's 0-cut system is as follows:

$$F': \begin{cases} \underline{x_1}(0) + \underline{x_1}(0)\ \ \underline{x_2}(0) = 3, \\ \overline{x_1}(0) + \overline{x_1}(0)\ \ \overline{x_2}(0) = 18, \\ 2\underline{x_1}(0) + \underline{x_1}(0)^2 + \underline{x_2}(0) = 5, \\ 4\overline{x_1}(0) + \overline{x_1}(0)^2 + 4\overline{x_2}(0) = 41. \end{cases}$$

Using Wu's algorithm, the set of characteristic sets for F' is

$$Z = \{z_1 = \{-6\underline{x_1}(0) + 3 + 2\underline{x_1}(0)^2 + \underline{x_1}(0)^3, -45\overline{x_1}(0) + 72 + 4\overline{x_1}(0)^2 + \overline{x_1}(0)^3, 2\underline{x_1}(0) + \underline{x_1}(0)^2 + \underline{x_2}(0) - 5, 4\overline{x_1}(0) + \overline{x_1}(0)^2 + 4\overline{x_2}(0) - 41\}\}.$$

By Wu's Well-ordering Principle Theorem, we have

$$V(F') = (V(z_1)\backslash V(a)) = (V(z_1)\backslash V(4)).$$

Therefore,

$$V(F') = \{\underline{x_1}(0) = 1, \overline{x_1}(0) = 3, \underline{x_2}(0) = 2, \overline{x_2}(0) = 5\}.$$

Therefore, the solution to the 0-cut system is as follows:

$$\tilde{x}_1{}^0 = [1,3], \tilde{x}_2{}^0 = [2,5].$$

We observe that the fully fuzzy polynomial equation's 0-cut system has a positive solution. Thus, by Theorem 1, the original system also has a positive solution. Let $\tilde{x}_1 = (x'_1, x_1, x''_1)$ and $\tilde{x}_2 = (x'_2, x_2, x''_2)$. The given system of FFPEs can be expressed as:

$$\begin{cases} (x'_1, x_1, x''_1) \oplus (x'_1, x_1, x''_1) \otimes (x'_2, x_2, x''_2) = (3,10,18), \\ (2,3,4) \otimes (x'_1, x_1, x''_1) \oplus (x'_1, x_1, x''_1)^2 \oplus (1,2,4) \otimes (x'_2, x_2, x''_2) = (5,18,41), \end{cases}$$

where (x'_1, x_1, x''_1) and (x'_2, x_2, x''_2) are positive triangular FNs and this system is equivalent to the following:

$$\begin{cases}(x'_1 + x'_1x'_2, x_1 + x_1x_2, x''_1 + x''_1x''_2) = (3,10,18),\\ (2x'_1 + x'^2_1 + x'_2, 3x_1 + x_1^2 + 2x_2, 4x''_1 + x''^2_1 + 4x''_2) = (5,18,41).\end{cases}$$

By employing the method described in Section 4, it is possible to transform the aforementioned system of fuzzy functional differential equations (FFPEs) into a clear-cut system.

$$F'': \begin{cases}x'_1 + x'_1x'_2 = 3,\\ x_1 + x_1x_2 = 10,\\ x''_1 + x''_1x''_2 = 18,\\ 2x'_1 + x'^2_1 + x'_2 = 5,\\ 3x_1 + x_1^2 + 2x_2 = 18,\\ 4x''_1 + x''^2_1 + 4x''_2 = 41.\end{cases}$$

Using Wu's algorithm, the set of characteristic sets for F'' is

$$Z = \{z_1 = \{-6x'_1 + 3 + 2{x'_1}^2 + {x'_1}^3, -20x_1 + 20 + 3{x_1}^2 + {x_1}^3, -45x''_1 + 72 + 4{x''_1}^2 + {x''_1}^3, 2x'_1 + {x'_1}^2 + x'_2 - 5, 3x_1 + {x_1}^2 + 2x_2 - 18, 4x''_1 + {x''_1}^2 + 4x''_2 - 41\}\}.$$

By Wu's Well-ordering Principle Theorem, we have

$$V(F'') = (V(z_1)\backslash V(a)) = (V(z_1)\backslash V(8)).$$

Therefore,

$$V(F'') = \{x'_1 = 1, x_1 = 2, x''_1 = 3, x'_2 = 2, x_2 = 4, x''_2 = 5\}.$$

Thus, the original system has the following solution

$$\tilde{x}_1 = (x'_1, x_1, x''_1) = (1,2,3), \tilde{x}_2 = (x'_2, x_2, x''_2) = (2,4,5).$$

Example 3. Consider the following system of FFPEs:

$$\begin{cases}(1,2,3) \otimes \tilde{x}_1 \oplus \tilde{x}_2 = (4,5,7),\\ \tilde{x}_1 \oplus (2,3,4) \otimes \tilde{x}_2 = (12,25,39).\end{cases}$$

The parametric form of the above system can be presented in the following form:

$$\begin{cases}[1 + r, 3 - r] \otimes [\underline{x_1}(r), \overline{x_1}(r)] \oplus [\underline{x_2}(r), \overline{x_2}(r)] = [4 + r, 7 - 2r],\\ [\underline{x_1}(r), \overline{x_1}(r)] \oplus [2 + r, 4 - r] \otimes [\underline{x_2}(r), \overline{x_2}(r)] = [12 + 13r, 39 - 14r].\end{cases}$$

0-cut system of the above system is:

$$\begin{cases} \underline{x_1}(0) + \underline{x_2}(0) = 4, \\ 3\overline{x_1}(0) + \overline{x_2}(0) = 7, \\ \underline{x_1}(0) + 2\underline{x_2}(0) = 12, \\ \overline{x_1}(0) + 4\overline{x_2}(0) = 39. \end{cases}$$

By applying Wu's method, we can derive the solution for the 0-cut system as follows: ${\tilde{x}_1}^0 = [-4, -1]$ and ${\tilde{x}_2}^0 = [8,10]$. However, since ${\tilde{x}_1}^0$ is an interval that does not contain any positive values, it follows that the system does not have a positive solution. This conclusion is supported by Theorem 1, which confirms that the original system also lacks a positive solution.

Example 4. Consider the following system of FFPEs:

$$\begin{cases} \tilde{x}_1^3 \oplus \tilde{x}_2^2 = (2,3,5), \\ (1,2,4) \otimes \tilde{x}_1^3 \oplus \tilde{x}_2^2 = (3,4,7). \end{cases}$$

The parametric form of the above system is as follows:

$$\begin{cases} [\underline{x_1}(r)^3, \overline{x_1}(r)^3] \oplus [\underline{x_2}(r)^2, \overline{x_2}(r)^2] = [2 + r, 5 - 2r], \\ [1 + r, 4 - 2r] \otimes [\underline{x_1}(r)^3, \overline{x_1}(r)^3] \oplus [\underline{x_2}(r)^2, \overline{x_2}(r)^2] = [3 + r, 7 - 3r]. \end{cases}$$

It's 0-cut system is as follows:

$$\begin{cases} \underline{x_1}(0)^3 + \underline{x_2}(0)^2 = 2, \\ \overline{x_1}(0)^3 + \overline{x_2}(0)^2 = 5, \\ \underline{x_1}(0)^3 + \underline{x_2}(0)^2 = 3, \\ 4\overline{x_1}(0)^3 + \overline{x_2}(0)^2 = 7. \end{cases}$$

Wu's algorithm produces an output of $Z = \emptyset$ for the 0-cut system, indicating that the system has no solution.

Example 5. A shipping company transports two types of shipping packs, with volumes of approximately 2 and 4 m^3 per weight unit. The first type occupies volumes of approximately (1.75,2,2.25) for each weight unit, while the second type occupies volumes of approximately (3.25,4,4.25). The values of each weight unit of the first type of pack are about 19 dollars (18,19,20), while the values of each weight unit of the second type of pack are about 12 dollars (11,12,12.5). The total cost of the packs is approximately 1540 dollars (1359,1540,1677.5), and they occupy a total volume of about 340 m^3 (269.5,340,379.25). To determine the weight of each type of pack, we can set up a system of linear equations using FNs $\tilde{x}_1$ and $\tilde{x}_2$ to represent the weight of each unit of the first and second type of pack, respectively. These equations form a system of Fuzzy Functional Polynomial Equations (FFPEs).

$$\begin{cases}(1.75,2,2.25)\otimes\tilde{x}_1\oplus(3.25,4,4.25)\otimes\tilde{x}_2=(269.5,340,379.25),\\(18,19,20)\otimes\tilde{x}_1\oplus(11,12,12.5)\otimes\tilde{x}_2=(1359,1540,1677.5).\end{cases}$$

The system described above can be expressed in parametric form as follows:

$$\begin{cases}[1.75+0.25r,2.25-0.25r]\otimes\left[\underline{x_1}(r),\overline{x_1}(r)\right]\oplus[3.25+0.75r,4.25-0.25r]\otimes\left[\underline{x_2}(r),\overline{x_2}(r)\right]=[269.5+70.5r,379.25-39.25r],\\[18+r,20-r]\otimes[\underline{x_1}(r),\overline{x_1}(r)]\oplus[11+r,12-0.5r]\otimes[\underline{x_2}(r),\overline{x_2}(r)]=[1359+181r,1677.5-137.5r].\end{cases}$$

0-cut system of the above system is:

$$F':\begin{cases}1.75\underline{x_1}(0)+3.25\underline{x_2}(0)=269.5,\\2.25\overline{x_1}(0)+4.25\overline{x_2}(0)=379.25,\\18\underline{x_1}(0)+11\underline{x_2}(0)=1359,\\20\overline{x_1}(0)+12.5\overline{x_2}(0)=1677.5.\end{cases}$$

Using Wu’s algorithm, the set of characteristic sets for the above system is

$$Z=\{z_1=\{\underline{x_1}(0)-37,\overline{x_1}(0)-42,\underline{x_2}(0)-63,\overline{x_2}(0)-67\}\}.$$

By Wu’s Well-ordering Principle Theorem, we have

$$V(F')=(V(z_1)\backslash V(a))=(V(z_1)\backslash V(1)).$$

Therefore,

$$V(F')=\{\underline{x_1}(0)=37,\overline{x_1}(0)=42,\underline{x_2}(0)=63,\overline{x_2}(0)=67\}.$$

Therefore, we can find the solution to the 0-cut system as follows:

$$\tilde{x}_1{}^0=[37,42],\tilde{x}_2{}^0 63,57].$$

We observe that the 0-cut system of the fully fuzzy polynomial equation above has a positive solution. Therefore, according to Theorem 1, the original system also has a positive solution.

Let $\tilde{x}_1=(x'_1,x_1,x''_1)$ and $\tilde{x}_2=(x'_2,x_2,x''_2)$. Then given system of FFPEs may be written as:

$$\begin{cases}(1.75,2,2.25)\otimes(x'_1,x_1,x''_1)\oplus(3.25,4,4.25)\otimes(x'_2,x_2,x''_2)=(269.5,340,379.25),\\(18,19,20)\otimes(x'_1,x_1,x''_1)\oplus(11,12,12.5)\otimes(x'_2,x_2,x''_2)=(1359,1540,1677.5),\end{cases}$$

Here, (x'_1,x_1,x''_1) and (x'_2,x_2,x''_2) are positive triangular FNs and this system is equal to the following system:

$$\begin{cases}(1.75x'_1 + 3.25x'_2, 2x_1 + 4x_2, 2.25x''_1 + 4.25x''_2) = (269.5,340,379.25),\\ (18x'_1 + 11x'_2, 19x_1 + 12x_2, 20x''_1 + 12.5x''_2) = (1359,1540,1677.5).\end{cases}$$

By applying the method discussed in Section 4, we can convert the above system of Fuzzy Functional Polynomial Equations (FFPEs) into the following crisp system:

$$F'': \begin{cases}1.75x'_1 + 3.25x'_2 = 269.5,\\ 2x_1 + 4x_2 = 340,\\ 2.25x''_1 + 4.25x''_2 = 379.25,\\ 18x'_1 + 11x'_2 = 1359,\\ 19x_1 + 12x_2 = 1540,\\ 20x''_1 + 12.5x''_2 = 1677.5.\end{cases}$$

Applying Wu's algorithm, we obtain the set of characteristic sets for F'' as follows:

$$Z = \{z_1 = \{767x'_1 - 27686, x_1 - 40, 709x'' - 29258, 767x'_2 - 49455, x_2 - 65, 2836x''_2 - 194725\}\}.$$

By Wu's Well-ordering Principle Theorem, we have

$$V(F'') = (V(z_1)\backslash V(a)) = (V(z_1)\backslash V(1182886811236)).$$

Therefore,

$$V(F'') = \{x'_1 = 36.09647979, x_1 = 40, x''_1 = 41.30465444, x'_2 = 64.47848761, x_2 = 65, x''_2 = 68.66184767\}.$$

Thus, the original system has the following solution

$$\tilde{x}_1 = (x'_1, x_1, x''_1) = (36.09647979,40,41.30465444),$$
$$\tilde{x}_2 = (x'_2, x_2, x''_2) = (64.47848761,65,68.66184767).$$

6. Conclusion

Since there is no additive inverse for an arbitrary FN, finding solutions to systems of Fuzzy Functional Polynomial Equations (FFPEs) is a significant challenge. This paper presents a novel approach based on Wu's algorithm for obtaining all positive fuzzy solutions to systems of FFPEs. This algorithm allows us to solve triangular systems, which are relatively easy to solve. The proposed method is not dependent on a suitable starting point, and all solutions can be obtained simultaneously. Numerical results demonstrate the effectiveness of the proposed algorithm in obtaining all positive solutions to systems of FFPEs. The proposed method can be applied to a system in a family of polynomial systems. Future research in this area could focus on developing more efficient and accurate methods for solving FFPEs, as well as exploring new applications in various fields.

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