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Sensitivity Analysis Algorithm to Measure Fuzzy Efficiency Security Margin of DMUs: A New FDEA Approach

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A B S T R A C T

The calculated efficiencies of Decision Making Units (DMUs) in Data Envelopment Analysis (DEA) are relative, prompting each DMU to strive for performance improvement to avoid falling behind in efficiency compared to other DMUs. Generally, DMU performance can be assessed from two perspectives: optimistic and pessimistic. A segment of DEA involves exploring the sensitivity of the set of efficient DMUs to variations in input and output values. In real-world scenarios, DEA encounters challenges when dealing with fuzzy and interval inputs and/or outputs. This paper focuses on a crucial aspect of sensitivity analysis and introduces an algorithm utilizing classic fuzzy DEA models. This algorithm is designed to determine the relative efficiency security margin of DMUs with fuzzy and interval inputs and outputs, along with their simultaneous perturbation. Beyond the optimistic frontier, we also consider the pessimistic frontier for the observed DMUs, terming it the inefficiency improvement margin. This information enables company managers to pinpoint their closest threats and enhance performance to maintain their position in rankings. Numerical examples are provided for illustration purposes.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric method in operations research for the estimation of production frontiers, was first put forward by Charnes, Cooper and Rhodes [7]. Also it is a linear programming based technique for measuring the relative performance of organizational units where the presence of multiple inputs and outputs makes comparison difficult.

Data envelopment analysis originated as a nonparametric approach in operations research, pioneered by Charnes, Cooper, and Rhodes [7], aimed at estimating production frontiers. Additionally, it serves as a linear programming-based technique for evaluating the relative performance of organizational units, particularly when multiple inputs and outputs complicate comparisons.

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DEA proves beneficial for assessing the performance of units with numerous inputs and outputs, with the objective of identifying efficient and inefficient Decision Making Units (DMUs). Standard DEA models operate under the assumption of complete data precision, which may not always hold true. In real-world scenarios, some inputs and outputs may exist in the form of interval or random data. Introducing imprecise data to the standard linear CCR model transforms the DEA model into a nonlinear one, known as imprecise DEA.

Imprecise data introduces uncertainties, highlighting that observed values cannot be considered as definitively unique. Fuzzy set theory emerges as a valuable tool for describing situations involving imprecise or vague data. Fuzzy sets navigate such uncertainties by assigning a degree to which a certain object belongs to a set.

In addition to uncertainties, interval data, is defined as a data type measured along a scale, where each point is equidistant from one another. This type of data is represented by numerical values, ensuring standardized and equal distances between points. An illustrative example of interval data is temperature, where the difference between 80 degrees Fahrenheit and 70 degrees Fahrenheit adheres to a standardized and equal scale.

In recent years, DEA researchers have explored the sensitivity of efficiency and inefficiency to data perturbations [5–6-11]. Traditional data envelopment analysis models assume exact values on a ratio scale and don't handle imprecise data. In real-world situations, especially with unknown decision variables like interval, ordinal, or ratio-bounded data, the DEA model is termed imprecise DEA (IDEA) [11].

Imprecise data refers to information known within specified bounds or satisfying certain ordinal relations. This paper introduces a sensitivity analysis algorithm for determining the efficiency security margin of decision-making units with interval and fuzzy data, covering two efficiency and inefficiency frontiers. The algorithm is extended to rank sets of interval and fuzzy data.

Efficiency, a relative measure, varies across different ranges in DEA. Traditional DEA evaluates best relative efficiencies within the range of less than or equal to one, establishing an efficiency frontier. Conversely, worst relative efficiencies within the range of greater than or equal to one indicate the poorest performances, establishing an inefficiency frontier. The original DEA model optimistically evaluates each DMU with desirable weights, resulting in optimistic efficiencies, while the pessimistic view yields pessimistic efficiencies using undesirable weights.

Cooper et al. [10] addressed imprecise data in DEA, emphasizing the need for upper and lower bounds when efficiency scores are not constant. Jahanshahloo et al. [17] proposed the interval DEA model, obtaining efficiency intervals from optimistic and pessimistic viewpoints. To enhance the lower bound, different ideal points are defined, leading to a ranked model extended to interval data. Azizy [3] suggests considering both optimistic and pessimistic efficiencies simultaneously, integrating them into an interval for a comprehensive view. Jahanshahloo et al. [16] find the stability radius for decision-making units with interval data, and Haghighat and Khorram [12] focus on identifying efficient units under different data settings. Yannis et al. [38] introduce an interval DEA approach for units with missing values, providing upper and lower bounds for efficiency scores. The efficiency analysis is extended by estimating new values for initial interval bounds, potentially improving a unit's efficiency. Jahed et al. [18] addressed conflicting outcomes from optimistic and pessimistic views in DEA evaluations, proposing a new measure for overall performance to reconcile biases. They also introduced fuzzy DEA models for pessimistic evaluations in a fuzzy context. Wang and Yang [36] developed a bounded DEA model, measuring DMU efficiencies within an interval by introducing an anti-ideal DMU with a known lower bound. This model incorporates preference information on input and output weights.
Esmaeili [11] introduced an approach using the Enhanced Russell Measure (ERM) to handle interval data in DEA. Azizy and Wang [5] identified limitations in bounded DEA models and proposed improvements, particularly addressing zero values for outputs. He et al. [14] aimed to enhance interval efficiencies for inefficient DMUs by using ideal points to determine potential improvements. Specific programs were established to adjust inputs and outputs to approach target values, ensuring the final adjusted data remain in interval format. Azizy and Jahed [4] proposed improved interval DEA models allowing DEA analysis using concepts of best and worst relative efficiencies.

Wang et al. [37] propose two fuzzy DEA models using fuzzy arithmetic to handle input and output data fuzziness. The models, formulated as linear programming, determine fuzzy efficiencies for decision-making units (DMUs). An analytical fuzzy ranking approach is developed to compare and rank DMUs based on fuzzy efficiencies. Chen et al. [8] discuss Fuzzy-DEA, emphasizing its necessity when dealing with fuzzy input or output variables. Their research focuses on applying the Fuzzy Slack-Based Measurement (Fuzzy SBM) model under risk uncertainty. Fuzzy SBM provides efficiency scores within a confidence range, aligning with risk anticipation, and evaluates Taiwan banking management achievement under market risk. Mugera [28] integrates fuzzy set theory into DEA to compute technical efficiency scores with imprecise input and output data. The approach measures efficiency within specified intervals, presenting fuzzy efficiency scores as interval bounds, offering decision-makers insights at different possibility levels. Barak and Dahooei [6] propose a hybrid method using fuzzy DEA and fuzzy multi-attribute decision-making (F-MADM) to rank airline safety. Fuzzy DEA calculates criteria weights, and MADM methods (Fuzzy SAW, Fuzzy TOPSIS, Fuzzy VIKOR, ARAS-F, COPRAS-F, and Fuzzy MULTIMOORA) are employed to assess and rank airlines based on obtained weights.

Zhou and Xu [40] conduct a comprehensive investigation into Fuzzy Data Envelopment Analysis (FDEA). Their paper covers a literature retrospective analysis, method review of basic and extended FDEAs, and an application review in real-life situations. The results offer insights for further research on FDEA theory and practical applications. Liu and Lee [21] propose a novel method for calculating fuzzy cross-efficiency that considers all possible weights of Decision Making Units (DMUs) simultaneously, eliminating the need for weight selection. Utilizing the α-level-based approach, they formulate the fuzzy cross-efficiency evaluation using a pair of linear programs, generating lower and upper bounds for the fuzzy efficiency score. Rouyendegh et al. [31] aim to enhance the business performance of healthcare firms through a DEA-based fuzzy multi-criteria decision-making model. The study, designed primarily for hospitals, employs Data Envelopment Analysis and fuzzy analytic hierarchy process to quantify data and structure the decision-making model.

Arya and Yadav [2] propose an IF slack-based measure (IFSBM) model for DMU efficiency and an IF super efficiency SBM (IFSESBM) model for efficient DMUs. They also introduce a ranking method for intuitionistic fuzzy interval numbers (IFINs) based on α and β-cuts. Han et al. [13] present an efficiency analysis method using FDEACM (fuzzy DEA cross-model) with Fuzzy Data. This method, based on fuzzified multi-criteria ethylene energy consumption data, provides objective benchmarks for effective production situations and improvement directions for ethylene plants. Amindoust [1] utilizes DEA to introduce a model for evaluating and ranking suppliers from a sustainable perspective. The model integrates fuzzy set theory and DEA, incorporating decision makers' preferences and addressing ambiguity and uncertainty in supplier selection. Linguistic values in the form of triangular fuzzy numbers are used for criteria and sub-criteria weights, and a fuzzy-DEA model is developed using the α-cut approach. Ji et al. [19] develop a novel fuzzy DEA model using fuzzy Choquet integral to evaluate DMU efficiencies with interactive fuzzy variables. The proposed model assesses fuzzy efficiency and introduces a ranking method for fuzzy efficiency.

He et al. [15] present an effective approach to continuously improve production quality, addressing the puzzle of infant failure root causes. They introduce a novel root cause identification method using an associated tree and fuzzy DEA. The associated tree guides the analysis process based on axiomatic domain mapping, and fuzzy DEA clusters potential factors from big data regarding product life cycle, ranking the weight of each node in the associated tree. Liu et al. [20] investigate a novel group decision-making approach based on DEA cross-efficiency with intuitionistic fuzzy preference relations, avoiding information distortion. They define an interval transform function to convert intuitionistic fuzzy preference relations into interval multiplicative preference relations. An interval transform function-based DEA model is developed to obtain the ranking vector of consistent intuitionistic fuzzy preference relations. Zadmirzaei et al. [39] propose a novel environmental efficiency DEA model, FUNSBM, incorporating predicting artificial intelligence algorithms. This model measures environmental efficiency using the directional distance function and weak disposability. Liu et al. [22] apply the credibility distribution of fuzzy variables in uncertainty theory to formulate a fuzzy optimization model for studying the coordination mechanism of a distribution supply chain system with fuzzy demand. Using a modified sequence quadratic programming algorithm, they obtain optimal order quantity, retailer sales volume, and the maximum expected return for the distribution supply chain.

Mohanta and Sharanappa [26] propose an Intuitionistic Fuzzy DEA (IFDEA) model based on triangular intuitionistic fuzzy numbers (TIFNs). The weighted possibility mean for TIFN is used to compare and rank them, and the IFDEA model is converted into a crisp DEA model to assess relative efficiencies. Cinaroglu [9] evaluates Turkish health system efficiency using a multistep fuzzy stochastic procedure, comparing crisp and stochastic efficiency estimates. Conventional, bias-corrected, and fuzzy DEA estimates explore province-based health systems' efficiency scores. Mehrasa and Behzadi [24] introduce the random fuzzy DEA (Ra-Fu DEA) model, considering fuzzy data with a skew-normal distribution. The model investigates one state of possibilityprobability in the presence of a skew-normal distribution with a fuzzy mean and threshold level. Majdi et al. [23] propose a common-weight method in a fuzzy environment to determine common sets of weights (CWS) for the best and worst fuzzy efficiencies of all DMUs, considering optimistic and pessimistic perspectives. Pourbabagol et al. [30] use agile supply chain (ASC) theory to introduce (Nec) and possibility (Pos) equality constraints in a fuzzy network DEA slack-based model, measuring the efficiency of different agility levels in a dairy supply chain. Shiang and Yueh [32] present a method for calculating fuzzy cross-efficiency directly without the need for weight selection. Linear programs at specific α-levels generate lower and upper bounds of the fuzzy efficiency score. Meng and Shi [25] transform the fuzzy DEA model to the conventional DEA model using the center of mass formula. An algorithm based on the golden section method is employed to locate the standard that the evaluated DMU best fits. Mu et al. [27] demonstrate how fuzzy DEA evaluates the eco-efficiency of dairy farming, accounting for uncertainty around environmental and economic indicators. Omrani et al. [29] propose an integrated fuzzy clustering cooperative game DEA approach, using fuzzy C-means and game theory to evaluate DMUs within each cluster. Song et al. [33] present a fuzzy slacks-based measure model incorporating a confidence coefficient, solving input slacks and efficiency evaluation problems in the presence of left–right fuzzy numbers for environmental efficiency evaluation.

The concept of the security margin of efficiency for decision-making units (DMUs) is a novel and unexplored topic, particularly in the context of imprecise DEA. Referred to as the performance security margin of DMUs, this method addresses the potential threat to the efficiency of efficient units posed by their improving but initially inefficient competitors. In essence, each efficient unit must establish a security margin to safeguard its position in the competitive landscape.

The paper is structured as follows: In Section 2, Interval and fuzzy DEA models are introduced to measure the optimistic and pessimistic efficiencies of DMUs. Sections 3 and 4 present a sensitivity analysis algorithm designed to determine the efficiency interval and fuzzy security margin of DMUs considering both optimistic and pessimistic frontiers. Additionally, for a comprehensive ranking of DMUs with imprecise data using the proposed sensitivity analysis algorithm, we employ an approach based on the preference degree matrix for interval efficiency. The paper concludes in Section 5.

2. Preliminaries

In this section, we review DEA models designed to measure both optimistic and pessimistic efficiencies of decision-making units, providing illustrative examples for clarity.

Consider n DMUs for evaluation, each with m inputs $x_{ij} = (i = 1 \dots m)$ and s outputs $y_{rj} = (r = 1 \dots s)$. In interval DEA, some exact input and output values are unknown, residing within specified intervals $[x_{ij}^L, x_{ij}^U]$and $[y_{rj}^L, y_{rj}^U]$, where $x_{ij}^L \geq 0$ and $y_{rj}^L \geq 0$. Addressing this uncertainty, Wang et al. [34] introduced a pair of linear programming models (1) and (2) to determine upper and lower bounds on efficiency, focusing on optimistic efficiency:

$$\begin{aligned} \max \quad & \theta_0^U = \sum_{r=1}^{s} u_r\, y_{ro}^U \\ s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^L = 1, \\ & \sum_{r=1}^{s} u_r\, y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \leq 0, \quad j = 1, \dots n, \\ & u_r, v_i \geq 0, \;\; r = 1, \dots, s, \;\; i = 1, \dots, m \end{aligned} \tag{1}$$

$$\begin{aligned} \max \quad & \theta_0^L = \sum_{r=1}^{s} u_r\, y_{ro}^L \\ s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^U = 1, \\ & \sum_{r=1}^{s} u_r\, y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \leq 0, \quad j = 1, \dots n, \\ & u_r, v_i \geq 0, \;\; r = 1, \dots, s, \;\; i = 1, \dots, m \end{aligned} \tag{2}$$

In the expressions, DMU_o represents the DMU under evaluation, while v_i ,$(i = 1, \dots, m)$ and u_r ,$(r = 1, \dots, s)$ are decision variables. θ_o^U represents optimistic efficiency under the most favorable situation, and θ_o^L represents optimistic efficiency under the most unfavorable situation for DMU_o. These values together define the optimistic efficiency interval$[\theta_o^L, \theta_o^U]$. If there exists a set of positive weights v_i^* ,$(i = 1, \dots, m)$and u_r^* ,$(r = 1, \dots, s)$ which leads to $\theta_o^{U*} = 1$, then DMU_o is considered optimistic efficient.

Without loss of generality, let's assume that all input and output data, x_{ij} and y_{rj}, are uncertain and defined by by $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ as triangular fuzzy numbers. In this case, it holds true that x_{ij}^L >0 and y_{rj}^L>0 for i =1,…,m, r =1,…, s and j =1,…,n.

It is crucial to note that existing fuzzy DEA models either stem directly from the defuzzification of crisp DEA models or arise from interval DEA models. The former neglects the fact that a fuzzy fractional programming cannot be transformed into an LP model in the traditional manner, unlike a crisp fractional programming. On the other hand, the latter requires solving a series of linear programming models based on different α-level sets, leading to significant computational efforts in obtaining fuzzy efficiencies of DMUs. To address these challenges, our sensitivity analysis algorithm utilizes fuzzy DEA models (3) - (5) proposed by Wang et al. [37] from an optimistic viewpoint. Consequently, the efficiency of DMUj is measured as follows:

$$
\begin{aligned}
\max \quad & \theta_0^U = \sum_{r=1}^{s} u_r\, y_{ro}^U \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^L = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \le 0, j = 1, \dots n, \\
& u_r, v_i \ge 0, \quad i = 1, \dots, m; \quad r = 1, \dots, s
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\max \quad & \theta_0^M = \sum_{r=1}^{s} u_r\, y_{ro}^M \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^M = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \le 0, \quad j = 1, \dots n, \\
& u_r, v_i \ge 0, \quad i = 1, \dots, m; \quad r = 1, \dots, s
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\max \quad & \theta_0^L = \sum_{r=1}^{s} u_r\, y_{ro}^L \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^U = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \le 0, \quad j = 1, \dots n, \\
& u_r, v_i \ge 0, \quad r = 1, \dots, s, \quad i = 1, \dots, m
\end{aligned} \tag{5}
$$

The optimal objective function values of the three LP models (3)-(5) collectively yield the optimistic fuzzy efficiency of DMUo, denoted as $\tilde{\theta}_0^* \approx (\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*})$. This result can be interpreted as a triangular fuzzy number. If there exists a set of positive weights u_r^* ,$(r = 1, \dots, s)$ and v_i^* ,$(i = 1, \dots, m)$that satisfies $\theta_0^{U*} = 1$, the DMUo is termed optimistic fuzzy efficient. It can be easily shown that $\theta_0^L \le \theta_0^M \le \theta_0^U$.

From a pessimistic viewpoint, efficiencies can be ascertained using the following interval DEA models, which gauge the worst performance of DMUo relative to the other DMUs [4]:

$$
\begin{aligned}
\min \quad & \varphi_0^L = \sum_{r=1}^{s} u_r\, y_{ro}^L \\
s.t. \quad & \sum_{r=1}^{s} u_r\, y_{rj}^L - \sum_{i=1}^{m} v_i\, x_{ij}^U \ge 0, j = 1, \dots n, \\
& \sum_{i=1}^{m} v_i\, x_{io}^U = 1, \\
& u_r, v_i \ge 0, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\min \quad & \varphi_0^U = \sum_{r=1}^{s} u_r\, y_{ro}^U \\
s.t. \quad & \sum_{r=1}^{s} u_r\, y_{rj}^L - \sum_{i=1}^{m} v_i\, x_{ij}^U \ge 0, j = 1, \dots n, \\
& \sum_{i=1}^{m} v_i\, x_{io}^L = 1, \\
& u_r, v_i \ge 0, \ \ r = 1, \dots, s, \ \ i = 1, \dots, m.
\end{aligned} \tag{7}
$$

In models (6) and (7), φ_0^L represents the pessimistic efficiency under the most unfavorable situation, and φ_0^U is the pessimistic efficiency under the most favorable situation for DMUo. Together, they represent the pessimistic efficiency interval $[\varphi_0^L, \varphi_0^U]$ for DMUo. When there is a set of positive weight v_i^* , $(i = 1, \dots, m)$ and u_r^* , $(r = 1, \dots, s)$ to achieve $\varphi_0^{L*} = 1$, it is termed pessimistic inefficient DMUo.

Based on the pessimistic viewpoint, the optimum objective function values of the three LP models (8)-(10) form the pessimistic fuzzy efficiency of DMUo, i.e., $\tilde{\varphi}_0^* \approx (\varphi_0^{L*}, \varphi_0^{M*}, \varphi_0^{U*})$, which can be interpreted as a triangular fuzzy number. When there is a set of positive weights u_r^* , $(r = 1, \dots, s)$ and v_i^* , $(i = 1, \dots, m)$ to achieve $\varphi_0^{U*} = 1$, it is termed pessimistic fuzzy efficient [18].

$$
\begin{aligned}
\min \quad & \varphi_0^L = \sum_{r=1}^{s} u_r\, y_{ro}^L \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^U = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^L - \sum_{i=1}^{m} v_i\, x_{ij}^U \ge 0, j = 1, \dots n, \\
& u_r, v_i \ge 0, \ \ i = 1, \dots, m; \ \ \ r = 1, \dots, s
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\min \quad & \varphi_0^M = \sum_{r=1}^{s} u_r\, y_{ro}^M \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^M = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^L - \sum_{i=1}^{m} v_i\, x_{ij}^U \ge 0, \quad j = 1, \dots n, \\
& u_r, v_i \ge 0, \ \ i = 1, \dots, m; \ \ \ r = 1, \dots, s
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\min \quad & \varphi_0^U = \sum_{r=1}^{s} u_r\, y_{ro}^U \\
s.t. \quad & \sum_{i=1}^{m} v_i\, x_{io}^L = 1, \\
& \sum_{r=1}^{s} u_r\, y_{rj}^L - \sum_{i=1}^{m} v_i\, x_{ij}^U \ge 0, \quad j = 1, \dots n, \\
& u_r, v_i \ge 0, \ \ r = 1, \dots, s, \ \ i = 1, \dots, m.
\end{aligned} \tag{10}
$$

It can be easily shown that $\varphi_0^L \le \varphi_0^M \le \varphi_0^U$. To illustrate the distinction between DMUs on optimistic and pessimistic frontiers, we consider an example with one input and two outputs, as presented in Table 1. The efficiency and inefficiency frontiers of this example are depicted in Figure 1. As shown in the figure, DMUs can be categorized into three groups: optimistic efficient, pessimistic inefficient, and unspecified. In this example, four DMUs (DMU A, DMU E, DMU G, and DMU J) are labelled as optimistic efficient. Additionally, three DMUs (DMU A, DMU B, and DMU J) are identified as pessimistic inefficient, while five DMUs (DMU C, DMU H, DMU D, DMU F, and DMU I) fall into the unspecified category.

Table 1. Data for ten unts with one input and two outputs

DMU	Input 1	Output 1	Output 2
A	1	[0.8, 1.2]	[7.5, 8.5]
B	1	[1.8, 2.2]	[2.4, 3.6]
C	1	[1.7, 2.3]	[5.7, 6.3]
D	1	[2.5, 3.5]	[2.7, 3.3]
E	1	[2.8, 3.2]	[6.7, 7.3]
F	1	[3.8, 4.2]	[1.8, 2.2]
G	1	[3.4, 4.6]	[4.6, 5.4]
H	1	[4.7, 5.3]	[1.5, 2.5]
I	1	[5.6, 6.4]	[1.7, 2.3]
J	1	[6.7, 7.3]	[0.8, 1.2]

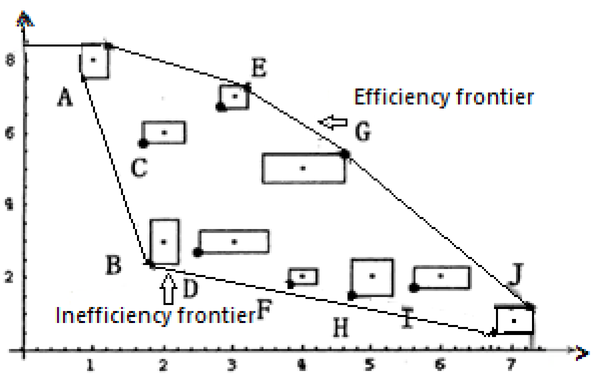

Figure 1. Efficiency and inefficiency frontier for ten DMUs

The optimistic and pessimistic interval efficiency for ten DMUs are obtained with the models (1) and (2) are shown in Table 2.

Table 2. Interval optimistic and pessimistic efficiency for ten DMUs

DMUs	Input	Output1	Output2	Optimistic interval efficiency $[\theta_j^{L*}, \theta_j^{U*}]$	Pessimistic interval efficiency$[\varphi_j^{L*}, \varphi_j^{U*}]$
A	1	[0.8, 1.2]	[7.5, 8.5]	[0.8824, 1.0000]	[1.0000, 1.2625]
B	1	[1.8, 2.2]	[2.4, 3.6]	[0.4160, 0.5656]	[1.0000, 1.2798]
C	1	[1.7, 2.3]	[5.7, 6.3]	[0.7289, 0.8330]	[1.2409, 1.5570]
D	1	[2.5, 3.5]	[2.7, 3.3]	[0.5248, 0.6965]	[1.1769, 1.4870]
E	1	[2.8, 3.2]	[6.7, 7.3]	[0.9089, 1.0000]	[1.8117, 1.0397]
F	1	[3.8, 4.2]	[1.8, 2.2]	[0.6142, 0.6956]	[1.0178, 1.1954]
G	1	[3.4, 4.6]	[4.6, 5.4]	[0.7914, 1.0000]	[1.8946, 2.3101]
H	1	[4.7, 5.3]	[1.5, 2.5]	[0.7018, 0.8558]	[1.0157, 1.4160]
I	1	[5.6, 6.4]	[1.7, 2.3]	[0.8292, 0.9761]	[1.1810, 1.4693]
J	1	[6.7, 7.3]	[0.8, 1.2]	[0.9178, 1.0000]	[1.0000, 1.1995]

From Table 2, it is evident that four DMUs—DMU A, DMU E, DMU G, and DMU J—based on models (1) and (2) are classified as optimistic efficient. Similarly, according to models (6) and (7), three DMUs—DMU A, DMU B, and DMU J—are recognized as pessimistic efficient.

3. Novel Sensitivity Analysis Algorithm for Evaluating Interval and Fuzzy Optimistic and Pessimistic Efficiency Security Margin

In general, the primary focus of sensitivity analysis is the impact of various factors on the efficiency of DMUs and the identification of conditions that influence efficiency variations. Original DEA models typically assume precise values for inputs and outputs. However, real-world scenarios often involve uncertainties. This section introduces a sensitivity analysis algorithm designed for sets of interval and fuzzy data. Additionally, we extend the proposed algorithm to rank DMUs when dealing with interval and fuzzy data.

Enhancing the performance of DMUs is a constant goal. Consequently, each DMU must consider the performance of other competitive units. The efficiency distance between a DMU and its peers establishes a security margin for its overall performance.

Now, the critical questions emerge: What defines the security margin for the DMU under evaluation to enhance and sustain its performance? How much can the improvement of other inefficient DMUs impact the decrease in performance of efficient DMUs? Answering these questions requires introducing the concepts of absolute and relative efficiency security margins. The algorithm presented in this section focuses on determining the relative security margin of DMUs.

To identify the nearest DMU posing a threat to DMU K, subsequent to calculating the relative (interval or fuzzy) efficiency security margin of DMU K (for $1 \leqslant t \leqslant n$, $t \neq k$), we need to pinpoint the minimum value among them. This minimum value is termed the absolute (interval or fuzzy) efficiency security margin of DMU K and is defined as follows:

$$ARESM^{*}(k)=\min\{RESM(k,t);\ 1\leq t\leq n\ ,t\neq k\}$$

In the first step, the efficiency of all decision-making units is determined using one of the classic DEA models. Here, DMU k represents the unit under performance evaluation, seeking to identify the safety margin for its performance, while DMU t is referred to as the comparison unit, which can be any of the inefficient units. To ascertain the absolute efficiency margin, each inefficient unit is chosen as the benchmark, and through the ARESM(k) formula, the nearest unit posing a threat to the efficiency of efficient DMU k is selected.

Step 1: Utilize one of the DEA models to compute the interval/fuzzy efficiency of all DMUs, including DMU k and DMU t.

Step 2: Record the value of E(k) in $\tilde{E}(k)$ as the initial interval/fuzzy efficiency of DMU k.

Step 3: Store the values of y(r,t) and x(i,t) for r=1,…,s and i=1,…,m in $\tilde{y}(r,t)$ and $\tilde{x}(i,t)$, respectively. This step ensures the preservation of the initial input and output values for the measurement unit, with the index t representing the compared unit.

Step 4: Set the values of α and δ as α=δ=0.01. Here, α represents the coefficient of variations for inputs and outputs, and δ denotes the percentage of data variations.

Step 5: Compute the new values of inputs and outputs for r=1,…,s and i=1,…,m using the following formulas:

$$y(r,t) = \tilde{y}(r,t).(1+\alpha),\ r=1,\ldots,s$$

$$x(i,t)= \tilde{x}(i,t).(1-\alpha),\ i=1,\ldots,m$$

In this step, the perturbation values for the inputs and outputs of the compared unit are obtained simultaneously, and the new values are recorded. Subsequently, the efficiency score of the compared unit is recalculated with these new values. This process is repeated until it jeopardizes the efficiency of the unit under evaluation.

Step 6: Compute the interval/fuzzy efficiency of DMU k and save it in $\tilde{E}(k)$.

Step 7: If E(k) is less than $\tilde{E}(k)$, proceed to Step 9; otherwise, move to the next step.

Step 8: Set α= α +δ and repeat the fifth step.

* Absolute Efficiency Security Margin

Step 9: Express the efficiency security margin of DMU k relative to DMU t as a percentage:

$$RESM(k,t)=100.(\alpha-\delta)$$

In Step 9, the efficiency security margin of DMU k relative to DMU t is calculated as a percentage, denoted as RESM(k,t). The algorithm is then stopped.

3.1. Sensitivity Analysis of the Optimistic Interval Efficiency Security Margin of DMUs

In this section, we determine the optimistic interval efficiency security margin of decision-making units with the proposed algorithm and subsequently rank them. For ranking DMUs, we adopt the approach based on the preference degree matrix for interval efficiency introduced by Wang et al. [34].

Let's consider DMUs in Table 1 with one input and two outputs. DMU C and DMU E are designated as the compared unit and the unit under evaluation, respectively. We then execute the steps of the algorithm. The algorithm iterates until the efficient DMU E recedes from the efficient frontier. Results are obtained after 5 stages, as shown in Table 3.

From Table 3, we observe that after the fifth iteration of the algorithm, the performance security margin of DMU E is jeopardized, and its position is taken over by DMU C. Once DMU E loses its efficiency, the algorithm stops. The first stage where the efficiency of DMU E diminishes corresponds to the interval performance security margin, which is calculated as a percentage using the following formula:

$$RESM\ (E, C) = 100(\alpha-\delta) = 100(0.06-0.01) = 5\%$$

Moreover, from the last column, we discern that the improved performance of DMU C has an impact on the performance of the other DMUs. Subsequently, for the comprehensive ranking of DMUs following the execution of the sensitivity analysis algorithm, as outlined in Table 3, we employ the approach based on the preference degree matrix for interval efficiency, as presented in Wang et al. [35]. The results are displayed in Table 4.

Table 3. Optimistic relative interval efficiency security margin $[\theta_j^{L*}, \theta_j^{U*}]$

DMUs	Stage1	Stage2	Stage3	Stage4	Stage5
A	[0.8824, 1.0000]	[0.8824, 1.0000]	[0.8824, 1.0000]	[0.8824, 1.0000]	[0.8818, 0.9975]
B	[0.4160, 0.5656]	[0.4160, 0.5656]	[0.4160, 0.5656]	[0.4160, 0.5656]	[0.3978, 0.5341]
C	[0.7436, 0.8498]	[0.7739, 0.8845]	[0.8218, 0.9392]	[0.8792, 1.0000]	[0.9048, 1.0000]
D	[0.5248, 0.6965]	[0.7739, 0.8845]	[0.5248, 0.6965]	[0.5248, 0.6965]	[0.5070, 0.6753]
E	[0.9089, 1.0000]	[0.9089, 1.0000]	[0.9089, 1.0000]	[0.9047, 1.0000]	[0.8321, 0.9253]
F	[0.6142, 0.6956]	[0.6142, 0.6956]	[0.6142, 0.6956]	[0.6142, 0.6956]	[0.6050, 0.6838]
G	[0.7914, 1.0000]	[0.7914, 1.0000]	[0.7914, 1.0000]	[0.7914, 1.0000]	[0.7562, 0.9639]
H	[0.7018, 0.8558]	[0.7018, 0.8558]	[0.7018, 0.8558]	[0.7018, 0.8558]	[0.6961, 0.8431]
I	[0.8292, 0.9761]	[0.8292, 0.9761]	[0.8292, 0.9761]	[0.8292, 0.9761]	[0.8231, 0.9664]
J	[0.9178, 1.0000]	[0.9178, 1.0000]	[0.9178, 1.0000]	[0.9178, 1.0000]	[0.9178, 1.0000]

Table 4. Preference degree matrix for optimistic interval efficiency based on sensitivity analysis algorithm and their ranks

DMU	A	B	C	D	E	F	G	H	I	J	Rank after algorithm	Rank before algorithm
A	-	1.0000	0.4395	1.0000	**0.7918**	1.0000	0.7461	1.0000	0.6734	0.4027	4	3
B	0.0000	-	0.0000	0.0890	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	10	10
C	0.5605	1.0000	-	1.0000	0.8912	1.0000	0.8049	1.0000	**0.7417**	0.4634	**2**	**6**
D	0.0000	**0.9110**	0.0000	-	0.0000	0.0285	0.0000	0.0000	0.0000	0.0000	9	9
E	0.2082	1.0000	0.1088	1.0000	-	1.0000	**0.5620**	0.9542	0.4321	0.0428	**5**	**2**
F	0.0000	1.0000	0.0000	**0.7155**	0.0000	-	0.0000	0.0000	0.0000	0.0000	8	8
G	0.2539	1.0000	0.1951	1.0000	0.4380	1.0000	-	**0.7550**	0.4011	0.1590	6	5
H	0.0000	1.0000	0.0000	0.9546	0.0458	**1.0000**	0.2450	-	0.0689	0.0000	7	7
I	**0.3266**	1.0000	0.2583	1.0000	0.5679	1.0000	0.5989	0.9311	-	0.3123	3	4
J	0.5973	1.0000	**0.5366**	1.0000	0.9572	1.0000	0.8410	1.0000	0.7845	-	1	1

Following the establishment of the preference degree matrix in Table 4, derived from the algorithm results presented in Table 3, it is evident that DMU C enhanced its performance, securing the second rank. Conversely, DMU E experienced a decline, now occupying the fifth rank. Based on the information in the last column of Table 4, the performance of the ten DMUs is assessed as follows:

$$DMU_J \succ^{53.66\%} DMU_C \succ^{74.17\%} DMU_I \succ^{32.66\%} DMU_A \succ^{79.18\%} DMU_E \succ^{56.20\%}$$
$$DMU_G \succ^{75.50\%} DMU_H \succ^{100\%} DMU_F \succ^{71.55\%} DMU_D \succ^{91.10\%} DMU_B$$

$DMU_J \succ^{53.66\%} DMU_C$ means that the performance DMU_J "is better than" DMU_C by 53.66%. It is clear that the best performance is related to DMU_J. The detailed results of ranking are presented in Table 4.

3.2. Sensitivity Analysis of the Pessimistic Interval Efficiency Security Margin of DMUs

Continuing with the analysis, we now determine the interval inefficiency improvement margin of DMUs from a pessimistic viewpoint. For this illustration, consider DMU B and DMU D as the compared unit and unit under evaluation, respectively. The results of the algorithm's steps are displayed in Table 5.

Based on the proposed algorithm, we assess the relative interval inefficiency improvement margin of DMU B concerning DMU D. Here, DMU B and DMU D represent the unit under evaluation and compared unit, respectively. After five stages, the inefficiency score of DMU B changes to [1.0351, 1.3573]. This initial stage indicates the improvement of the inefficient DMU B when it is distant from the inefficiency frontier, prompting the algorithm to halt. The results of the sensitivity analysis algorithm are detailed in Table 5. The interval inefficiency improvement margin of DMU B regarding DMU D is derived as follows:

$$\text{RESM (D, B)} = 100(\alpha-\delta) = 100(0.05-0.01) = 4\%$$

Table5. Pessimistic interval relative inefficiency improvement margin $[\varphi_j^{L*}, \varphi_j^{U*}]$

DMU	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
A	[1.0000,1.2588]	[1.0000,1.2515]	[1.0000,1.2410]	[1.0000,1.2281]	[1.0000,1.2182]
B	[1.0000,1.2819]	[1.0000,1.2862]	[1.0000,1.2930]	[1.0000,1.3027]	[1.0351,1.3573]
C	[1.2269,1.5361]	[1.1997,1.4956]	[1.1609,1.4377]	[1.1127,1.3658]	[1.0759,1.3110]
D	[1.1551,1.4609]	[1.1164,1.4178]	[1.0627,1.3596]	[1.0000,1.2917]	[1.0000,1.2917]
E	[1.7850,2.0087]	[1.7331,1.9484]	[1.6590,1.8624]	[1.5669,1.7554]	[1.4967,1.6739]
F	[1.0056,1.1800]	[1.0000,1.1689]	[1.0000,1.1615]	[1.0000,1.1528]	[1.0000,1.1528]
G	[1.8573,2.2661]	[1.7849,2.1849]	[1.6815,2.0707]	[1.5530,1.9335]	[1.4551,1.8843]
H	[1.0072,1.3991]	[1.0033,1.3914]	[1.0033,1.3914]	[1.0033,1.3914]	[1.0033,1.3914]
I	[1.1717,1.4553]	[1.1674,1.4490]	[1.1674,1.4490]	[1.1674,1.4490]	[1.1674,1.4490]
J	[1.0000,1.1965]	[1.0000,1.1951]	[1.0000,1.1951]	[1.0000,1.1951]	[1.0000,1.1951]

Additionally, from the last column, it is observed that the improved performance of DMU C has influenced the performance of other DMUs, including DMU H, DMU I, and DMU J, causing them to enhance their positions and move further from the inefficiency frontier. Subsequently, following the execution of the sensitivity analysis algorithm, for the comprehensive ranking of DMUs, we employ the approach grounded in the preference degree matrix for interval inefficiency proposed by Wang et al. [35], adopting a pessimistic perspective. The outcomes of this ranking process are displayed in Table 6.

Table 6. Preference degree matrix for pessimistic interval efficiency based on sensitivity analysis algorithm and their rankings

DMU	A	B	C	D	E	F	G	H	I	J	Rank without algorithm	Rank with algorithm
A	-	0.3388	0.3139	0.4279	0.0000	0.5881	0.0000	0.3544	0.1016	**0.5279**	8	8
B	0.6612	-	**0.5049**	0.5821	0.0000	0.7522	0.0000	0.4984	0.3145	0.6907	7	5
C	0.6861	0.4951	-	**0.5904**	0.0000	0.8018	0.0000	0.4938	0.2779	0.7229	3	6
D	**0.5721**	0.4180	0.4096	-	0.0000	0.6563	0.0000	0.4242	0.2168	0.5992	4	7
E	1.0000	1.0000	1.0000	1.0000	-	1.0000	0.3608	1.0000	**1.0000**	1.0000	2	2
F	0.4042	0.2478	0.1982	0.3438	0.0000	-	0.0000	0.2739	0.0000	0.4392	9	10
G	1.0000	1.0000	1.3717	1.0000	**0.6392**	1.0000	-	1.0000	1.0000	1.0000	1	1
H	0.6456	**0.5016**	0.5063	0.5757	0.0000	0.7236	0.0000	-	0.0000	0.6711	6	4
I	0.8984	0.6855	0.7221	0.7832	0.0000	1.0000	0.0000	**0.6655**	-	0.9419	5	3
J	0.4721	0.3093	0.2771	0.4008	0.0000	**0.5608**	0.0000	0.3289	0.0581	-	10	9

Table 6 illustrates that after the fifth stage of executing the algorithm, DMU B has increased its distance from the inefficiency frontier and achieved a rank of 5. The compared unit, DMU D, is positioned close to the inefficiency frontier, securing rank 7. Additionally, the ranks of DMUs I, J and H have also seen improvements. Through the assessment from the pessimistic perspective using the sensitivity analysis algorithm, it is evident that the best performance is associated with DMU G. The ranking of the ten DMUs in terms of their pessimistic efficiency intervals, based on the results in the last column of Table 6, is as follows:

$$DMU_G \succ^{63.92\%} DMU_E \succ^{100\%} DMU_I \succ^{66.55\%} DMU_H \succ^{50.16\%} DMU_B \succ^{50.49\%} DMU_C \succ^{59.04\%} DMU_D \succ^{57.21\%} DMU_A \succ^{52.79\%} DMU_J \succ^{56.08\%} DMU_F$$

4. Sensitivity Analysis for Determining Fuzzy Efficiency Security Margin of DMUs from Both Efficiency and Inefficiency Frontiers

In this section, we apply the proposed sensitivity algorithm to ascertain the fuzzy performance security margin of DMUs from both optimistic and pessimistic viewpoints. Subsequently, we extend the algorithm to rank DMUs, and provide an illustrative numerical example.

4.1. Sensitivity analysis to determine the optimistic fuzzy efficiency security margin of DMUs

A fuzzy number is a convex fuzzy set, characterized by a given interval of real numbers, each with a grade of membership between 0 and 1. The most used fuzzy numbers, are triangular and trapezoidal fuzzy numbers. Suppose there are n DMUs to be evaluated, each with m inputs and s outputs. Let x_{ij} $(i = 1, \dots, m)$ and y_{rj} $(r = 1, \dots, s)$ be the input and output data of $DMU_j (j = 1, \dots, n)$. Without loss of generality, all input and output data x_{ij} and y_{rj} are assumed to be uncertain and characterized by triangular fuzzy numbers $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$, where $x_{ij}^L > 0$ and $y_{rj}^L > 0$ for $(i = 1, \dots, m)$, $(r = 1, \dots, s)$ and $(j = 1, \dots, n)$. Crisp input and output data can be considered a special case of fuzzy input and output data $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ with $x_{ij}^L, = x_{ij}^M, = x_{ij}^U$ and $y_{rj}^L = y_{rj}^M, = y_{rj}^U$. Therefore, DMU_j efficiency is measured by (3)-(5) models.

Examining a performance assessment scenario in China involves the evaluation of eight manufacturing enterprises concerning two inputs and two outputs [37]. The considered outputs include the gross output value (GOV) and product quality (PQ), while manufacturing cost (MC) and the number of employees (NOE) serve as inputs. The input and output data for these manufacturing enterprises are outlined in Table 7. Employing models (3)-(5) and (8)-(10) for each manufacturing enterprise yields two fuzzy efficiencies for each from an optimistic and pessimistic perspective, respectively. Subsequently, we proceed to determine the optimistic fuzzy performance security margin of each decision-making unit and then rank them following the completion of the sensitivity analysis algorithm.

Table 7. Input and output data and optimistic efficiency for eight manufacturing enterprises

	Input 1	Input 2	Output 1	Output 2	Optimistic Fuzzy Efficiency
DMUs	MC	NOE	GOV	PQ	
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)	(0.8124,0.9033,1.0000)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)	(0.9750,0.9945,1.0000)
C	(2510, 2570, 1500)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)	(0.7946,0.8122,0.9036)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)	(0.7764,0.8049,0.9070)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)	(0.9603,0.9872,1.0000)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)	(0.8352,0.8518,0.8852)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)	(0.8752,0.8926,1.0000)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)	(0.8195,0.8363,0.8861)

By applying models (3) -(5) to each manufacturing enterprise, we obtained the optimistic fuzzy efficiency for each, as illustrated in Table 7. The last column of Table 7 reveals that manufacturing enterprises A, B, and E collectively establish an efficient frontier, with B being the enterprise showcasing the best performance, followed by enterprise E. Considering Table 7 with fuzzy data and selecting DMU B and DMU G as the compared unit and unit under evaluation, respectively, we then execute the algorithm, the results of which are presented in Table 8.

Table 8. Optimistic fuzzy relative efficiency security margin with sensitivity analysis algorithm

DMUs	Stage1 Optimistic Fuzzy Efficiency $(\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*})$	Stage2 Optimistic Fuzzy Efficiency $(\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*})$	Stage3 Optimistic Fuzzy Efficiency $(\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*})$	Stage4 Optimistic Fuzzy Efficiency $(\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*})$
A	(0.8124,0.9033,1.0000)	(0.8124,0.9006,1.0000)	(0.8010,0.8632,1.0000)	(0.7394,0.8076,0.9649)
B	(0.9750,0.9945,1.0000)	(0.9750,0.9945,1.0000)	(0.9613,0.9864,1.0000)	(0.8873,0.9202,0.9470)
C	(0.7946,0.8122,0.9045)	(0.7946,0.8121,0.9045)	(0.7846,0.8041,0.8786)	(0.7261,0.7504,0.8153)
D	(0.7764,0.8049,0.9070)	(0.7764,0.8038,0.9055)	(0.7655,0.7809,0.8842)	(0.7066,0.7208,0.8387)
E	(0.9603,0.9872,1.0000)	(0.9603,0.9872,1.0000)	(0.9603,0.9872,1.0000)	(0.9341,0.9735,1.0000)
F	(0.8352,0.8518,0.8852)	(0.8352,0.8518,0.8828)	(0.8234,0.8399,0.8452)	(0.7601,0.7753,0.7802)
G	(0.8929,0.9107,0.9648)	(0.9293,0.9478,0.8864)	(0.9729,0.9923,1.0000)	(0.9729,0.9923,1.0000)
H	(0.8195,0.8363,0.8864)	(0.8195,0.8368,0.8864)	(0.8080,0.8286,0.8598)	(0.7459,0.7715,0.7977)

In Table 8, it is evident that after the fourth stage of the algorithm, with the improvement in the efficiency of DMU G, the performance security margin of DMU B is jeopardized, and DMU G takes its place. Once DMU B loses its efficiency, the algorithm comes to a halt. The initial stage, where the efficiency of DMU B decreases, can be interpreted as the interval efficiency security margin, calculated by the formula:

$$\text{RESM (B, G)} = 100.\ (\alpha - \delta)=100(0.04\text{-}0.01) =3\%$$

To provide a comprehensive ranking of performances for the eight DMUs, Table 9 displays matrices of an analytical ranking approach based on degree preference. This approach is developed for comparing and ranking fuzzy efficiencies of DMUs, offering a full ranking and indicating to what extent a fuzzy efficiency is greater than another [37].

Table 9. Preference degree matrix for optimistic fuzzy efficiency based on sensitivity analysis algorithm and their rankings

DMU	A	B	C	D	E	F	G	H	Rank without algorithm	Rank with algorithm
A	-	0.1107	0.8616	0.8511	0.0157	**0.9056**	0.8702	0.0000	3	4
B	**0.8893**	-	1.0000	1.0000	0.0241	1.0000	0.0000	1.0000	1	3
C	0.1384	0.0000	-	**0.5968**	0.0000	0.3409	0.0000	0.3750	7	7
D	0.1489	0.0000	0.4032	-	0.0000	0.3054	0.0000	0.3258	8	8
E	0.9843	**0.9759**	1.0000	1.0000	-	1.0000	0.0043	1.0000	2	2
F	0.0944	0.0000	0.6591	0.6946	0.0000	-	0.0000	**0.5333**	5	5
G	1.0000	1.0000	1.0000	1.0000	**0.8372**	1.0000	-	1.0000	4	1
H	0.1298	0.0000	**0.6364**	0.6742	0.0000	0.4667	0.0000	-	6	6

After forming the preference degree matrix in Table 9 based on the results of the algorithm in Table 8, it is evident that DMU G has improved its performance and achieved rank 1. Conversely, DMU B has fallen to rank 3. The positions of the other DMUs, except for DMU A, remain unchanged. According to the last column of Table 9, the performance of the ten DMUs is ranked as follows:

$$DMU_G \succ^{83.72\%} DMU_E \succ^{97.59\%} DMU_B \succ^{88.93\%} DMU_A \succ^{90.56\%} DMU_F \succ^{53.33\%}$$
$$DMU_H \succ^{63.64\%} DMU_C \succ^{59.68\%} DMU_D \succ$$

$DMU_G \succ^{83.72\%} DMU_E$ signifies that the performance of DMU G is better than that of DMU E by 83.72%. It is evident that the best performance is associated with DMU G. The detailed results of the full ranking are presented in Table 9.

4.2. Sensitivity Analysis to Determine Pessimistic Fuzzy Efficiency Improvement Margin of DMUs

Next, we conduct a sensitivity analysis algorithm from a pessimistic perspective to identify the performance improvement margin of decision-making units. Consider Table 7 with fuzzy data involving two inputs and two outputs. Identify DMU A and DMU D as the compared unit and unit under evaluation, respectively. The results of the algorithm's steps are presented in Table 10:

Table 10. Pessimisitic relative fuzzy inefficiency improvement margin ($\varphi_0^{L*}, \varphi_0^{M*}, \varphi_0^{U*}$)

DMUs	Pessimistic Fuzzy Efficiency	Stage1 Pessimistic Fuzzy Efficiency	Stage 2 Pessimistic Fuzzy Efficiency
A	(1.0463,1.0672,1.0723)	(1.0256,1.0461,1.0511)	(1.0000,1.0263,1.0392)
B	(1.1106,1.2602,1.2880)	(1.1106,1.2406,1.2625)	(1.1106,1.2349,1.2584)
C	(1.0235,1.0441,1.0521)	(1.0032,1.0234,1.0312)	(1.0000,1.0201,1.0279)
D	(1.0000,1.0200,1.0287)	(1.0000,1.0200,1.0287)	(1.0034,1.0407,1.0571)
E	(1.0000,1.1702,1.2476)	(1.0000,1.1657,1.2229)	(1.0000,1.1643,1.2189)
F	(1.0000,1.0971,1.1040)	(1.0000,1.0754,1.0822)	(1.0000,1.0520,1.0709)
G	(1.0851,1.1497,1.1586)	(1.0746,1.1269,1.1356)	(1.0683,1.1117,1.1303)
H	(1.0190,1.0771,1.0855)	(1.0084,1.0558,1.0640)	(1.0052,1.0524,1.0606)

Based on the proposed algorithm and the information presented in Table 10, we assess the relative fuzzy inefficiency improvement margin of DMU D with respect to DMU A. DMU A and DMU D are denoted as the unit under evaluation and the compared unit, respectively. After two stages, the inefficiency score of DMU D changes to (1.0034, 1.0407, 1.0571). This represents the initial stage of improving the inefficient DMU D when it is distant from the inefficiency frontier. We can now conclude the algorithm. The results of the sensitivity analysis algorithm are detailed in Table 10. The fuzzy inefficiency improvement margin of DMU D concerning DMU A is determined as follows:

$$\text{RESM (A, D)} = 100.\ (\alpha - \delta) = 100(0.02-0.01) = 1\%$$

Continuing with the same approach, we utilize the preference degree matrix for ranking DMUs from a pessimistic perspective, as depicted in Table 11.

Table 11. Preference degree matrix for pessimistic fuzzy efficiency based on sensitivity analysis algorithm and their rankings

DMU	A	B	C	D	E	F	G	H	Rank without algorithm	Rank with algorithm
A	-	0.0203	**0.6522**	0.0718	0.0328	0.2113	0.0000	0.2105	3	7
B	1.0000	-	1.0000	1.0000	**0.8216**	1.0000	0.9867	1.0000	1	1
C	0.3478	0.0000	-	0.1622	0.0188	0.1356	0.0000	0.1087	5	8
D	**0.7234**	0.0000	0.8378	-	0.0669	0.3882	0.0000	0.3913	7	6
E	0.9672	0.1784	0.9812	0.9331	-	0.9058	**0.6693**	0.9218	2	2
F	0.7887	0.0000	0.8678	0.6118	0.0942	-	0.0008	**0.5181**	8	4
G	1.0000	0.0133	1.0000	1.0000	0.3307	**0.9916**	-	1.0000	4	3
H	0.7895	0.0000	0.8913	**0.6087**	0.0782	0.4819	0.0000	-	6	5

To provide a comprehensive ranking of performances for the eight DMUs, Table 11 displays matrices of an analytical ranking approach based on degree preference, developed for comparing and ranking fuzzy efficiencies. Following the formation of the preference degree matrix based on the results of the algorithm in Table 10, it is evident that DMU D has improved its performance, securing the sixth rank. This marks the initial stage of improvement for the previously inefficient DMU D when it is still far from the inefficiency frontier. Conversely, the performance of DMU A worsens, obtaining the seventh rank. According to the last column of Table 11, the performance of ten DMUs is rated as follows:

$$DMU_B \succ^{82.16\%} DMU_E \succ^{66.93\%} DMU_G \succ^{99.16\%} DMU_F \succ^{51.81\%} DMU_H \succ^{60.87\%}$$
$$DMU_D \succ^{72.34\%} DMU_A \succ^{65.22\%} DMU_C$$

The detailed results of the full ranking are presented in Table 11.

5. Conclusion

In conclusion, this paper contributed to the field of performance assessment by introducing interval and fuzzy DEA models to gauge optimistic and pessimistic efficiencies of DMUs. The novel sensitivity analysis algorithm proposed in this study provides a comprehensive evaluation of the performance security margin of DMUs, especially in scenarios where data is imprecise. This algorithm was effectively applied to investigate optimistic interval and fuzzy efficiency security margins, as well as pessimistic interval relative efficiency improvement margins of DMUs.

Upon closer examination of the outcomes presented in Table 8, a noteworthy observation was made after the fourth stage of the algorithm. The improvement in the efficiency of DMU G emerged as a potential threat to the performance security margin of DMU B, leading to a notable shift in their positions. The calculated interval efficiency security margin was quantified at 3%, shedding light on the vulnerability of DMU B in the face of efficiency changes.

Furthermore, recognizing the need to rank DMUs based on their performance, the proposed sensitivity analysis algorithm was extended. This extension allowed for the ranking of DMUs in an illustrative example involving eight manufacturing enterprises in China. Each enterprise was evaluated using two inputs and two outputs, and rankings were determined from both optimistic and pessimistic perspectives. These rankings provide actionable insights for managers to identify potential threats and opportunities for improvement in their respective enterprises.

In essence, the adaptability of this approach makes it suitable for a wide range of scenarios, accommodating situations involving both desirable and undesirable data. The combined use of interval and fuzzy DEA models, along with the sensitivity analysis algorithm, offers a robust framework for performance assessment and decision-making in the presence of uncertainties.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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