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The Relation Between some Polynomials and the Wiener Index of Fuzzy Graph

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A B S T R A C T

In this paper, the distance between two vertices in a fuzzy graph is defined in a new way. In addition, some new degree-based fuzzy graph polynomials are introduced. By using this definition, fuzzy graph polynomials, and a special lower triangular matrix, the Wiener index and the generalized Wiener index of a fuzzy graph are computed, which coincide with the Wiener index and the generalized Wiener index in the crisp graph. The result is used to compute the Wiener index of the sum, products, and composition of two fuzzy graphs.

1. Introduction

The first definition of a fuzzy graph was given by Kaufmann [5]. His definition was under Zadeh's fuzzy relation [13]. After that, Rosenfeld [9] introduced the fuzzy graph theory by considering fuzzy relations on fuzzy sets. He established some relations concerning the properties of path graph, trees, and various graphs. The generalization of a crisp graph is a fuzzy graph. Therefore, there are many similar properties between them.

Let $G=(V, E)$ be a graph. We denote the distance between two vertices u and v of G by $d(u,v)$ and defined as the number of edges in the shortest path connecting u and v. The oldest and most studied distance-based structure descriptor is the Wiener index introduced as early as in 1947 [11] and defined as the sum of the distances between all vertex pairs of the underlying graph:

$$W(G)=\sum_{\{u,v\}\subseteq V} d(u,v)$$

In [10], the authors introduced two new degree-based graph polynomials, and established their relations to the Wiener index and the degree distance. In this paper, we intend to define a polynomial similar to one of them

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for fuzzy graphs, and use it to obtain the Wiener index for fuzzy graphs. But before defining the Wiener index in a fuzzy graph, we need to define the distance between two vertices in a fuzzy graph. In the research literature, the distance between two vertices in fuzzy graphs is defined in such a way that, in a special case, it does not correspond to the concept of distance in crisp graphs (see [2], [4], and [6]). Thus, in the special case, the Wiener index of a crisp graph does not follow from the Wiener index of a fuzzy graph.

We define the distance between vertices in a fuzzy graph in such a way that the distance between two vertices in a crisp graph can be obtained in a particular case. Based on this definition, it is easy to see that the Wiener index in crisp graphs results from the Wiener index in fuzzy graphs. We will also see that using the polynomial mentioned above and the new distance concept, the Wiener index can be calculated as a fuzzy graph. Using a special lower triangular matrix, we calculate a generalized Wiener index in the fuzzy graph, and finally, we calculate the Wiener index in the fuzzy graphs of certain binary operations on pairs of graphs, and we will see that in a particular case, they are equal to crisp graphs.

2. Preliminaries

In this section, we collect some of the definitions and the results that will be required afterward. To begin with, we go through a few essential definitions from [7].

Definition 1. A fuzzy subset of a non-empty set S is a map $\sigma : S \to [0,1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in $[0,1]$ such that $0 \le \sigma(x) \le 1$.

If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. If V is a nonempty set, then we define the relation $\sim$ on $V \times V$ by for all $(x,y),(u,v) \in V \times V$, $(x,y) \sim (u,v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. It is easy to show that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x,y)]$ denote the equivalence class of (x,y) with respect to $\sim$. Then $[(x,y)] = \{(x,y),(y,x)\}$. Let $\mathcal{E}_V = \{[(x,y)] \mid x, y \in V, x \ne y\}$. Often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V,E). The elements of V are the vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x,y)]$. Clearly, we have $xy = yx$. We note that graph (V,E) has no loops or parallel edges.

Definition 2. A fuzzy graph $G = (V, \sigma_G, \mu_G)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma := \sigma_G : V \to [0,1]$ and $\mu := \mu_G : \mathcal{E} \to [0,1]$ such that for all $x, y \in V$, $\mu(xy) \le \sigma(x) \wedge \sigma(y)$, where $\wedge$ denote minimum.
The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G.

Definition 3. [7, pp.15] A path P in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}x_i) > 0$ for $i = 1, \cdots, n$. Here n is called the length of the path. We call P a cycle if $x_0 = x_n$ and $n \ge 3$.

Definition 4. (See [7, Definition 3.3.1]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree $x \in V$ is denoted by $d_G(x)$ and defined as $d_G(x) = \sum_{y \in V} \mu(xy)$.

Definition 5. [8, Definition 3.1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The size of G is denoted by $S(G)$ and defined as $\sum_{xy \in \mathcal{E}} \mu(xy)$.

Definition 6. [4, Definition 3.1] A fuzzy star graph $F(S_n) = (V_1 \cup V_2, \sigma, \mu)$ consists of two vertex sets $V = \{v_1\}$ *and* $V_2 = \{u_2, u_3, \ldots, u_n\}$ such that $\mu(v_1 u_i) > 0$ for $2 \le i \le n$ and $\mu(u_j u_k) = 0$ for $2 \le j, k \le n$ and $j \ne k$.

In 1989, Bhutani [1], introduced the concept of a complete fuzzy graph as follows. A complete fuzzy graph is a fuzzy graph $G = (V, \sigma, \mu)$ such that $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. In this paper, we present a new definition of complete fuzzy graph.

Definition 7. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. We say that G is a complete fuzzy graph **(CFG)** in which

$0 < \mu(uv) = k \le \min\{\sigma(v); v \in V\}$, for all $u, v \in V$.

Definition 8. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Sum of two fuzzy graphs G_1 and G_2 is denoted by $G_1 \vee G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & , \quad v \in V_1 \\ \sigma_2(v) & , \quad v \in V_2 \end{cases}, \quad \mu(uv) = \begin{cases} \mu_1(uv) & , \quad u, v \in V_1 \\ \mu_2(uv) & , \quad u, v \in V_2 \\ k & , \quad u \in V_1, v \in V_2 \end{cases},$$

where $k = \min\{\sigma_1(u), \sigma_2(v)\}$ for every $u \in \mathrm{V}_1$ and $v \in V_2$.

Definition 9. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. The Cartesian product of graphs G_1 and G_2 is denoted by $G_1 \times G_2 = (V, \sigma, \mu)$ is a fuzzy graph such that $V = V_1 \times V_2$, $\sigma((u,v)) = \sigma_1(u) \vee \sigma_2(v)$, where $\vee$ is denoted maximum and

$$\mu((u,v)(u',v')) = \begin{cases} \mu_2(vv') & , \quad \text{if } u = u' \\ \mu_1(uu') & , \quad \text{if } v = v' \\ 0 & , \quad o.w \end{cases}$$

We have $d_{G_1 \times G_2}(u,v) = d_{G_1}(u) + d_{G_2}(v)$.

Definition 10. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. The composition $G_1 o G_2 = (V, \sigma, \mu)$ is a fuzzy graph such that $V = V_1 \times V_2$, $\sigma((u,v)) = \sigma_1(u) \vee \sigma_2(v)$ for every $u \in V_1$, $v \in V_2$ and

$$\mu((u,v)(u',v')) = \begin{cases} \mu_2(vv'), \textit{ if } & u = u', vv' \in \mathcal{E}(G_2) \\ \mu_1(uu'), & uu' \in \mathcal{E}(G_1) \\ 0, & o.w \end{cases},$$

We have $d_{G_1 o G_2}(u,v) = 2d_{G_1}(u) + d_{G_2}(v)$.

3. Main results

In this section, we will introduce a lower triangular matrix and prove an important theorem by using this matrix which we will use to compute the generalized Wiener index in a fuzzy graph. Furthermore, we present two polynomials of a fuzzy graph and get some applications. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 & \cdots \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

where entry a_{ij} of lower triangle of A is as follows:

$$a_{ij} = \begin{cases} 1 & , \quad j = 1 \text{ or } j = i \\ a_{(i-1)(j-1)} + j a_{(i-1)j} & , \quad 1 < j < i \\ 0 & , \quad o.w \end{cases}.$$

In the next lemma, the inverse of matrix A is presented.

Lemma 1. The inverse of the matrix A is as follows:

$$B := A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & -3 & 1 & 0 & 0 & 0 & 0 & \dots \\ -6 & 11 & -6 & 1 & 0 & 0 & 0 & \dots \\ 24 & -50 & 35 & -10 & 1 & 0 & 0 & \dots \\ -120 & 274 & -225 & 85 & -15 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

where entry b_{ij} of lower triangle of A^{-1} is as follows:

$$b_{ij} = \begin{cases} 1 & , \quad j = i \\ (-1)^{i-1}(i-1)! & , \quad j = 1 \\ b_{(i-1)(j-1)} - (i-1)b_{(i-1)j} & , \quad 1 < j < i \\ 0 & , \quad o.w \end{cases}.$$

Proof. Suppose that $AB = C = [c_{ij}]_{n\times n}$. We compute c_{ij} for all i, j separately. We first consider the special case $i = j = 1$.

$$c_{11} = \sum_{k=1}^{n} a_{1k} b_{k1} = a_{11} b_{11} = 1.$$

The case $i = j$ will prove similarly; that is, $c_{ii} = 1$. Now consider the case $i > 1$ and $j = 1$.

$$c_{i1} = \sum_{k=1}^{n} a_{ik} b_{k1} = a_{i1} b_{11} + \sum_{k=2}^{n} a_{ik} b_{k1} = 1 + \sum_{k=2}^{n} (a_{(i-1)(k-1)} + k a_{(i-1)k})((-1)^{k-1}(k-1)!) =$$
$$1 + \sum_{k=2}^{n} (-1)^{k-1}(k-1)! a_{(i-1)(k-1)} + \sum_{k=2}^{n} (-1)^{k-1} k(k-1)! a_{(i-1)k} =$$
$$1 + \sum_{k=1}^{n-1} (-1)^{k} k! a_{(i-1)k} + \sum_{k=2}^{n-1} (-1)^{k-1} k! a_{(i-1)k} + (-1)^{n-1} n! a_{(i-1)n} =$$
$$1 + (-1)^1 1! a_{(i-1)1} + \sum_{k=2}^{n-1} ((-1)^k + (-1)^{k-1}) k! a_{(i-1)k} + 0 = 0.$$

Now let $1 < i < j$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} 0 + a_{i2} 0 + \dots + a_{ii} 0 + 0 b_{(i+1)j} + \dots + 0 b_{nj} = 0.$$

Finally we consider the case $1 < j < i$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} 0 + a_{i2} 0 + \dots + a_{i(j-1)} 0 + a_{ij} b_{jj} + a_{i(j+1)} b_{(j+1)j} +$$
$$a_{i(j+1)} b_{(j+1)j} + \dots + a_{ii} b_{ij} + 0 b_{(i+1)j} + \dots + 0 b_{nj} = \sum_{k=j}^{i} a_{ik} b_{kj} =$$
$$\sum_{k=j}^{i} (a_{(i-1)(k-1)} + k a_{(i-1)k}) b_{kj} = \sum_{k=j}^{i} a_{(i-1)(k-1)} b_{kj} + \sum_{k=j}^{i} k a_{(i-1)k} b_{kj} =$$
$$\sum_{k=j-1}^{i-1} a_{(i-1)k} b_{(k+1)j} + \sum_{k=j}^{i-1} k a_{(i-1)k} b_{kj} + i a_{(i-1)i} b_{ij} =$$

$$\sum_{k=j-1}^{i-1} a_{(i-1)k}(b_{k(j-1)} - kb_{kj}) + \sum_{k=j}^{i-1} ka_{(i-1)k}b_{kj} =$$
$$\sum_{k=j-1}^{i-1} a_{(i-1)k}b_{k(j-1)} - \sum_{k=j-1}^{i-1} ka_{(i-1)k}b_{kj} + \sum_{k=j}^{i-1} ka_{(i-1)k}b_{kj} =$$
$$\sum_{k=j-1}^{i-1} a_{(i-1)k}b_{k(j-1)} - (j-1)a_{(i-1)(j-1)}b_{(j-1)j} - \sum_{k=j}^{i-1} ka_{(i-1)k}b_{kj} + \sum_{k=j}^{i-1} ka_{(i-1)k}b_{kj} =$$
$$\sum_{k=j-1}^{i-1} a_{(i-1)k}b_{k(j-1)} = \sum_{k=j-2}^{i-2} a_{(i-2)k}b_{k(j-2)} = \ldots = \sum_{k=1}^{i-j+1} a_{(i-j+1)k}b_{k1} = c_{(i-j+1)1} = 0.$$

Therefore, $AB = C = I_{n\times n}$ and $B = A^{-1}$.

Lemma 2. Let $\{b_n\}$ and $\{t_n\}$ be two sequences of real numbers and

$$f(x) = \sum_{n=1}^{m} b_n x^{t_n},$$

where x is a positive real number. Then

$$\sum_{n=1}^{m} b_n t_n^k x^{t_n - 1} = \sum_{j=1}^{k} a_{kj} x^{j-1} f^{(j)}(x),$$

where a_{kj} is the entry in the kth row and the jth column in matrix A, k is a positive integer, and $f^{(j)}$ is j^{th} order derivative of f. In particular,

$$\sum_{n=1}^{m} b_n t_n^k = \sum_{j=1}^{k} a_{kj} f^{(j)}(1).$$

Proof. We proceed by induction on k, with the case $k = 1$ being trivial. Let $i > 1$ and the theorem is true for $k = i$. By the induction hypothesis

$$\sum_{n=1}^{m} b_n t_n^i x^{t_n - 1} = \sum_{j=1}^{i} a_{ij} x^{j-1} f^{(j)}(x).$$

multiplying the preceding equation by x gives

$$\sum_{n=1}^{m} b_n t_n^i x^{t_n} = \sum_{j=1}^{i} a_{ij} x^{j} f^{(j)}(x).$$

By differentiating the preceding equation with respect to x, we obtain

$$\sum_{n=1}^{m} b_n t_n^{i+1} x^{t_n - 1} = \sum_{j=1}^{i} ja_{ij} x^{j-1} f^{(j)}(x) + \sum_{j=1}^{i} a_{ij} x^{j} f^{(j+1)}(x)$$
$$= a_{i1} f'(x) \sum_{j=2}^{i} ja_{ij} x^{j-1} f^{(j)}(x) + \sum_{j=1}^{i} -1 a_{ij} x^{j} f^{(j+1)}(x) + a_{ii} x^{i} f^{(i+1)}(x)$$
$$= a_{(i+1)1} f'(x) \sum_{j=2}^{i} ja_{ij} x^{j-1} f^{(j)}(x) + \sum_{j=2}^{i} a_{i(j-1)} x^{j-1} f^{(j)}(x) + a_{(i+1)(i+1)} x^{i} f^{(i+1)}(x)$$
$$= a_{(i+1)1} f'(x) \sum_{j=2}^{i} (ja_{ij} + a_{i(j-1)}) x^{j-1} f^{(j)}(x) + a_{(i+1)(i+1)} x^{i} f^{(i+1)}(x)$$
$$= \sum_{j=1}^{i+1} a_{(i+1)j} x^{j-1} f^{(j)}(x).$$

By the above Lemma, we will calculate the generalized Wiener index in a fuzzy graph (See Theorem 2). For another application of the above lemmas, suppose that $\{b_n\}$ *and* $\{t_n\}$ are two sequences of real numbers and

$$f(x) = \sum_{n=1}^{m} b_n x^{t_n},$$

where x is a positive real number. In addition, Let $X = (f'(1), f''(1), \ldots, f^{(k)}(1))$ and

$$Y = (\sum_{n=1}^{m} b_n t_n, \sum_{n=1}^{m} b_n t_n^2, \ldots, \sum_{n=1}^{m} b_n t_n^k).$$

Then $X = A^{-1}Y$.

Now, we present two polynomials of the fuzzy graph and get some applications.

Definition 11. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and let the sequence

$$(d_1, d_2, \cdots, d_n) := (d_G(v_1), d_G(v_2), \cdots, d_G(v_n))$$

be fuzzy degree sequence of G such that $d_1 > d_2 > \ldots > d_n$. We define its *fuzzy degree polynomial* by:

$$S_G(x) := \sum_{i=1}^{n} x^{d_i}.$$

Lemma 3. Let $(d_1, d_2, \ldots, d_n)$ be the fuzzy degree sequence of fuzzy graph $G = (V, \sigma, \mu)$ such that $d_i \geq 1$. Then

$$\int_0^1 \frac{1}{x} S_G(x) dx = \sum_{i=1}^{n} \frac{1}{d_i}.$$

Proof. The proof of this fact is straightforward.

By the definition, if $G = (V, \sigma, \mu)$ be a fuzzy graph, then we have $S_G(1) = |V| = n$; and $S'_G(1) = 2S(G)$.

Theorem 1. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $|V_1| = n_1$ and $|V_2| = n_2$. Then,

1) $S_{G_1 \cup G_2}(x) = S_{G_1}(x) + S_{G_2}(x)$;

2) $S_{G_1 \vee G_2}(x) = x^{kn_2} S_{G_1}(x) + x^{kn_1} S_{G_2}(x)$;

3) $S_{G_1 \times G_2}(x) = S_{G_1}(x) \times S_{G_2}(x)$.

Proof. We prove these assertions directly.

1) $$S_{G_1 \cup G_2}(x) = \sum_{v \in V_1 \cup V_2} x^{d_{G_1 \cup G_2}(v)} = \sum_{v \in V_1} x^{d_{G_1}(v)} + \sum_{v \in V_2} x^{d_{G_2}(v)} = S_{G_1}(x) + S_{G_2}(x).$$

2) $$\begin{aligned} S_{G_1 \vee G_2}(x) &= \sum_{v \in V_1 \cup V_2} x^{d_{G_1 \vee G_2}(v)} = \sum_{v \in V_1} x^{d_{G_1}(v) + kn_2} + \sum_{v \in V_2} x^{d_{G_2}(v) + kn_1} \\ &= x^{kn_2} S_{G_1}(x) + x^{kn_1} S_{G_2}(x). \end{aligned}$$

3) $$\begin{aligned} S_{G_1 \times G_2}(x) &= \sum_{(a,b) \in V_1 \times V_2} x^{d_{G_1 \times G_2}(a,b)} = \sum_{(a,b) \in V_1 \times V_2} x^{d_{G_1}(a) + d_{G_2}(b)} \\ &= \sum_{a \in V_1,\, b \in V_2} x^{d_{G_1}(a)} \times x^{d_{G_2}(b)} = \sum_{a \in V_1} x^{d_{G_1}(a)} \times \sum_{b \in V_2} x^{d_{G_2}(b)} = S_{G_1}(x) \times S_{G_2}(x). \end{aligned}$$

Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph. Rosenfeld [19] has defined $\mu-$length of any $u-v$ path P as the sum of reciprocals of arc weights in P and distance between u and v called the $\mu-$distance denoted by $d_\mu(u,v)$, as the the smallest $\mu-$length of P. When G is a crisp graph, Rosenfeld's definition of distance between u *and* v is not the length of the shortest path. Here we introduce a new definition for distance between two vertices u *and* v in a fuzzy graph G.

Definition 12. Suppose that r is the length of the shortest path between two vertices $u, v \in V$. The distance between two vertices u *and* v of G is denoted by $d(u,v)$ and is defined as

$$d(u,v) = \min\{\sum_{i=0}^{r-1} \mu(u_i u_{i+1}); u_0 = u, \cdots, u_r = v\}.$$

By our definition, if $\mu(uv) = 1$ for all $u, v \in V$, then $d(u,v)$ is the length of the shortest path as in the crisp graph. In what follows, when G is a fuzzy graph we consider $d(u,v)$ according to the above equation.

We define Wiener index in the connected fuzzy graph $G = (V, \sigma, \mu)$ as

$$W(G)=\sum_{\{u,v\}\subseteq V} d(u,v).$$

We also define generalized Wiener index in the connected fuzzy graph as follows

$$W^k(G)=\sum_{\{u,v\}\subseteq V} d(u,v)^k.$$

Example 1. Let $F(S_n)$ be a fuzzy star graph of order n. Then $W\left(F\left(S_n\right)\right)=\left(n-1\right)S\left(F\left(S_n\right)\right)$.

Definition 13. Let $G=(V,\sigma,\mu)$ be a fuzzy graph. We define the polynomial $\mathcal{T}(G,x)$ as:

$$\mathcal{T}(G,x)=\sum_{\{u,v\}\subseteq V} x^{d(u,v)}.$$

Assume that $|V|=n,$ by the above definition we have:

$$\mathcal{T}(G,1)=\binom{n}{2} \text{ and } \mathcal{T}'(G,1)=W(G).$$

Example 2. Let $G=(V,\sigma,\mu)$ be a **CFG** of order n (See definition 7). Then

$$\mathcal{T}(G,x)=\sum_{\{u,v\}\subseteq V} x^k=\binom{n}{2}x^k.$$

Theorem 2. Let $G=(V,\sigma,\mu)$ be a connected fuzzy graph. Then

$$W^k(G)=\sum_{j=1}^{k} a_{kj}\mathcal{T}^{(j)}(G,1).$$

Proof. It follows from Lemma 2.

Theorem 3. Let $G_1=(V_1,\sigma_1,\mu_1)$ and $G_2=(V_2,\sigma_2,\mu_2)$ be two connected fuzzy graphs such that $|V_1|=p_1$ and $|V_2|=p_2$. Also assume that $d(u,v)\geq 2k$ for every $u,v\in V_1$ or $u,v\in V_2$. Then we have

$$\mathcal{T}(G_1\vee G_2,x)=\sum_{\substack{uv\in\mathcal{E}(G_1)\\ u,v\in V_1}} x^{\mu_{G_1}(u,v)}+(\binom{p_1}{2}-q_1)x^{2k}+\sum_{\substack{uv\in\mathcal{E}(G_2)\\ u,v\in V_2}} x^{\mu_{G_2}(u,v)}$$
$$+(\binom{p_2}{2}-q_2)x^{2k}+x^k p_1p_2$$

Proof. By the definition, we have

$$\mathcal{T}(G_1\vee G_2,x)=\sum_{\{u,v\}\subseteq V_1\cup V_2} x^{d_{G_1\vee G_2}(u,v)}=\sum_{\substack{uv\in\mathcal{E}(G_1)\\ u,v\in V_1}} x^{\mu_{G_1}(u,v)}+\sum_{\substack{uv\notin\mathcal{E}(G_1)\\ u,v\in V_1}} x^{2k}$$
$$+\sum_{\substack{uv\in\mathcal{E}(G_2)\\ u,v\in V_2}} x^{\mu_{G_2}(u,v)}+\sum_{\substack{uv\notin\mathcal{E}(G_2)\\ u,v\in V_2}} x^{2k}+\sum_{\substack{u\in V_1\\ v\in V_2}} x^k$$
$$=\sum_{\substack{uv\in\mathcal{E}(G_1)\\ u,v\in V_1}} x^{\mu_{G_1}(u,v)}+(\binom{p_1}{2}-q_1)x^{2k}+\sum_{\substack{uv\in\mathcal{E}(G_2)\\ u,v\in V_2}} x^{\mu_{G_2}(u,v)}$$
$$+(\binom{p_2}{2}-q_2)x^{2k}+x^k p_1p_2.$$

From the above theorem, we can compute the Wiener index in the fuzzy graph $G_1\vee G_2$.

$$\mathcal{T}'(G_1\vee G_2,x)=\sum_{\substack{uv\in\mathcal{E}(G_1)\\ u,v\in V_1}} \mu_{G_1}(u,v)x^{\mu_{G_1}(u,v)-1}+2k(\binom{p_1}{2}-q_1)x^{2k-1}$$
$$+\sum_{\substack{uv\in\mathcal{E}(G_2)\\ u,v\in V_2}} \mu_{G_2}(u,v)x^{\mu_{G_2}(u,v)-1}+2k(\binom{p_2}{2}-q_2)x^{2k-1}+kx^{k-1}p_1p_2.$$

So,

$$\mathcal{T}'(G_1 \vee G_2, 1) = W(G_1 \vee G_2) = S(G_1) + S(G_2) + 2k(\binom{p_1}{2} + \binom{p_2}{2} - q_1 - q_2) + kp_1p_2.$$

Corollary 1. When G_1 and G_2 are two connected crisp graphs, we have $S(G_1) = q_1$, $S(G_2) = q_2$ and $k = 1$. Therefore,

$$W(G_1 \vee G_2) = 2\binom{p_1}{2} + 2\binom{p_2}{2} + p_1p_2 - (q_1 + q_2).$$

The above result coincides with [12, Theorem 2].

Theorem 4. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two connected fuzzy graphs such that $|V_1| = p_1$ and $|V_2| = p_2$. Then

$$\mathcal{T}(G_1 \times G_2, x) = p_1\mathcal{T}_{G_2}(x) + p_2\mathcal{T}_{G_1}(x) + 2\mathcal{T}_{G_2}(x)\mathcal{T}_{G_1}(x)$$

Proof. It is easy to prove that

$$d_{G_1 \times G_2}((u,v),(u',v')) = d_{G_1}(u,u') + d_{G_2}(v,v').$$

So, by the definition we have

$$\begin{aligned}
\mathcal{T}(G_1 \times G_2, x) &= \sum_{(u,v),(u',v')\}\subseteq V_1\times V_2} x^{d_{G_1\times G_2}((u,v),(u',v'))} \\
&= \sum_{\{(u,v),(u',v')\}\subseteq V_1\times V_2} x^{d_{G_1}(u,u')+d_{G_2}(v,v')} = \sum_{\substack{u=u' \\ \{v,v'\}\subseteq V_2}} x^{d_{G_2}(v,v')} \\
&+ \sum_{\substack{v=v' \\ \{u,u'\}\subseteq V_1}} x^{d_{G_1}(u,u')} + \sum_{u\neq u', v\neq v'} x^{d_{G_1}(u,u')+d_{G_2}(v,v')} = p_1\mathcal{T}(G_2, x) \\
&+ p_2\mathcal{T}(G_1, x) + 2\sum_{\{u,u'\}\subseteq V_1} x^{d_{G_1}(u,u')} \times \sum_{\{v,v'\}\subseteq V_2} x^{d_{G_2}(v,v')} \\
&= p_1\mathcal{T}(G_2, x) + p_2\mathcal{T}(G_1, x) + 2\mathcal{T}(G_1, x) \times \mathcal{T}(G_2, x).
\end{aligned}$$

From the above theorem we have

$$\begin{aligned}
\mathcal{T}'(G_1 \times G_2, 1) &= W(G_1 \times G_2) = p_1W(G_2) + p_2W(G_1) + 2W(G_1)\binom{p_2}{2} \\
&+ 2W(G_2)\binom{p_1}{2} = p_1W(G_2) + p_2W(G_1) + W(G_1)(p_2^2 - p_2) + W(G_2)(p_1^2 - p_1) \\
&= p_1^2W(G_2) + p_2^2W(G_1).
\end{aligned}$$

By [12, Theorem 1], if G_1 and G_2 are two connected graphs, then

$$W(G_1 \times G_2) = p_1^2W(G_2) + p_2^2W(G_1).$$

So, we can see that our result is true in the crisp graphs.

Remark 1. Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two connected fuzzy graphs. By our definition of distance between two vertices in a fuzzy graph G, we have

$$d_{G_1 o G_2}((u,v)(u',v')) = \begin{cases} d_{G_1}(u,u'), & \text{if } v = v', \{u,u'\} \subseteq V_1 \\ d_{G_2}(v,v'), & \text{if } u = u', \{v,v'\} \subseteq V_2 \\ d_{G_1}(u,u'), & o.w \end{cases}.$$

Theorem 5. Let $G_1 = (V_1, \sigma_1, \mu_1)$ be a connected fuzzy graph and $G_2 = (V_2, \sigma_2, \mu_2)$ be a fuzzy graph such that $\mu_2(uv) > 0$ for every u, v in V_2. Then

$$W(G_1 o G_2) = p_2^2W(G_1) + p_1S(G_2).$$

Proof. By our definition, we have

$$\mathcal{T}(G_1oG_2,x)=\sum_{\{(u,v),(u',v')\}\subseteq V_1\times V_2} x^{d_{G_1oG_2}((u,v),(u',v''))} = \sum_{\substack{v=v'\\ \{u,u\}\subseteq V_1}} x^{d_{G_1}(u,u')} + \sum_{\substack{u=u'\\ \{v,v'\}\subseteq V_2}} x^{d_{G_1}(v,v')}$$

$$+2\sum_{\{u,u'\}\subseteq V_1,\ \{v,v'\}\subseteq V_2} x^{d_{G_1}(u,u')} = \sum_{\substack{v=v'\\ \{u,u'\}\subseteq V_1}} x^{d_{G_1}(u,u')} + \sum_{\substack{u=u'\\ vv'\in\mathcal{E}(G_2)}} x^{\mu_2(vv')}$$

$$+2\sum_{\{u,u'\}\subseteq V_1,\ \{v,v'\}\subseteq V_2} x^{d_{G_1}(u,u')}.$$

So,

$$\mathcal{T}'(G_1oG_2,1)=W(G_1oG_2)=p_2W(G_1)+p_1S(G_2)+2\binom{p_2}{2}W(G_1)=p_2^2W(G_1)+p_1S(G_2).$$

Example 3. Let $F(S_p)$ be a fuzzy star graphs and let $G=(V,\sigma,\mu)$ be a **CFG** of order n and vertex degrees k. Then

$$W(F(S_p)oG)=n^2(p-1)S(F(S_p))+pk\binom{n}{2}.$$

4. Conclusion

In this paper, the distance between two vertices in a fuzzy graph is defined in a different way. Also, some new degree-based fuzzy graph polynomials are introduced. By utilizing a special lower triangular matrix, the Wiener index and the generalized Wiener index of a fuzzy graph are computed, which coincide with the Wiener index and the generalized Wiener index in the crisp graph. The result is used to compute the Wiener index of the sum, products, and composition of two fuzzy graphs. It seems that by defining other suitable polynomials, it is possible to calculate other topological indices such as the Zagreb index in the fuzzy graph.

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