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Uncertain Location of Network Structured Production Units

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A B S T R A C T

Facility location problems are one of the most important issues for healthcare organizations and centers to achieve social welfare and respond to customer needs. Proper distribution of health and treatment facilities in cities is vital to minimize costs and improve the efficiency of health centers. The main contribution of the current article is dealing with the uncertainty issue in the p-median location-efficient problem. In this article, the p-median location problem along with network data envelopment analysis (Network DEA) is used in parallel mode to calculate the efficiency of health and treatment centers. In this issue, health centers are considered as parallel networks with two departments that operate independently. Due to the precision of the input and output values, triangular fuzzy numbers and the α-level fuzzy method have been used. The primary results that consider the uncertainty provide efficient solution and suggestions for the potential location of health centers in our case study.

1. Introduction

Today, proper distribution of health facilities in cities and improving the efficiency of health and treatment centers is an important issues for organizations and healthcare centers. This could be done to achieve social welfare and respond to the needs of customers. Therefore, it is very important to evaluate the performance of health service provider units as well as the location of these units to minimize costs. To improve the quality of these services and increase the efficiency of units, a method for optimal allocation of resources and facilities is needed. For this purpose, the problem of locating the p-median along with data envelopment analysis (DEA) is used to calculate the efficiency of health and treatment centers. DEA is a set of mathematical models based on linear programming, which is a non-parametric method for evaluating the efficiency or calculating the efficiency of a limited number of decision-making units or DMUs in multi-input and multi-output modes. DEA measures the relative efficiency of organizations, ranks them, identifies the strengths and weaknesses of each organization, and provides suggestions for improving the efficiency of each organization. In 1957, Farrell was

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the first to present a nonparametric method for determining efficiency in the case of two inputs and one output and presented the linear segment-segment convex hull method for boundary approximation. Farrell suggested that to determine efficiency, an efficiency frontier should first be defined and then the distance from the efficient frontier should be interpreted as a measure of inefficiency. In 1978, Charnes, Cooper, and Rhodes [5] extended Farrell's nonparametric method to a system with multiple inputs and outputs using mathematical programming and named it "DEA". So far, more than a thousand references, including articles, specialized reports, etc., have been published in this field, and the importance and capabilities of data coverage analysis are becoming more visible every day [4, 9]. Because conventional DEA models consider systems as a closed set and ignore the process within the system, performance, and relationships between them, Fare and Grosskopf [9] in an article pointing out the weakness of the conventional DEA model, introduced "envelopment analysis of network data" and expressed its importance in analyzing the performance of DMUs more precisely. In this model, a DMU is considered a network structure with all internal parts and relationships between them. Since in network models, the internal processes of each DMU can be examined, these models provide a more accurate picture of the efficiency of DMUs. In these models, the limitation of internal processes is added to the limitation of the total process, as a result, the total efficiency score will be lower than the efficiency score of internal processes. To calculate the efficiency of each DMU taking into account their internal processes, we will come across a network structure. Several models have been developed to measure the performance of a network system. Some models measure DMU and process efficiency simultaneously to derive mathematical relationships between them and identify the most effective way to improve the efficiency of a DMU based on them. In this research, we use Kao's parallel network model [13] and decompose the efficiency of each DMU into the efficiency of the internal processes of that DMU. In Kao's parallel network model [13], in each DMU, several parts operate independently and each one consumes several externally supplied inputs and produces several final outputs.

Crisp input and output data are fundamentally indispensable in conventional DEA. However, the observed values of the input and output data in real-world problems are sometimes imprecise or vague. For example, the data is not an exact value due to the uncertain environment or measurement inaccuracy or lack of knowledge and unavailability of data in practical problems. It is also possible that the level of service provided by the DMU cannot be evaluated by any criteria and can only be expressed subjectively with linguistic terms such as excellent, good, acceptable, and unacceptable. For this reason, fuzzy DEA models can play a more important role in evaluating the efficiency of health systems in real problems. The principle of fuzzy theory was first presented in 1965 by Professor Lotfi Asgarzadeh [24]. The use of fuzzy concepts allows us to introduce flexibility to achieve more practical solutions by incorporating concepts such as knowledge, experience, and human judgment into crisp classical models. Many researchers have proposed various fuzzy methods to deal with fuzzy and ambiguous data in DEA. Among the research conducted in the field of fuzzy efficiency of health care centers, the following can be mentioned. Arya and Yadav [2] proposed intuitionistic fuzzy slack-based measure models and intuitionistic fuzzy super-efficiency slack-based measure models to evaluate hospital efficiency, defuzzifying these models using a- and b-cuts. Ebrahimnejad [7] and Go´mez-Gallego et al. [11] have demonstrated the usefulness of stochastic DEA estimates such as fuzzy models by incorporating trapezoidal fuzzy variables into the health system efficiency estimates. In [11], the fuzzy DEA approach developed by Kao and Liu is used to evaluate the performance of a sample of European countries in the management of their health systems. Cinaroglu [6] has applied a multi-stage fuzzy stochastic method to evaluate the efficiency of the Turkish health system by comparing explicit and stochastic efficiency estimates with the combination of machine learning predictors. Amini and Rezaeenour [1] ranked the health centers of Golestan province using fuzzy hierarchical analysis and TOPSIS.

Recently, research has also been done in the field of network data envelopment analysis in fuzzy environments. For example, Nasseri and Niksefat [19] formulated a DEA model that handles the three-stage process and adverse outputs in a fuzzy stochastic environment. Tavassoli and Farzipoor [22] developed a new non-radial network data envelopment analysis (NDEA) model to evaluate railway performance in terms of overall efficiency (OE), technical efficiency (TE), service effectiveness (SE) and technical effectiveness (TEF).

Rostamy-Malkhalifeh et al. [21] have used all-fuzzy models of network data envelopment analysis (fuzzy input-output and fuzzy input price) to evaluate the efficiency of fuzzy revenue by using the function ranking method. Among the conducted research, no study has been conducted on network DEA of healthcare centers in a fuzzy environment. In this article, we consider the evaluation of the fuzzy network efficiency of healthcare centers. Also, to find the optimal location of health centers to achieve the minimum cost and the shortest possible time for customers, we use the p-median location problem. Recently, Azodi et al. [3] have conducted a study on the combination of positioning problems with data envelopment analysis with fuzzy variables. This study deals with the problem of customer allocation to facilities concerning some fuzzy parameters. Each facility has a fuzzy efficiency which is calculated by the data envelopment analysis method with fuzzy parameters. In the continuation of this study, by defining health care centers as parallel and independent networks, we will locate the p-median problem and calculate the efficiency of fuzzy network data envelopment analysis of health care centers. In this way, we show the values of inputs and outputs of each center with triangular fuzzy numbers and used of α-level set to convert triangular fuzzy numbers to interval numbers, then we use the extension principle Zadeh [25] to solve the model. Therefore, the contribution of this paper can be summarized as a theoretical empirical parts. From theoretical view, we deal with the uncertainty issue via fuzzy concept for the efficient-location problem. Empirically, we analyzed the possibility of location and relocation of health centers in Shahrood, considering the existing uncertainty in the analysis.

In Section 2, we present definitions and concepts of fuzzy sets. In Section 3, we formulate the parallel network data envelopment analysis model and specify the fuzzy input and output variables. In Section 4, we present the model of the P-median location problem. In Section 5, we will investigate the location of healthcare service centers in Shahrood City by determining the fuzzy network efficiency of these centers. The conclusion is given in Section 6.

2. Preliminaries

This section contains some basic concepts and definitions of fuzzy set theory and fuzzy ranking function.

2.1 Fuzzy sets and fuzzy numbers

The following fuzzy concepts are taken from [8,17].

Definition 1. Let X be the universal set. $\tilde{A}$ is called a fuzzy set in X if $\tilde{A}$ is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$; where $\mu_{\tilde{A}}(x)$ is the membership function of $x \in \tilde{A}$. Note that the membership function of $\tilde{A}$ is a characteristic (indicator) function for $\tilde{A}$ and it shows the degree of belonging $x\ to\ \tilde{A}$.

Definition 2. The α-level set of $\tilde{A}$ is the set $\tilde{A}_\alpha = \{x \in R | \mu_{\tilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0,1]$. The lower and upper bounds of α-level set of $\tilde{A}$ are finite numbers represented by inf $x \in \tilde{A}_\alpha$ and sup $x \in \tilde{A}_\alpha$, respectively.

Definition 3. The support of a fuzzy set $\tilde{A}$ is a set of elements in X for which $\mu_{\tilde{A}}(x)$ is positive, that is, supp $\tilde{A}_\alpha = \{x \in R | \mu_{\tilde{A}}(x) > 0\}$.

To calculate the α-level of the triangular fuzzy number $\tilde{A} = (a^1, a^2, a^3)$, we use the following equation:

$$a_\alpha^2 = [a_\alpha^1, a_\alpha^3] = [(a^2 - a^1)\alpha + a^1, -(a^3 - a^2)\alpha + a^3]$$

Definition 4. A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}\ \forall x, y \in X$ and $\lambda \in [0,1]$.

Definition 5. A convex fuzzy set $\tilde{A}$ on R is a fuzzy number if the following conditions hold:

I. Its membership function is piecewise continuous.

II. There exist three intervals $[a, b]$, $[b, c]$ and $[c, d]$ such that $\mu_{\tilde{A}}$ is increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

Definition 6. The fuzzy number $\tilde{A} = (a^1, a^2, a^3)$ is called a triangular fuzzy number with the membership function as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a^1}{a^2-a^1}, & a^1\le x\le a^2,\\ \dfrac{a^3-x}{a^3-a^2}, & a^2\le x\le a^3.\end{cases}$$

Definition 7. A triangular fuzzy number $\tilde{A}=(a^1,a^2,a^3)$ is said to be a non-negative (respectively, positive) fuzzy number if, and only if, $a^1\ge 0$, (respectively, $a^1>0$).

Definition 8. Let $\tilde{A}=(a^1,a^2,a^3)$ and $\tilde{B}=(b^1,b^2,b^3)$ be two non-negative triangular fuzzy numbers. The arithmetic on fuzzy numbers is defined as follows

I. $x\tilde{A}=(xa^1,xa^2,xa^3), \quad \forall x\in R;\quad x\ge 0,$
II. $x\tilde{A}=(xa^3,xa^2,xa^1), \quad \forall x\in R;\quad x<0,$
III. $\tilde{A}\oplus\tilde{B}=(a^1+b^1,a^2+b^2,a^3+b^3),$
IV. $\tilde{A}\ominus\tilde{B}=(a^1-b^3,a^2-b^2,a^3-b^1),$
V. $\tilde{A}\otimes\tilde{B}\approx(a^1b^1,a^2b^2,a^3b^3),\ \ a^1,b^1\ge 0,$
VI. $\frac{\tilde{A}}{\tilde{B}}\approx\left(\frac{a^1}{b^3},\frac{a^2}{b^2},\frac{a^3}{b^1}\right), \quad a^1\ge 0,\ b^1>0.$

2.2 Ranking function

Among the various fuzzy ranking methods, the most appropriate method is based on the concept of comparison of fuzzy numbers by using the ranking functions (see e.g. [10,18]). An efficient approach for ordering the elements of $F(\mathfrak{R})$ is to define a ranking function $R: F(\mathfrak{R})\to\mathfrak{R}$ which maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers on $\mathfrak{R}$, then the function R is called a fuzzy ranking function if

$$\tilde{A}\succeq\tilde{B} \iff R(\tilde{A})\ge R(\tilde{B}),$$
$$\tilde{A}\succ\tilde{B} \iff R(\tilde{A})>R(\tilde{B}),$$
$$\tilde{A}\simeq\tilde{B} \iff R(\tilde{A})=R(\tilde{B}).$$

In this article, we use the ranking function defined by Khodaqoli et al. [16] under the title of average rank for [a,b]. This ranking function is defined as follows:

$$R([a,b])=\frac{a+b}{2}.$$

3. DEA network parallel system with fuzzy parameters

To calculate the efficiency of each DMU, taking their internal processes, we will face a network structure. Several models have been developed to measure the performance of a network system. Some models measure DMU and process efficiency simultaneously to derive mathematical relationships between them and identify the most effective way to improve the efficiency of a DMU based on them. In this research, we use Kao's parallel network model [14] and decompose the efficiency of each DMU into the efficiency of the internal processes of that DMU. Due to the inaccuracy of measurement and the unavailability of data or the level of service provided by DMU cannot be evaluated with any criteria, we express the input and output values with triangular fuzzy numbers.

As shown in Figure 1, DMU_j has sections or q processes and each section applies the same inputs $\tilde{X}_i$, $i=1,\dots,m$ to produce identical outputs $\tilde{Y}_r$, $r=1,\dots,s$. The total inputs used by all q processes of the DMU_j are equal to $\sum_{k=1}^{q}\tilde{X}_{ij}^{(k)}=\tilde{X}_{ij}$. Also, the sum of all outputs produced by all q processes is equal to the system outputs, i.e. $\sum_{k=1}^{q}\tilde{Y}_{rj}^{(k)}=\tilde{Y}_{rj}$. In the parallel system, there are no intermediate products to connect the sections and all the sections operate independently.

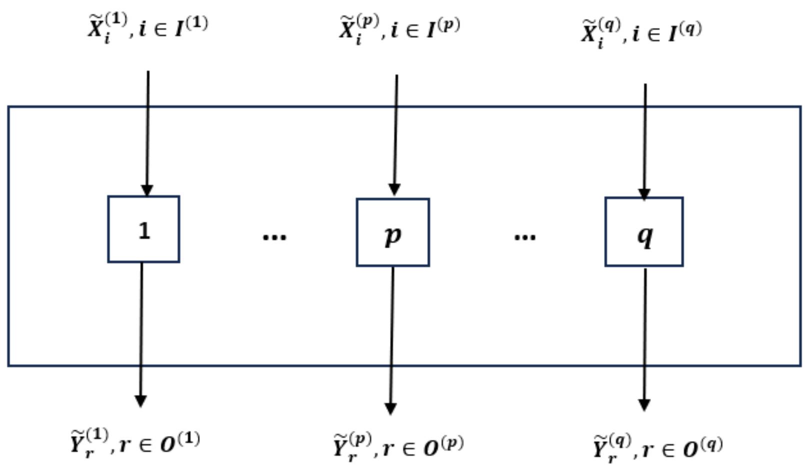

Figure 1. Structure of a parallel DMU with fuzzy parameters

To measure the relative efficiency of a DMU in a network environment, Kao and Lin [15] proposed a relational model that requires the same factor to have the same coefficient to maintain the relationship of the sections in the system. For example, the inputs of $\tilde{X}_{ij}$ have the same coefficient of v_i and the outputs of $\tilde{Y}_{rj}$ have the same coefficient of u_r. Since crisp values can be represented as degenerated fuzzy numbers with only one value in their domain, we will assume all observations to be fuzzy for ease of expression. Conceptually, the BCC input model with fuzzy observations can be formulated as:

$$P(1): \quad \tilde{E}_k = \max \sum_{r=1}^{s} u_r \tilde{Y}_{rk}$$

$$\begin{aligned} s.t. \quad & \sum_{i=1}^{m} v_i \tilde{X}_{ik} = 1 \\ & \sum_{r=1}^{s} u_r \tilde{Y}_{rj} - \sum_{i=1}^{m} v_i \tilde{X}_{ij} \leq 0 \qquad j = 1, \dots, n \\ & \sum_{r \in O^{(p)}} u_r \tilde{Y}_{rj}^{(p)} - \sum_{i \in I^{(p)}} v_i \tilde{X}_{ij}^{(p)} \leq 0 \quad p = 1, \dots, q, \quad j = 1, \dots, n \\ & u_r, v_i \leq \varepsilon, \qquad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned}$$

Since the observations $\tilde{X}_{ik}$ and $\tilde{Y}_{rk}$ are fuzzy numbers, the resulting efficiency $\tilde{E}_k$ is also a fuzzy number.

In model $P(1)$, by placing triangular fuzzy numbers instead of inputs and outputs, we have:

$$P(2): \quad \tilde{E}_k = \left(E_k^1, E_k^2, E_k^3\right) = \max \sum_{r=1}^{s} u_r \left(Y_{rk}^1, Y_{rk}^2, Y_{rk}^3\right)$$

$$\begin{aligned} s.t. \quad & \textstyle\sum_{i=1}^{m} v_i \left(X_{ik}^1, X_{ik}^2, X_{ik}^3\right) = 1 \\ & \textstyle\sum_{r=1}^{s} u_r \left(Y_{rj}^1, Y_{rj}^2, Y_{rj}^3\right) - \sum_{i=1}^{m} v_i \left(X_{ij}^1, X_{ij}^2, X_{ij}^3\right) \leq 0 \qquad j = 1, \dots, n \\ & \textstyle\sum_{r \in O^{(p)}} u_r \left(Y_{rj}^{1(p)}, Y_{rj}^{2(p)}, Y_{rj}^{3(p)}\right) - \sum_{i \in I^{(p)}} v_i \left(X_{ij}^{1(p)}, X_{ij}^{2(p)}, X_{ij}^{3(p)}\right) \leq 0 \quad p = 1, \dots, q, \quad j = 1, \dots, n \\ & u_r, v_i \leq \varepsilon, \qquad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned}$$

3.1 Fuzzy approach for solving the problem

Considering that the fuzzy number $\tilde{E}_k$ in model $P(2)$ is the result of mathematical operations of other fuzzy numbers, for example, $\tilde{X}_{ik}$ and $\tilde{Y}_{rk}$, the membership function $\tilde{E}_k$ can be obtained through the expansion principle of the membership function $\tilde{X}_{ik}$ and $\tilde{Y}_{rk}$ (Presented by Zadeh and Yager [23]) as follows

$$\tilde{\mu}_{\tilde{E}_k}(e) = sup_{x,y}\ min\left\{\tilde{\mu}_{\tilde{X}_{ij}}\left(x_{ij}\right), \tilde{\mu}_{\tilde{Y}_{rj}}\left(y_{rj}\right), \forall i,j,r \mid e = E_k(x,y)\right\}$$

where $E_k(x,y)$ is the conventional DEA model for measuring efficiency using observations of (x,y).

Using Kao and Liu's [15] two-level mathematical programming model to find the fuzzy efficiency $\tilde{E}_k$, we provide the following solution.

The above relationship states that given a set of the (x,y) observations that can yield the efficiency largest of the minimum membership grades corresponding to all the sets of (x,y) observations that can produce the efficiency score e. If we assume $\tilde{\mu}_{\tilde{E}_k}(e) = \alpha$, for those sets of Special Types of Data (x,y) observations that have an efficiency score e, one must have $\tilde{\mu}_{\tilde{X}_{ij}}\left(x_{ij}\right) \geq \alpha$, $\tilde{\mu}_{\tilde{Y}_{rj}}\left(y_{rj}\right) \geq \alpha$.

Using α-level triangular fuzzy numbers $[(X_{ij})_\alpha^L, (X_{ij})_\alpha^U]$, $[(Y_{rj})_\alpha^L, (Y_{rj})_\alpha^U]$, $[(X_{ij}^{(p)})_\alpha^L, (X_{ij}^{(p)})_\alpha^U]$, $[(Y_{rj}^{(p)})_\alpha^L, (Y_{rj}^{(p)})_\alpha^U]$ for $p = 1, \dots, q$, We can calculate the α-level triangular fuzzy number $\tilde{E}_k$. Since all values in the α-cut have a membership grade greater than or equal to α, $(E_k)_\alpha^L$ and $(E_k)_\alpha^U$ are the smallest and largest efficiency values, respectively, that have a membership grade α calculated from the values of $X_{ij}, X_{ij}^{(p)}$ $(p = 1, \dots, q)$ and Y_{rj}, $Y_{rj}^{(p)}$ $(p = 1, \dots, q)$ in their respective α-cuts, with at least one occurring at the lower or upper bound of the α-cut, to have a membership grade α. Thus we have:

$$\begin{aligned}
(E_k)_\alpha^L = \min\ & E_k(x,y) \\
s.t.\ & \left(X_{ij}\right)_\alpha^L \leq x_{ij} \leq \left(X_{ij}\right)_\alpha^U, \quad \forall\, i,j \\
& \left(Y_{rj}\right)_\alpha^L \leq y_{ij} \leq \left(Y_{rj}\right)_\alpha^U, \quad \forall\, r,j \\
& \left(X_{ij}^{(p)}\right)_\alpha^L \leq x_{ij}^{(p)} \leq \left(X_{ij}^{(p)}\right)_\alpha^U, \quad \forall\, i,j, \quad \forall\, p = 1,\dots,q \\
& \left(Y_{rj}^{(p)}\right)_\alpha^L \leq y_{rj}^{(p)} \leq \left(Y_{rj}^{(p)}\right)_\alpha^U, \quad \forall\, r,j, \ \ \forall\, p = 1,\dots,q
\end{aligned}$$

$$\begin{aligned}
(E_k)_\alpha^U = \max\ & E_k(x,y) \\
s.t.\ & \left(X_{ij}\right)_\alpha^L \leq x_{ij} \leq \left(X_{ij}\right)_\alpha^U, \quad \forall\, i,j \\
& \left(Y_{rj}\right)_\alpha^L \leq y_{ij} \leq \left(Y_{rj}\right)_\alpha^U, \quad \forall\, r,j \\
& \left(X_{ij}^{(p)}\right)_\alpha^L \leq x_{ij}^{(p)} \leq \left(X_{ij}^{(p)}\right)_\alpha^U, \quad \forall\, i,j, \quad \forall\, p = 1,\dots,q \\
& \left(Y_{rj}^{(p)}\right)_\alpha^L \leq y_{rj}^{(p)} \leq \left(Y_{rj}^{(p)}\right)_\alpha^U, \quad \forall\, r,j, \ \ \forall\, p = 1,\dots,q
\end{aligned}$$

Similar to the case of the interval data, in which the minimum and maximum efficiencies occur at the least and most favorable conditions, respectively, the former occurs at the largest inputs and smallest outputs of the DMU being evaluated, the smallest inputs and largest outputs of the other DMUs, and the latter occurs at the smallest inputs and largest outputs of the DMU being evaluated, and the largest inputs and smallest outputs of the other DMUs.

Therefore, model $P(2)$ is divided into two sub-models to calculate α-level $[(E_k)_\alpha^L, (E_k)_\alpha^U]$ as follows:

$$P(3): \qquad (E_k)_\alpha^L = \max \sum_{r=1}^{s} u_r (Y_{rk})_\alpha^L$$

$$s.t. \quad \sum_{i=1}^{m} v_i (X_{ik})_\alpha^U = 1$$

$$\sum_{r=1}^{s} u_r (Y_{rk})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ik})_\alpha^U \le 0$$

$$\sum_{r=1}^{s} u_r \left(Y_{rj}\right)_\alpha^U - \sum_{i=1}^{m} v_i \left(X_{ij}\right)_\alpha^L \le 0 \qquad j = 1,\dots,n$$

$$\sum_{r\in O^{(p)}} u_r \left(Y_{rj}^{(p)}\right)_\alpha^U - \sum_{i\in I^{(p)}} v_i \left(X_{ij}^{(p)}\right)_\alpha^L \le 0 \quad p = 1,\dots,q, \quad j = 1,\dots,n$$

$$u_r, v_i \le \varepsilon, \qquad r = 1,\dots,s, \quad i = 1,\dots,m.$$

$$P(4): \qquad (E_k)_\alpha^U = \max \sum_{r=1}^{s} u_r (Y_{rk})_\alpha^U$$

$$s.t. \quad \sum_{i=1}^{m} v_i (X_{ik})_\alpha^L = 1$$

$$\sum_{r=1}^{s} u_r (Y_{rk})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ik})_\alpha^L \le 0$$

$$\sum_{r=1}^{s} u_r \left(Y_{rj}\right)_\alpha^L - \sum_{i=1}^{m} v_i \left(X_{ij}\right)_\alpha^U \le 0 \qquad j = 1,\dots,n$$

$$\sum_{r\in O^{(p)}} u_r \left(Y_{rj}^{(p)}\right)_\alpha^L - \sum_{i\in I^{(p)}} v_i \left(X_{ij}^{(p)}\right)_\alpha^U \le 0 \quad p = 1,\dots,q, \quad j = 1,\dots,n$$

By solving models $P(3)$ and $P(4)$, we calculate $[(E_k)_\alpha^L, (E_k)_\alpha^U]$, then using the ranking function provided by Khodaqoli et al. [16], we calculate the $[(E_k)_\alpha^L, (E_k)_\alpha^U]$ rank as follow:

$$R([(E_k)_\alpha^L, (E_k)_\alpha^U]) = \frac{(E_k)_\alpha^L + (E_k)_\alpha^U}{2}.$$

4. P-median problem

The p-median problem seeks to find the p midpoint among the candidate points for the facility so that the total costs be minimized. The goal of p-median problems is to find the location of p facilities so that the weighted sum of the distance of customers to the nearest facility is minimized.

To formulate p-median problems, we introduce the following notation:

I: the set of demand points,

J: set of possible locations for the facilities,

d_{ij} : distance between customer i and potential facility j,

p: total number of facilities to be located,

w_i : weight associated with each demand point.

We define the allocation decisions, namely which facility j satisfies the demand expressed by customer i, through the following x-variables:

$$x_{ij} = \begin{cases} 1 & if\ demand\ point\ i\ is\ allocated\ to\ facility\ j \\ 0 & otherwise, \end{cases} \qquad \forall\ i, j\ \in J,$$

and location decisions are represented with:

$$y_j = \begin{cases} 1 & if\ a\ facility\ is\ located\ at\ point\ j \\ 0 & otherwise, \end{cases} \qquad \forall\ j \in J.$$

The p-median problems are formulated by Hakimi [12], ReVelle, and Swain [20] as follows:

$$\text{P(5):} \quad \min \sum_{i \in I, j \in J} w_i d_{ij} x_{ij}$$

$$s.t. \quad \sum_{j \in J} x_{ij} = 1 \qquad \forall i\ \in I,$$

$$x_{ij} \leq y_j \qquad \forall i\ \in I,\ \forall j\ \in J,\ \ i \neq j$$

$$\sum_{j \in J} y_j = p$$

$$x_{ij}, y_j \in \{0,1\} \qquad \forall i\ \in I,\ \forall j\ \in J\ .$$

In this model, the first constraints ensure that all the demand points are allocated. The second constraints guarantee that a point receives allocation only if it is a plant. The third constraint fixes the number of plants to p. The last constraints states that all variables are binary.

5. Locating and analyzing the efficiency of health centers in Shahrood city

Shahrood City is located in Semnan province and has 11 health centers that cover a population of 165,789 people. We consider each of these centers as a decision-making unit (DMU), which includes two healthcare departments and doctors who provide services to clients independently in each DMU, there is an input for each department, which produces different outputs. The input of each department is the space allocated to personnel. Since the space required by the personnel of each department is not precisely known, therefore, we consider these values as fuzzy numbers. Also, the output produced by each department is the level of customer satisfaction. Because customer satisfaction is expressed qualitatively in terms of excellent, good, acceptable, or poor, therefore, these values are also shown in the form of triangular fuzzy numbers.

In this work, we have used the p-median problem for locating the health centers of Shahrood City to select five centers as health service centers to minimize the weighted sum of the centers, Also, maximize the total efficiency of the centers. Considering the efficiency in locating the p-median problem of health centers helps the decision-maker to reach more effective information and better analysis of the problem, which is realized by Data Envelopment Analysis methods. Since the conventional DEA methods ignore the efficiency of internal processes in calculating the total efficiency, therefore, in this article, we have used the Network Data Envelopment Analysis method. In this method, the efficiency of the internal parts of each base affects the efficiency of the whole center, and by knowing the strengths and weaknesses of each part, the efficiency of each center can be increased. Table 1 shows the fuzzy input and output values of each of the DMUs and departments.

Table 1. Fuzzy input and output values of each of the DMUs and departments

Output	Input	Departments	DMUs
(0.5,0.7,0.8)	(30,40,50)	Health Care	A
(0.9,1,1)	(10,30,50)	Doctor	
(0.9,1,1)	(35,65,80)	Health Care	B
(1,1,1)	(20,40,50)	Doctor	
(0.7,0.9,1)	(10,25,55)	Health Care	C
(0.6,0.8,1)	(15,25,35)	Doctor	
(1,1,1)	(7,18,30)	Health Care	D
(0.1,0.2,0.4)	(5,10,15)	Doctor	
(0,0.3,0.5)	(17,30,48)	Health Care	E
(0,0.3,0.4)	(5,12,25)	Doctor	
(1,1,1)	(40,60,80)	Health Care	F
(0.5,0.7,0.8)	(25,35,50)	Doctor	
(0.6,0.7,0.8)	(7,18,30)	Health Care	G
(0.5,0.6,0.8)	(5,10,15)	Doctor	
(0.8,0.9,1)	(25,40,90)	Health Care	H
(0.7,0.9,1)	(20,37,40)	Doctor	
(0.7,0.8,1)	(40,55,90)	Health Care	I
(0.7,0.8,0.9)	(10,16,24)	Doctor	
(0.4,0.5,0.6)	(7,18,35)	Health Care	J
(0.4,0.5,0.7)	(5,10,35)	Doctor	
(0.4,0.6,0.7)	(12,27,35)	Health Care	K
(0.5,0.6,0.7)	(4,8,24)	Doctor	

By solving models $P(3)$ and $P(4)$, we calculate $[(E_k)_\alpha^L, (E_k)_\alpha^U]$ .Table 2 shows the efficiency values of each of the DMUs for different alphas. In Figure 2, the comparison between the efficiency values in the α-level method is done.

Table 2. Efficiency values of each DMU for different alpha

Efficiency			DMUs
$\alpha = 0.9$	$\alpha = 0.5$	$\alpha = 0.2$	
0.3	0.44	0.46	A
0.32	0.34	0.51	B
0.41	0.59	0.56	C
0.53	0.64	0.6	D
0.16	0.3	0.43	E
0.21	0.31	0.37	F
0.57	0.65	0.6	G
0.28	0.42	0.54	H
0.26	0.4	0.49	I
0.42	0.57	0.54	J
0.42	0.6	0.56	K

Figure 2. Comparison between efficiency values in α-level method

The advantage of α-level method compared to other fuzzy methods is that in this method, you can choose the best option based on different answers for different alphas.

After calculating the efficiency of the health centers, we set the values of the efficiencies equal to the w_i weights assigned to each center in model P(5) and solve the p-median problem to find the optimal solution. Table 3 shows the distance between health centers. The optimal solution of the p-median problem with efficiency values $\alpha = 0.2$ equal to C, G, I, J, K. As you can see, the health centers with the highest efficiency have been selected to provide health services with the best performance and lowest cost and access time to customers.

Table 3. Distance between centers

K	J	I	H	G	F	E	D	C	B	A	
2.7	2.2	1.7	2.7	3.6	2.3	2.1	4.3	2	3.5	0	**A**
2.7	5.2	3	1.8	6.4	4	4.8	2.6	3	0	3.5	**B**
3.4	3.1	2.8	1.7	4.1	3.7	3.6	4.6	0	3	2	**C**
2.4	6	3.3	3.7	7.4	3.7	4.9	0	4.6	2.6	4.3	**D**
3	2.2	2.3	4.1	3.6	1.8	0	4.9	3.6	4.8	2.1	**E**
1.9	3.3	1.5	3.8	4.8	0	1.8	3.7	3.7	4	2.3	**F**
5.6	2	4.7	5.3	0	4.8	3.6	7.4	4.1	6.4	3.6	**G**
2.9	4.2	2.7	0	5.3	3.8	4.1	3.7	1.7	1.8	2.7	**H**
1.4	3.2	0	2.7	4.7	1.5	2.3	3.3	2.8	3	1.7	**I**
4.1	0	3.2	4.2	2	3.3	2.2	6	3.1	5.2	2.2	**J**
0	4.1	1.4	2.9	5.6	1.9	3	2.4	3.4	2.7	2.6	**K**

6. Conclusion

In this article, to properly distribute health and treatment facilities in cities to minimize costs and improve the efficiency of health centers, we used the p-median location problem along with network data envelopment analysis (Network DEA). For this purpose, we considered health centers as DMUs with parallel networks, so that each DMU includes two healthcare departments and doctors that operate independently. Due to the accuracy of the input and output values, we have used triangular fuzzy numbers. To calculate the efficiency of each center, α-level fuzzy method and the extension principle Zadeh have been used. By putting efficiency values instead of w_i weights in the pi-median problem, the optimal solution to the problem was obtained. Solving the -p-median problem by considering the efficiency of DMUs makes it possible to choose health centers with the highest efficiency and provide health services with the lowest cost and access time to their customers. We performed our case study with the most possible data that we accessed. For having more deep analysis we needed more data that we could not find properly which was the main limitation of our study. In many real-world problems including ours, we may face different types of uncertainty in different data sets that should be considered. We left it for future research where we get more access and more information about them. This could be a potential and interesting future research line.

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References

1. Amini, F., & Rezaeenour, J. (2016). Ranking healthcare centers using fuzzy analytic hierarchy process and TOPSIS: Iranian experience. *International Journal of Operational Research*, *6*(1), 25-39.
2. Arya, A., & Yadav, S. P. (2018). Development of intuitionistic fuzzy super-efficiency slack based measure with an application to health sector. *Computers & Industrial Engineering*, *115*, 368-380.
3. Azodi, A., Fathali, J., Ghiyasi, M., & Sayar, T. (2023). Fuzzy balanced allocation problem with efficiency on facilities. *Soft Computing*, *27*(10), 6573-6586.
4. Castelli, L., Pesenti, R., & Ukovich, W. (2004). DEA-like models for the efficiency evaluation of hierarchically structured units. *European journal of operational research*, *154*(2), 465-476.
5. Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429-444.
6. Cinaroglu, S. (2023). Fuzzy efficiency estimates of the Turkish Health System: A comparison of interval, Bias-Corrected, and fuzzy data envelopment analysis. *International Journal of Fuzzy Systems*, *25*(6), 2356-2379.
7. Ebrahimnejad, A. (2012). Cost efficiency measures with trapezoidal fuzzy numbers in data envelopment analysis based on ranking functions: application in insurance organization and hospital. *International Journal of Fuzzy System Applications (IJFSA)*, *2*(3), 51-68.
8. Ebrahimnejad, A., & Amani, N. (2021). Fuzzy data envelopment analysis in the presence of undesirable outputs with ideal points. *Complex & intelligent systems*, *7*(1), 379-400.
9. Färe, R., Grosskopf, S., & Whittaker, G. (2007). Network dea. *Modeling data irregularities and structural complexities in data envelopment analysis*, 209-240.
10. Garcia-Aguado, C., & Verdegay, J. (1993). On the sensitivity of membership functions for fuzzy linear programming problems. *Fuzzy sets and systems*, *56*(1), 47-49.
11. Gómez-Gallego, J. C., Gómez-Gallego, M., García-García, J. F., & Faura-Martinez, U. (2021). Evaluation of the efficiency of European health systems using fuzzy data envelopment analysis. Healthcare,
12. Hakimi, S. L. (1964). Optimum locations of switching centers and the absolute centers and medians of a graph. *Operations research*, *12*(3), 450-459.
13. Kao, C. (2009a). Efficiency decomposition in network data envelopment analysis: A relational model. *European journal of operational research*, *192*(3), 949-962.
14. Kao, C. (2009b). Efficiency measurement for parallel production systems. *European journal of operational research*, *196*(3), 1107-1112.
15. Kao, C., & Liu, S.-T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy sets and systems*, *113*(3), 427-437.
16. Khodagholi, M., Dolati, A., & Hoseinzadeh, A. (2018). The solving an inverse 1-median problem by using alpha-cut fuzzy. *Journal of Decisions and Operations Research*, *3*(1), 58-71.
17. Mahdavi-Amiri, N., & Nasseri, S. (2006). Duality in fuzzy number linear programming by use of a certain linear ranking function. *Applied mathematics and computation*, *180*(1), 206-216.

18. Maleki, H. (2002). Ranking functions and their applications to fuzzy linear programming. *Far East Journal of Mathematical Sciences*, *4*(3), 283-302.
19. Nasseri, S. H., & Niksefat Dogori, P. (2022). A Three-Stage Process for Fuzzy Stochastic Network Data Envelopment Analysis Models. *Iranian Journal of Operations Research*, *13*(1), 184-197.
20. ReVelle, C. S., & Swain, R. W. (1970). Central facilities location. *Geographical analysis*, *2*(1), 30-42.
21. Rostamy-Malkhalifeh, M., Poudineh, E., & Payan, A. (2018). A fully fuzzy method of network data envelopment analysis for assessing revenue efficiency based on ranking functions. *Control and Optimization in Applied Mathematics*, *3*(2), 77-96.
22. Tavassoli, M., & Saen, R. F. (2022). A new fuzzy network data envelopment analysis model for measuring efficiency and effectiveness: assessing the sustainability of railways. *Applied Intelligence*, *52*(12), 13634-13658.
23. Yager, R. R. (1986). A characterization of the extension principle. *Fuzzy sets and systems*, *18*(3), 205-217.
24. Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.
25. Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, *1*(1), 3-28.