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A Method for Solving Linear Systems of Fuzzy Differential Equations under Generalized Hukuhara Differentiability

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ABSTRACT

The paper proposes a procedure for solving a linear system of fuzzy differential equations from the point of view of the generalized Hukuhara derivative. First, the method is based on two functions of half-length and midpoint of fuzzy numbers and next it is implemented on the problem in two separate cases of generalized Hukuhara differentiability, in details. Two numerical examples are given to clarify the practical application of the results.

1. Introduction

Many phenomena in the world, whose parameters have ambiguity and inaccuracy, will be modelled as fuzzy differential equations (see [11, 17, 19, 20, 21] for example). The use of a suitable concept of the derivative in the fuzzy environment in order to solve a fuzzy differential equation has always been researched [2, 3, 10, 13, 15, 22, 23]. The generalized Hukuhara differentiability (GH-differentiability) concept proposed in [4, 5] is a powerful tool for interpreting fuzzy differential equations, so that usually two solutions with different behaviour of a fuzzy differential equation are sought [1, 6, 7, 8, 9].

In this paper, we study the following fuzzy linear differential equations system:

$$\begin{cases} x'(t) = Ax(t) + b(t), & t > 0, \\ x(0) = x_0 \end{cases} \tag{1}$$

where A is a real $n \times n$ matrix, x_0 is a vector containing fuzzy numbers and $b(t)$ is a vector containing fuzzy-valued functions. The system (1) in case $n = 1$ has been studied by many authors [1, 7, 14, 16, 22].

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In the present paper, first, we present a result of the effect of the GH-differentiability of the function $b(t)$ on the GH-differentiability of unknown function $x(t)$. Next, we introduce a method to solve system (1) based on the GH-differentiability concept. In the proposed method, we first introduce two operators that are called the midpoint and haft-length functions defined on the set of fuzzy numbers and next by using these operators, we convert system (1) into two systems of ordinary differential equations, considering two cases 1. The components of vector $x(t)$ are (1)-differentiable fuzzy-valued functions and 2. The components of vectors $x(t)$ are (2)-differentiable fuzzy-valued functions. Also, we present the necessary and sufficient conditions for the results of the proposed method to lead to the solution of the main system (1). Based on solutions obtained from the designed ordinary differential equations systems, we can form the structure of the solution as a vector of fuzzy-valued functions which is illustrated by solving two numerical examples.

2. Preliminaries

In this section, we recall some basic definitions of fuzzy arithmetic and some necessary results of generalized differentiability of fuzzy-valued functions.

Definition 1. [24] A fuzzy number u is an ordered pair of functions $u=(u_\alpha^-,u_\alpha^+)$ on interval $[0,1]$, as parametric form such that u_α^- is bounded, non-decreasing left continuous function in $(0,1]$ and right continuous at 0, u_α^+ is bounded, non-increasing left continuous function in $(0,1]$ and right continuous at 0 and $u_\alpha^- \le u_\alpha^+$ for all $\alpha \in [0,1]$.

The set of all fuzzy numbers is denoted by R_F. For $u,v \in R_F$ and $\lambda \in R$, the operations addition and multiplication are defined [24] by $u+v=(u_\alpha^-+v_\alpha^-,u_\alpha^++v_\alpha^+)$ and $\lambda u=(\lambda u_\alpha^-,\lambda u_\alpha^+)$ if $\lambda \ge 0$, and $\lambda u=(\lambda u_\alpha^+,\lambda u_\alpha^-)$ if $\lambda<0$.

Definition 2. [24]. Let $u,v \in R_F$. If there exists $w \in R_F$ such that, $u=v+w$, then w is called the Hukuhara difference (H-difference) of u,v and it is denoted as $w=u\Theta v$.

Definition 3. [24] The Hausdorff metric in R_F is defined as $D:R_F \times R_F \to [0,+\infty)$, by

$$D(u,v)=\sup_{0\le\alpha\le1} \max\{|u_\alpha^- - v_\alpha^-|,|u_\alpha^+ - v_\alpha^+|\}.$$

A fuzzy-valued function f is denoted by $f(t)=(f_\alpha^-(t), f_\alpha^+(t))$ for each t in the domain of f.

Definition 4. [5] Let I be an open interval in R and let $f:I\to R_F$. We say that f is GH-differentiable at point $t\in I$, if

(1) There exists an element $f'(t)\in R_F$ such that for all $h>0$ sufficiently close to 0, the H-differences $f(t+h)\Theta f(t)$, $f(t)\Theta f(t-h)$ exist and the limits (in the metric D),

$$\lim_{h\to0^+}\frac{f(t+h)\Theta f(t)}{h}=\lim_{h\to0^+}\frac{f(t)\Theta f(t-h)}{h}=f'(t).$$

(2) There exists an element $f'(t)\in R_F$ such that for all $h>0$ sufficiently close to 0, the H-differences $f(t)\Theta f(t+h)$, $f(t-h)\Theta f(t)$ exist and the limits (in the metric D),

$$\lim_{h\to0^+}\frac{f(t)\Theta f(t+h)}{-h}=\lim_{h\to0^+}\frac{f(t-h)\Theta f(t)}{-h}=f'(t).$$

We fixed $I=(0,T)$ and for simplicity, we say that f is (j)-differentiable at $t\in I$, if it is GH-differentiable in the sense of (j) of Definition 4, for $j\in\{1,2\}$.

Theorem 1. [12, 14] Let $f:I\to R_F$. Then,

(1) If f is (1)-differentiable on I, then the functions $f_\alpha^-(t)$ and $f_\alpha^+(t)$ are differentiable at $t\in I$ and

$$f'(t)=((f_\alpha^-)'(t),(f_\alpha^+)'(t)),\quad \forall\alpha\in[0,1].$$

(2) If f is (2)-differentiable on I, then the functions $f_\alpha^-(t)$ and $f_\alpha^+(t)$ are differentiable at $t\in I$ and

$$f'(t)=((f_\alpha^+)'(t),(f_\alpha^-)'(t)),\quad \forall\alpha\in[0,1].$$

Theorem 2. [4, 7] Let $j\in\{1,2\}$ be fixed and $\lambda\in R$. If $f,g:I\to R_F$ are (j)-differentiable on I, then $f+\lambda g$ is (j)-differentiable and we have

$$(f+\lambda g)'(t)=f'(t)+\lambda g'(t),\quad \forall t\in I.$$

Definition 5. Let $j\in\{1,2\}$ be fixed. Let $f:I\to R_F$ be the fuzzy-valued vector function $f(t)=(f_1(t),f_2(t),\ldots,f_n(t))$. We say that f is (j)-differentiable on I if each one of functions $f_i(t)$ is (j)-differentiable on I, for $i\in\{1,2,\ldots,n\}$. And, we define

$$f'(t)=(f_1'(t),f_2'(t),\ldots,f_n'(t)).$$

We now give a result of the effect of the GH-differentiability of function $b(t)$ on the GH-differentiability of the solution function $x(t)$ of system (1).

Theorem 3. Let $j\in\{1,2\}$ be fixed. Let $x(t)$ be (j)-differentiable solution of system (1). If $b(t)$ is (j)-differentiable on I of arbitrary integer order $n\geq 1$, then $x(t)$ is (j)-differentiable on I of order n.

Proof:

Since $x(t)$ and $b(t)$ are both (j)-differentiable on I, then, by Definition 5 and Theorem 2, we get $Ax(t)+b(t)$ is (j)-differentiable and further

$(Ax(t)+b(t))'=Ax'(t)+b'(t)$.

Since $x(t)$ satisfies the system (1), then $x'(t)$ is (j)-differentiable on I and we get

$x''(t)=Ax'(t)+b'(t)$,

which follows that

$x''(t)=A^2x(t)+Ab(t)+b'(t)$.

Similarly, since, for integer number $n\geq 2$, $b(t)$ is (j)-differentiable of order $n-1$, we infer that the (j)-derivative of $x(t)$ of order n, exists and it is as follows:

$$x^{(n)}(t)=A^{(n)}x(t)+\sum_{k=1}^{n}A^{n-k}b^{(k-1)}(t),$$

which ends the proof. □

Remark 1. As a conclusion of Theorem 3, if the GH-derivative type of $b(t)$ is the same as the GH-derivative type of the solution function $x(t)$ in the interval under study, then the system (1) will have more flexibility in the sense that it allows us to take derivatives from sides of equality in system (1).

3. Solution Method to Solve the Linear System of Fuzzy Differential Equations

We are going to detail the method to solve system (1).

Definition 6. Let $u\in R_F$ be a fuzzy number. We define the midpoint function $M_\cdot(u):[0,1]\to R$ and the half-length function $L_\cdot(u):[0,1]\to[0,+\infty)$ as follows:

$$M_\alpha(u):=(Mu)(\alpha)=\frac{u_\alpha^+ + u_\alpha^-}{2},\qquad L_\alpha(u):=(Lu)(\alpha)=\frac{u_\alpha^+ - u_\alpha^-}{2}.$$

Moreover, if f be a fuzzy-valued function then, for simplicity, we put $(M_\alpha f)(t)=M_\alpha(f(t))$ and $(L_\alpha f)(t)=L_\alpha(f(t))$.

As an immediate result of above definition, for $u \in R_F$, we have

$$u = \left(M_\alpha(u) - L_\alpha(u), M_\alpha(u) + L_\alpha(u)\right).$$

Proposition 1. Consider the fuzzy number $u = \sum_{i=1}^{n} \lambda_i u_i$ where $u_1, u_2, ..., u_n \in R_F$ and $\lambda_1, \lambda_2, ..., \lambda_n \in R$. The following properties hold

$$M_\alpha(u) = \sum_{i=1}^{n} \lambda_i M_\alpha(u_i), \qquad L_\alpha(u) = \sum_{i=1}^{n} |\lambda_i| L_\alpha(u_i)$$

Proof: By direct calculation we obtain

$$\begin{aligned} M_\alpha(u) &= \frac{u_\alpha^+ + u_\alpha^-}{2} \\ &= \frac{1}{2}\left(\sum_{\substack{i=1 \\ \lambda_i \geq 0}}^{n} \lambda_i u_{i\alpha}^+ + \sum_{\substack{i=1 \\ \lambda_i < 0}}^{n} \lambda_i u_{i\alpha}^- + \sum_{\substack{i=1 \\ \lambda_i \geq 0}}^{n} \lambda_i u_{i\alpha}^- + \sum_{\substack{i=1 \\ \lambda_i < 0}}^{n} \lambda_i u_{i\alpha}^+ \right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i u_{i\alpha}^+ + \sum_{i=1}^{n} \lambda_i u_{i\alpha}^- \right) \\ &= \sum_{i=1}^{n} \lambda_i \left(\frac{u_{i\alpha}^+ + u_{i\alpha}^-}{2}\right) \\ &= \sum_{i=1}^{n} \lambda_i M_\alpha(u_i). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} L_\alpha(u) &= \frac{u_\alpha^+ - u_\alpha^-}{2} \\ &= \frac{1}{2}\left(\sum_{\substack{i=1 \\ \lambda_i \geq 0}}^{n} \lambda_i u_{i\alpha}^+ + \sum_{\substack{i=1 \\ \lambda_i < 0}}^{n} \lambda_i u_{i\alpha}^- - \sum_{\substack{i=1 \\ \lambda_i \geq 0}}^{n} \lambda_i u_{i\alpha}^- - \sum_{\substack{i=1 \\ \lambda_i < 0}}^{n} \lambda_i u_{i\alpha}^+ \right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} |\lambda_i| u_{i\alpha}^+ - \sum_{i=1}^{n} |\lambda_i| u_{i\alpha}^- \right) \\ &= \sum_{i=1}^{n} |\lambda_i| \left(\frac{u_{i\alpha}^+ - u_{i\alpha}^-}{2}\right) \\ &= \sum_{i=1}^{n} \lambda_i L_\alpha(u_i). \end{aligned}$$

□

Proposition 2. Let $u = (u_1, u_2, ..., u_n) \in R_F^n$, then

$$M_\alpha(u) = (M_\alpha(u_1), M_\alpha(u_2), ..., M_\alpha(u_n)), \qquad L_\alpha(u) = (L_\alpha(u_1), L_\alpha(u_2), ..., L_\alpha(u_n)).$$

Proof: We have $u_\alpha^- = (u_{1\alpha}^-, u_{2\alpha}^-, ..., u_{n\alpha}^-)$ and $u_\alpha^+ = (u_{1\alpha}^+, u_{2\alpha}^+, ..., u_{n\alpha}^+)$. Then, based on vector arithmetic we get

$$\frac{1}{2}(u_\alpha^+ + u_\alpha^-) = \left(\frac{u_{1\alpha}^+ + u_{1\alpha}^-}{2}, \frac{u_{2\alpha}^+ + u_{2\alpha}^-}{2}, ..., \frac{u_{n\alpha}^+ + u_{n\alpha}^-}{2} \right),$$

and

$$\frac{1}{2}(u_\alpha^+ - u_\alpha^-) = \left(\frac{u_{1\alpha}^+ - u_{1\alpha}^-}{2}, \frac{u_{2\alpha}^+ - u_{2\alpha}^-}{2}, \ldots, \frac{u_{n\alpha}^+ - u_{n\alpha}^-}{2} \right).$$

So the results are presented. □

Theorem 4. Let $j \in \{1,2\}$ be fixed and $f : I \to R_F$ be (j)-differentiable on I. Then,

$$(M_\alpha f')(t) = (M_\alpha f)'(t), \quad \text{and} \quad (L_\alpha f')(t) = (-1)^{j-1}(L_\alpha f)'(t).$$

Proof: We take $j = 2$. The proof is similar to one for case $j = 1$. By Propositions 1 and 2, we directly obtain

$$(L_\alpha f')(t) = L_\alpha(f'(t)) = (L_\alpha(f_1'(t)), L_\alpha(f_2'(t)), \ldots, L_\alpha(f_n'(t)))$$
$$= \left(\frac{(f_{1\alpha}^-)'(t) - (f_{1\alpha}^+)'(t)}{2}, \frac{(f_{2\alpha}^-)'(t) - (f_{2\alpha}^+)'(t)}{2}, \ldots, \frac{(f_{n\alpha}^-)'(t) - (f_{n\alpha}^+)'(t)}{2} \right)$$
$$= \left(-(\frac{f_{1\alpha}^+ - f_{1\alpha}^-}{2})'(t), -(\frac{f_{2\alpha}^+ - f_{2\alpha}^-}{2})'(t), \ldots, -(\frac{f_{n\alpha}^+ - f_{n\alpha}^-}{2})'(t) \right)$$
$$= -\big((L_\alpha f_1)'(t), (L_\alpha f_2)'(t), \ldots, (L_\alpha f_n)'(t)\big)$$
$$= -(L_\alpha f)'(t).$$

Similarly, we obtain

$$(M_\alpha f')(t) = M_\alpha(f'(t)) = (M_\alpha(f_1'(t)), M_\alpha(f_2'(t)), \ldots, M_\alpha(f_n'(t)))$$
$$= \left((\frac{f_{1\alpha}^+ + f_{1\alpha}^-}{2})'(t), (\frac{f_{2\alpha}^+ + f_{2\alpha}^-}{2})'(t), \ldots, (\frac{f_{n\alpha}^+ + f_{n\alpha}^-}{2})'(t) \right)$$
$$= \big((M_\alpha f_1)'(t), (M_\alpha f_2)'(t), \ldots, (M_\alpha f_n)'(t)\big)$$
$$= (M_\alpha f)'(t).$$

Then, the statement of theorem is concluded. □

We now study system (1) by using of the properties of the midpoint and half-length functions. For this end, we consider the problem for two cases of the GH-differentiability of solution function.

Case 1. Using (1)-differentiability

Consider the fuzzy differential equations $n \times n$ system

$$\begin{cases} x'(t) = Ax(t) + b(t), & t \in I, \\ x(0) = x_0 \in R_F^n \end{cases} \tag{2}$$

where $x : I \to R_F^n$ and $b : I \to R_F^n$ are (1)-differentiable. By applying the half-length function on both sides of this system, by Proposition 1, we get

$$\begin{cases} L_\alpha(x'(t)) = |A| L_\alpha(x(t)) + L_\alpha(b(t)), & t \in I, \\ L_\alpha(x(0)) = L_\alpha(x_0) \end{cases}$$

for each $\alpha \in [0,1]$. Here $|A| = (|a_{ij}|)$ is the absolute of matrix A. Since $x(t)$ is (1)-differentiable, so, by Theorem 4, the last system can be written as follows:

$$\begin{cases} (L_\alpha x)'(t) = |A| (L_\alpha x)(t) + (L_\alpha b)(t), & t \in I, \\ (L_\alpha x)(0) = L_\alpha(x_0) \end{cases} \tag{3}$$

Again, this time, by applying the midpoint function on both sides of system (2), by Proposition 1, we get

$$\begin{cases} M_\alpha(x'(t)) = AM_\alpha(x(t)) + M_\alpha(b(t)), \quad t \in I, \\ M_\alpha(x(0)) = M_\alpha(x_0) \end{cases}$$

for each $\alpha \in [0,1]$. This system, by Theorem 4, can be written as follows:

$$\begin{cases} (M_\alpha x)'(t) = A(M_\alpha x)(t) + (M_\alpha b)(t), \quad t \in I, \\ (M_\alpha x)(0) = M_\alpha(x_0) \end{cases} \tag{4}$$

So that, to solve system (2), it is enough to solve two systems (3) and (4). We now present the following result.

Theorem 5. (a) If system (1) has (1)-differentiable unique solution $x(t)$, then $(L_\alpha x)(t)$ and $(M_\alpha x)(t)$ are the unique solutions of systems (3) and (4), respectively.
(b) Suppose that $p_\alpha(t)$ and $q_\alpha(t)$ are respectively unique solutions of (3) and (4), such that $x(t) = \left(q_\alpha(t) - p_\alpha(t), q_\alpha(t) + p_\alpha(t)\right)$ defines a (1)-differentiable fuzzy-valued function on I. Then $x(t)$ is the unique solution of (1).

Proof: The statement (a) is a straightforward. For (b), it is enough to show that $x(t)$ satisfies the system (1). Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. For $i \in \{1,2,\ldots,n\}$, we obtain

$$\begin{aligned}
&(x_i'(t))_\alpha^- = (x_{i\alpha}^-)'(t) \\
&= (q_{i\alpha}')(t) - (p_{i\alpha}')(t) \\
&= \sum_{j=1}^n a_{ij} q_{j\alpha}(t) + (M_\alpha b_i)(t) - \sum_{j=1}^n |a_{ij}| \, p_{j\alpha}(t) - (L_\alpha b_i)(t) \\
&= \sum_{\substack{j=1 \\ a_{ij} \ge 0}}^n a_{ij}(q_{j\alpha}(t) - p_{j\alpha}(t)) + \sum_{\substack{j=1 \\ a_{ij} < 0}}^n a_{ij}(q_{j\alpha}(t) + p_{j\alpha}(t)) + b_{i\alpha}^-(t) \\
&= \sum_{\substack{j=1 \\ a_{ij} \ge 0}}^n a_{ij} x_{j\alpha}^-(t) + \sum_{\substack{j=1 \\ a_{ij} < 0}}^n a_{ij} x_{j\alpha}^+(t) + b_{i\alpha}^-(t) \\
&= \left(\sum_{j=1}^n a_{ij} x_j(t) + b_i(t) \right)_\alpha^- .
\end{aligned}$$

Similarly, we obtain

$$(x_i'(t))_\alpha^+ = \left(\sum_{j=1}^n a_{ij} x_j(t) + b_i(t) \right)_\alpha^+ .$$

Therefore, the proof is completed. □

Case 2. Using (2)-differentiability

Now, we consider the system (2) under (2)-differentiability. Similar to the previous case, by using Theorem 4 for $j = 2$ and Propositions 1 and 2, we obtain two ordinary differential equations systems, one as (4) and other as follows:

$$\begin{cases} L_\alpha(x'(t)) = -|A| L_\alpha(x(t)) - L_\alpha(b(t)), \quad t \in I, \\ L_\alpha(x(0)) = L_\alpha(x_0) \end{cases} \tag{5}$$

Theorem 6. (a) If system (1) has (2)-differentiable unique solution $x(t)$, then $(L_\alpha x)(t)$ and $(M_\alpha x)(t)$ are unique solutions of systems (5) and (4), respectively.
(b) Suppose that $p_\alpha(t)$ and $q_\alpha(t)$ are respectively unique solutions of (5) and (4), such that $x(t)=\left(q_\alpha(t)-p_\alpha(t), q_\alpha(t)+p_\alpha(t)\right)$ defines a (2)-differentiable fuzzy-valued function on I. Then $x(t)$ is the unique solution of (1).

Proof: Part (a) is a straightforward result. For part (b), similar to the proof of Theorem 5, we obtain

$$\begin{aligned}
&(x_i'(t))_\alpha^- = (x_{i\alpha}^+)'(t) \\
&= (q_{i\alpha}')(t) + (p_{i\alpha}')(t) \\
&= \sum_{j=1}^{n} a_{ij} q_{j\alpha}(t) + (M_\alpha b_i)(t) - \sum_{j=1}^{n} |a_{ij}|\, p_{j\alpha}(t) - (L_\alpha b_i)(t) \\
&= \sum_{\substack{j=1 \\ a_{ij}\ge 0}}^{n} a_{ij}(q_{j\alpha}(t) - p_{j\alpha}(t)) + \sum_{\substack{j=1 \\ a_{ij}<0}}^{n} a_{ij}(q_{j\alpha}(t) + p_{j\alpha}(t)) + b_{i\alpha}^-(t) \\
&= \left(\sum_{j=1}^{n} a_{ij} x_j(t) + b_i(t)\right)_\alpha^- .
\end{aligned}$$

Similarly, we obtain

$$(x_i'(t))_\alpha^+ = \left(\sum_{j=1}^{n} a_{ij} x_j(t) + b_i(t)\right)_\alpha^+ .$$

This ends the proof. □

Remark 2. We note that all components of function $(L_\alpha x)(t)$ obtained of systems (3) or (5) must be non-negative. Then, if there exits $t \in I$, and $i \in \{1,2,\dots,n\}$ such that $(L_\alpha x_i)(t) < 0$, for some $\alpha \in [0,1]$, then system (1) does not solution.

Remark 3. In particular cases $A \ge 0$ and under (1)-differentiability, system (1) becomes the following two separate systems:

$$\begin{cases} (x_\alpha^-)'(t) = A x_\alpha^-(t) + b_\alpha^-(t), \quad t \in I, \\ x_\alpha^-(0) = x_{0\alpha}^- \end{cases} \tag{6}$$

$$\begin{cases} (x_\alpha^+)'(t) = A x_\alpha^+(t) + b_\alpha^+(t), \quad t \in I, \\ x_\alpha^+(0) = x_{0\alpha}^+ \end{cases} \tag{7}$$

Consequently, in this case, it is better to solve systems (6) and (7) instead of systems (3) and (4). Moreover, in particular case $A \le 0$ and under (2)-differentiability, system (1) becomes the following two separate systems:

$$\begin{cases} (x_\alpha^+)'(t) = A x_\alpha^+(t) + b_\alpha^-(t), \quad t \in I, \\ x_\alpha^+(0) = x_{0\alpha}^+ \end{cases} \tag{8}$$

$$\begin{cases} (x_\alpha^-)'(t) = A x_\alpha^-(t) + b_\alpha^+(t), \quad t \in I, \\ x_\alpha^-(0) = x_{0\alpha}^- \end{cases} \tag{9}$$

Consequently, in this case, it is better to solve systems (8) and (9) instead of systems (4) and (5).

4. Examples

To clarify the process of the proposed method, in this section, we solve two numerical examples.

Example 1: Consider the following system

$$\begin{cases} x_1'(t) = -x_1(t) + x_2(t), \quad t \in I, \\ x_2'(t) = 2x_1(t) + 4e^{-3t}(1+\alpha, 2.5-0.5\alpha), \\ x_1(0) = (1+2\alpha, 4-\alpha), \\ x_2(0) = (-6.5+1.5\alpha, -2-3\alpha). \end{cases} \tag{10}$$

Here $b(t) = (b_1(t), b_2(t))$ with $b_1(t) = 0$ and $b_2(t) = 4e^{-3t}(1+\alpha, 2.5-0.5\alpha)$. According to theorem 5, (b) in [7], these functions are (2)-differentiable. So, based on Theorem 3 and Remark 1, we consider the problem (10) under (2)-differentiability. We first solve system (5) related to (10), that is as follows:

$$\begin{cases} (L_\alpha x_1)'(t) = -(L_\alpha x_1)(t) - (L_\alpha x_2)(t), \quad t \in I, \\ (L_\alpha x_2)'(t) = -2(L_\alpha x_1)(t) + 3e^{-3t}(1-\alpha), \\ (L_\alpha x_1)(0) = 3(1-\alpha), \\ (L_\alpha x_2)(0) = 4.5(1-\alpha). \end{cases}$$

By solving this system, we obtain

$$(L_\alpha x_1)(t) = \frac{3}{4}(e^{-2t} + e^{-3t})(1-\alpha), \tag{11}$$

and

$$(L_\alpha x_2)(t) = \frac{3}{4}(e^{-2t} + 2e^{-3t})(1-\alpha). \tag{12}$$

It is clear that $(L_\alpha x_1)(t) \geq 0$ and $(L_\alpha x_1)(t) \geq 0$ for all $t \in I$ and $\alpha \in [0,1]$. We now solve system (4) related to (10), that is

$$\begin{cases} (M_\alpha x_1)'(t) = -(M_\alpha x_1)(t) + (M_\alpha x_2)(t), \quad t \in I, \\ (M_\alpha x_2)'(t) = 2(M_\alpha x_1)(t) + e^{-3t}(7+\alpha), \\ (M_\alpha x_1)(0) = 2.5 + 0.5\alpha, \\ (M_\alpha x_2)(0) = -4.25 - 0.75\alpha. \end{cases}$$

By solving this system, we obtain

$$(M_\alpha x_1)(t) = \frac{1}{4}e^{-2t}(3+\alpha) + \frac{1}{4}e^{-3t}(7+\alpha), \tag{13}$$

and

$$(M_\alpha x_2)(t) = -\frac{1}{4}e^{-2t}(3+\alpha) - \frac{1}{2}e^{-3t}(7+\alpha). \tag{14}$$

Therefore, by (11) and (13), it follows that

$$\begin{aligned} x_{1\alpha}^-(t) &= e^{-2t}\alpha + e^{-3t}(1+\alpha), \\ x_{1\alpha}^+(t) &= \frac{1}{2}e^{-2t}(3-\alpha) + \frac{1}{2}e^{-3t}(5-\alpha), \quad \forall t \geq 0, \quad \forall \alpha \in [0,1], \end{aligned} \tag{15}$$

and by (12) and (14), it follows that

$$x_{2\alpha}^{-}(t)=\frac{1}{2}e^{-2t}(-3+\alpha)+e^{-3t}(-5+\alpha),$$
$$x_{2\alpha}^{+}(t)=-e^{-2t}\alpha-2e^{-3t}(1+\alpha),\quad \forall t\geq 0,\quad \forall\alpha\in[0,1], \tag{16}$$

Finally, from (15) and (16), we infer that

$$x_1(t)=e^{-2t}u_0+e^{-3t}v_0,$$

and

$$x_2(t)=-e^{-2t}u_0-2e^{-3t}v_0,$$

where $u_0=(\alpha,1.5-0.5\alpha)$ and $v_0=(1+\alpha,2.5-0.5\alpha)$ are fuzzy numbers. These functions are shown in figure 1, for $\alpha=0,0.5,1$.

Example 2: Consider the following system

$$\begin{cases} x_1'(t)=x_1(t)+x_2(t), \quad t\in I, \\ x_2'(t)=2x_2(t)+2e^{3t}(\alpha,2-\alpha), \\ x_1(0)=(2\alpha,5-3\alpha), \\ x_2(0)=(3\alpha,7-4\alpha). \end{cases} \tag{17}$$

Here $b(t)=(b_1(t),b_2(t))$ with $b_1(t)=0$ and $b_2(t)=2e^{3t}(\alpha,2-\alpha)$. According to theorem 5, (a) in [7], these functions are (1)-differentiable. We so consider the problem (17) under (1)-differentiability. We also note that since $A=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ is a non-negative matrix, then, we deal with a particular case described in Remark 3. Therefore, problem (17) can be considered as two separate problems as follows:

$$\begin{cases} (x_{1\alpha}^{-})'(t)=x_{1\alpha}^{-}(t)+x_{2\alpha}^{-}(t), \\ (x_{2\alpha}^{-})'(t)=2x_{2\alpha}^{-}(t)+2e^{3t}\alpha, \\ x_{1\alpha}^{-}(0)=2\alpha, \\ x_{2\alpha}^{-}(0)=3\alpha, \end{cases}$$

$$\begin{cases} (x_{1\alpha}^{+})'(t)=x_{1\alpha}^{+}(t)+x_{2\alpha}^{+}(t), \\ (x_{2\alpha}^{+})'(t)=2x_{2\alpha}^{+}(t)+2e^{3t}(2-\alpha), \\ x_{1\alpha}^{+}(0)=5-3\alpha, \\ x_{2\alpha}^{+}(0)=7-4\alpha. \end{cases}$$

We thus obtain

$$x_{1\alpha}^{-}(t)=(e^{2t}+e^{3t})\alpha,$$
$$x_{1\alpha}^{+}(t)=e^{2t}(3-2\alpha)+e^{3t}(2-\alpha),\quad \forall t\geq 0,\quad \forall\alpha\in[0,1], \tag{18}$$

and

$$x_{2\alpha}^{-}(t)=(e^{2t}+2e^{3t})\alpha,$$
$$x_{2\alpha}^{+}(t)=e^{2t}(3-2\alpha)+2e^{3t}(2-\alpha),\quad \forall t\geq 0,\quad \forall\alpha\in[0,1]. \tag{19}$$

Finally, from (18) and (19), we infer that
$x_1(t) = e^{2t}u_1 + e^{3t}v_1,$
and
$x_2(t) = e^{2t}u_1 + 2e^{3t}v_1,$
with the fuzzy numbers $u_1 = (\alpha, 3-2\alpha)$ and $v_1 = (\alpha, 2-\alpha)$. These functions are shown in figure 2, for $\alpha = 0, 0.5, 1$.

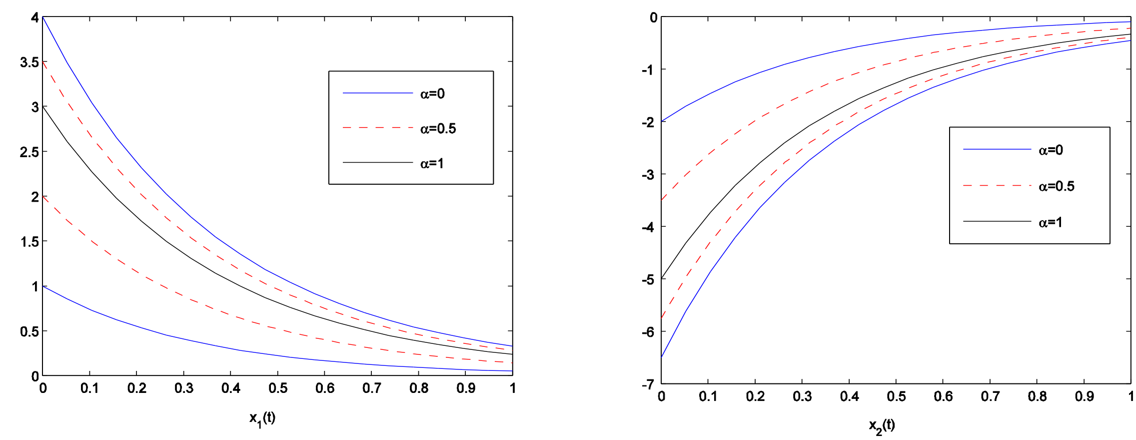

Figure 1. The Graphics of solutions from Example 1.

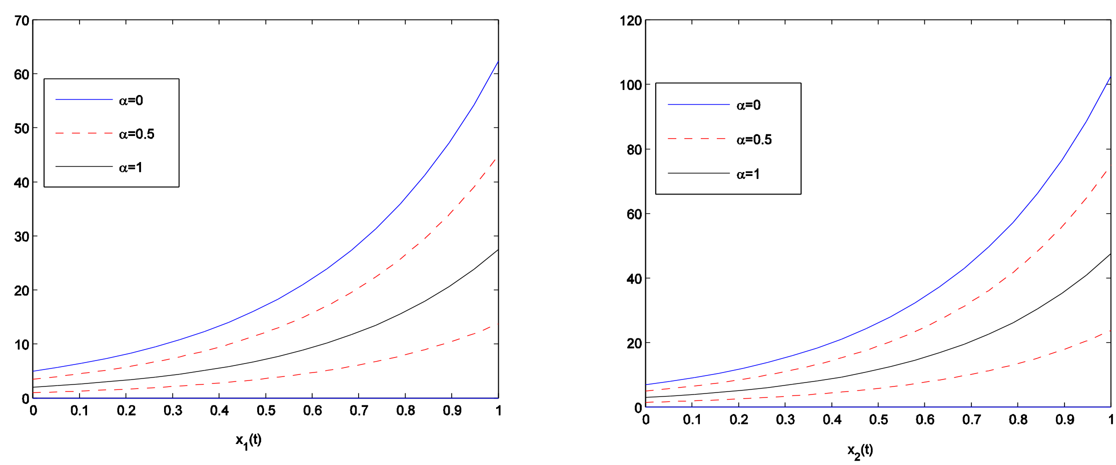

Figure 2. The Graphics of solutions from Example 2.

5. Conclusion and Further Research

In this paper, a linear system $n \times n$ consisting of first-order fuzzy differential equations is investigated under the concept of the GH-derivative of fuzzy-valued functions and a method to solve it is proposed. The direct transformation of the fuzzy main system under GH-derivative leads to an ordinary system $2n \times 2n$ while our method reduces this work by solving two ordinary systems $n \times n$ as a result, the calculation volume of the proposed method is less (see [18] for more details). Also, the necessary and sufficient conditions for the results of the proposed method to lead to the solution of the main system (1) as a fuzzy-valued vector function are given. For further research, we suggest studying the problem under other concepts of derivative, such as the principle of expansion and the fuzzy inclusion derivative.

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