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The Stability of Generalized Jordan Derivations Associated with Hochschild 2-Cocycles of Triangular Algebras

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ABSTRACT

In present paper, the stability of generalized Jordan derivations associated with Hochschild 2-cocycles of triangular algebras for the generalized Jensen-type functional equationis investigated. In fact, the main purpose of present paper is to prove the generalized Hyers-Ulam-Rassias stability of generalized Jordan derivation between algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$.

1. Introduction

In [9] Nakajima introduced a new type of generalized derivation. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let $\alpha: \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a bilinear (biadditive) mapping. α is called a Hochschild 2-cocycle if

$$x\alpha(y,z) - \alpha(xy,z) + \alpha(x,yz) - \alpha(x,y)z = 0. \tag{1}$$

A linear (additive) mapping $\delta: \mathcal{A} \to \mathcal{M}$ is called a linear (additive) generalized derivation if there is a 2-cocycle α such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x,y) \tag{2}$$

and δ is called a linear(additive) generalized Jordan derivation if

$$\delta(x^2) = \delta(x)x + x\delta(x) + \alpha(x,x) \tag{3}$$

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The stability of functional equations was first introduced by S. M. Ulam [13] in 1940. He posed the stability of group homomorphisms: Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if a function $f: G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$ then there exists a homomorphism $T: G_1 \longrightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$. If this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable or the functional equation $f(xy) = f(x)f(y)$ is stable.

In 1941, Hyers [6] gave a partial solution of Ulam's problem in the context of Banach spaces as the following: Suppose that X, Y are Banach spaces and $f: X \longrightarrow Y$ satisfies the following condition: there is $\varepsilon > 0$ such that $\| f(x+y) - f(x) - f(y) \| < \varepsilon$ for all $x, y \in X$. Then there is an additive mapping $T: X \longrightarrow Y$ such that $\| f(x) - T(x) \| < \varepsilon$ for all $x \in X$.

Let X and Y be Banach spaces with norms $\|.\|$ and $\|.\|$, respectively. Consider $f: X \longrightarrow Y$ to be a mapping such that f(tx) is continuous in $t \in R$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, \infty)\backslash\{1\}$ such that

$$\| f(x+y) - f(x) - f(y) \| < \theta(\| x \|^p + \| y \|^p),$$

for all $x, y \in X$. It was shown by Rassias [12] for $p \in [0,1)$ and Gajda [4] for $p > 1$ that there exists a unique R-linear mapping $T: X \longrightarrow X$ such that

$$\| f(x) - T(x) \| \leq \frac{2\theta}{|2 - 2^p|} \| x \|^p,$$

for all $x \in X$.

In 1992, a generalization of Rassias' theorem was obtained by Găvruta [5].

Jun and Lee [7] proved the following: Let X and Y be Banach spaces. Denote by $\varphi: X\backslash\{0\} \times X\backslash\{0\} \longrightarrow [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{n=0}^{\infty} 3^{-n} \varphi(3^n x, 3^n y) < \infty$$

for all $x, y \in X\backslash\{0\}$.

Suppose that $f: X \longrightarrow Y$ is a mapping satisfying

$$2f(\frac{x+y}{2}) = f(x) + f(y),$$

for all $x, y \in X\backslash\{0\}$.
Then there exists a unique additive mapping $T: X \longrightarrow Y$ such that

$$\| f(x) - f(0) - T(x) \| \leq \tfrac{1}{3}(\tilde{\varphi}(x, -x) - \tilde{\varphi}(x, -3x)),$$

for all $x \in X\backslash\{0\}$.

There are many interesting papers to consider the stability of any structures [1,2,3,4,8,10,11]. The main purpose of this paper is establishing the stability of a generalized Jordan derivations associated with Hochschild 2-cocycles of triangular algebras for the generalized Jensen-type functional equation

$$rf(\tfrac{x+y}{r}) = f(x) + f(y), \tag{4}$$

2. Main results

Theorem 1. Let $s > 1$, and let $f: \mathcal{A} \longrightarrow \mathcal{M}$ be a mapping satisfying $f(sa) = sf(a)$ for all $a \in \mathcal{A}$. Let there exist a function $\varphi: \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ such that $lim_{n\to\infty} \frac{\varphi(t^n a, t^n b, t^n c)}{t^n} = 0$, and a Hochschild 2-cocycle α such that

$$\| r\lambda f\left(\frac{a+b}{r}\right) - f(\lambda a) - f(\lambda b) + f(c^2) - f(c)c - cf(c) - \alpha(c,c) \| \leq \varphi(a,b,c), \tag{5}$$

for all $\lambda \in T^1 = \{z \in C: \| z \| = 1\}$ and all $a, b, c \in \mathcal{A}$. Then f is a generalized Jordan derivation.

Proof. Clearly $f(0) = 0$ because $f(0) = sf(0)$. Putting $a = b = 0$ in (5), we have

$$\| f(c^2) - f(c)c - cf(c) - \alpha(c,c) \| = \frac{1}{t^{2n}} \| f(t^{2n}c^2) - f(t^n c)t^n c - t^n x f(t^n c) \\ -\alpha(t^n c, t^n c) \| \leq \frac{\varphi(0,0,t^{2n}c)}{t^{2n}}, \tag{6}$$

for all $c \in \mathcal{A}$. Since $\frac{\varphi(0,0,t^{2n}c)}{t^n} \longrightarrow 0$ as $n \longrightarrow \infty$, therefore (6) leads to

$$f(c^2) = f(c)c + cf(c) + \alpha(c,c), \tag{7}$$

for all $c \in \mathcal{A}$. Now let $c = 0$ in (5), then

$$\| r\lambda f\left(\frac{a+b}{r}\right) - f(\lambda a) - f(\lambda b) \| = t^n \| r\lambda f\left(\frac{t^n a + t^n b}{r}\right) - f(\lambda t^n a) - f(\lambda t^n b)) \| \\ \leq \frac{\varphi(t^n a, t^n b, 0)}{t^n},$$

for all $a, b \in \mathcal{A}$. Since $\frac{\varphi(t^n a, t^n b, 0)}{t^n} \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain

$$r\lambda f\left(\frac{a+b}{r}\right) = f(\lambda a) + f(\lambda b), \tag{8}$$

which substituting $\lambda = 1$ we have

$$rf\left(\frac{a+b}{r}\right) = f(a) + f(b), \tag{9}$$

for all $a, b \in \mathcal{A}$. Thus the mapping f satisfies in (4).

It is not difficult to prove that f is additive. Clearly f is additive and R-linear. By putting $b = 0$ in (9) we obtain

$$rf\left(\frac{a}{r}\right) = f(a), \tag{10}$$

for all $a \in \mathcal{A}$. Now substituting $b = 0$ in (8) and using (10) formula we find

$$f(\lambda a) = \lambda f(a), \tag{11}$$

for all $a \in \mathcal{A}$ and $\lambda \in T^1$. Hence f is C-linear. ☐

Theorem 2. Suppose $r > 1$, and $g: \mathcal{A} \longrightarrow \mathcal{M}$ be a mapping with $g(0) = 0$ for which there exists a function $\varphi: \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ such that

$$\Phi(a,b,c) = \sum_{n=0}^{\infty} \frac{\varphi(t^n a, t^n b, t^n c)}{t^n} < \infty \tag{12}$$

$$\| r\lambda g\left(\frac{a+b}{r}\right) - g(\lambda a) - g(\lambda b) + g(c^2) - g(c)c - cg(c) - \alpha(c,c) \| \leq \Phi(a,b,c), \tag{13}$$

for all $\lambda \in T^1$ and all $a, b, c \in \mathcal{A}$.

Then there exists a unique generalized Jordan derivation $f: \mathcal{A} \longrightarrow \mathcal{M}$ such that

$$\| g(a) - f(a) \| \leq \Phi(a, 0,0), \tag{14}$$

for all $a \in \mathcal{A}$.

Proof. Putting $\lambda = 1$ and $b = c = 0$ in (13) leads to

$$\| g(a) - \frac{g(a)}{r} \| \leq \frac{\Phi(ra,0,0)}{r}, \tag{15}$$

Therefore by induction on n, we obtain

$$\| g(a) - g(a)r^n \| \leq \sum_{k=1}^{n} \frac{\Phi(r^k a,0,0)}{r^k}, \tag{16}$$

for all $a \in \mathcal{A}$.

Now we replace a by $r^m a$ in (16), hence we find

$$\| g(a) - \frac{g(r^{n+m}a)}{r^{n+m}} \| \leq \frac{1}{r^m} \sum_{k=m}^{n+m} \Phi(r^k a, 0,0), \ \forall a \in \mathcal{A}. \tag{17}$$

Thus $\left\{\frac{g(r^n a)}{r^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Put

$$f(x) = \lim_{n\to\infty} \frac{g(r^n x)}{r^n}. \tag{18}$$

Since $\mathcal{A}$ is complete, $f(x)$ in (18) exists for all $x \in \mathcal{A}$. It is easy to obtain the (14) formula from (16). Now since

$$\begin{aligned} \| rf\left(\frac{a+b}{r}\right) - f(a) - f(b) \| &= \lim_{n\to\infty} \frac{1}{r^n} \| rg(r^{n-1}(a+b)) - g(r^n a) - g(r^n b) \| \\ &\leq \lim_{n\to\infty} \frac{1}{r^n} \varphi(r^n a, r^n b, 0) = 0 \end{aligned}$$

for all $a, b \in \mathcal{A}$ thus we have

$$rf\left(\frac{a+b}{r}\right) = f(a) + f(b)$$

for all $a, b \in \mathcal{A}$.

Hence, f is a Jensen type function. For $\alpha \in T^1$ we have

$$\| \alpha f(a) - f(\alpha a) \| = \lim_{n\to\infty} \frac{1}{r^n} \| \alpha g(r^n a) - g(\alpha r^n a) \| \leq \lim_{n\to\infty} \frac{1}{r^n} \varphi(r^n a, r^n a, 0) = 0$$

Then $f(\alpha a) = \alpha f(a)$ for $\alpha \in T^1$ therefore f is C-linear. Also

$$\begin{aligned} \| g(c^2) - g(c)c - cg(c) - \alpha(c,c) \| &= \lim_{n\to\infty} \| \frac{1}{r^{2n}} g(r^{2n}c^2) - g(r^n c) r^n c - r^n c g(r^n c) - \frac{1}{r^{2n}} \alpha(r^n c, r^n c) \| \\ &\leq \lim_{n\to\infty} \frac{1}{r^{2n}} \varphi(0,0,r^n c) \\ &= 0, \end{aligned}$$

for all $c \in \mathcal{A}$.

Thus f is a unique generalized Jordan derivation satisfied (14). ☐

Theorem 3. Let $g: \mathcal{A} \longrightarrow \mathcal{M}$ is a mapping with $g(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in (0,1)$ such that

$$\| r\lambda g\left(\frac{a+b}{r}\right) - g(\lambda a) - g(\lambda b) + g(c^2) - g(c)c - cg(c) - \alpha(c,c) \| \leq \lambda(\| a \|^p + \| b \|^p + \| c \|^p), \quad (19)$$

for all $\theta \in T^1$ and all $a, b, c \in \mathcal{A}$.

Then there exists a unique generalized Jordan derivation $f: \mathcal{A} \longrightarrow \mathcal{M}$ such that

$$\| f(a) - g(a) \| \leq \frac{\theta}{1 - r^{p-1}} \| a \|^p. \quad (20)$$

for all $a \in \mathcal{A}$.

Proof. It is easy to prove by defining the function $\varphi: \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow R$ by

$$(a, b, c) \mapsto \theta(\| a \|^p + \| b \|^p + \| c \|^p)$$

Now, applying Theorem 2, one can find (20) inequality.**□**

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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