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On Fixed Points of Soft Set-Valued Maps

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ABSTRACT

Conventional mathematical tools which require all inferences to be exact, are not always sufficient to handle imprecisions in a wide variety of practical fields. Thus, among various developments in fuzzy mathematics, enormous efforts have been in process to produce new fuzzy analogues of the classical fixed point results and their various applications. Following this line in this paper, a new type of set-valued mapping whose range set lies in a family of soft sets is examined. To this effect, we introduce a few fixed point theorems which are generalizations of several significant fixed point results of point-to-point and point-to-set valued mappings in the comparable literature. Some of these particular cases are noted and analyzed. Moreover, nontrivial examples are provided to support the assumptions of our main results.

1. Introduction

One of the first well-known fixed point theorems in metric space structure appeared explicitly in Banach's thesis in 1922 [4], where it was applied to obtain the existence of a solution to an integral equation. The theorem is now popularly known as Banach fixed point theorem (or the contraction mapping principle). As a matter of fact, Banach contraction principle [4] is a reformulation of the successive approximation techniques originally used by some earlier mathematicians, namely Cauchy, Liouville, Picard, Lipschitz, and so on. The original idea of fixed point theorem due to Banach has been developed and applied in different directions. In some generalizations of the contraction mapping principle, the contractive inequality is weakened; see, for example, [7, 14, 17, 29], and in other, the topology of the ground space is weakened; for example, see [8, 12, 15, 16]. For a comprehensive survey in this direction, the interested reader may consult Rhoades [24], Smart [27] or Taskovic [28].

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The real world is filled with uncertainty, vagueness and imprecision. The notions we meet in everyday life are vague rather than precise. In recent time, researchers have developed keen interests in modelling vagueness due to the fact that many practical problems within fields such as biology, economics, engineering, environmental sciences, medical sciences, philosophy and so on, involve data containing various forms of uncertainties. To handle the complexity of vagueness, one cannot successfully employ classical mathematical methods due to the presence of different kinds of incomplete knowledge, typical of these mix-ups. Earlier in the literature, there were four known theories for dealing with imperfect knowledge, namely, Probability Theory (PT), Fuzzy Set Theory (FST) [30] and Rough Set Theory (RST) [23]. All the aforementioned tools require pre-assignment of some parameters; for example membership function in FST, probability density function in PT and equivalent relation in RST. Such pre-specifications, viewed in the backdrop of incomplete knowledge, give rise to everyday problems. With this concern, Molodstov [21] initiated the concept of Soft Set Theory (SST) with the aim of handling phenomena and notions of ambiguous, undefined and imprecise environments. Hence, SST does not need the pre-specifications of a parameter; rather, it accommodates approximate descriptions of objects. In other words, one can use any suitable parametrization tool with the help of words, sentences, real numbers, mappings, and so on; thereby, making SST an adequate formalism for approximate reasoning. Consequently, the arena of applications of mathematics gained tremendous developments as a result of the introduction of soft set by Molodstov [21]. Recall that in classical mathematics, to describe any system or object, we first construct its mathematical model and then attempt to obtain the exact solution. If the exact solution is too complicated, then we define the notion of approximate solution. On the other hand, in soft set theory, the initial description of an object takes an approximate nature with no restriction, and the notion of exact solution is not essential. In [21], Moldstov pointed out several directions for possible applications of soft set, such as in smoothness of functions, game theory, Riemann-integration, operation research, probability and so on. Presently, the concept of soft set is receiving more than a handful of extensions in different perspectives. For example, see [6, 9, 20, 22, 25] and the references therein. It is well-known that set-valued analysis has enormous applications in control theory, game theory, biomathematics, qualitative physics, viability theory, and so on. In this continuation, not long ago, Mohammed and Azam [3, 18, 26] studied the concept of soft set-valued maps and introduced the notions of e-soft fixed points and E-soft fixed points of maps whose range set is a family of soft sets. Applications in game theoretic approach and investigation of existence of solutions of some integro-differential equations have been proposed in [18, 26]. Moreover, it is shown in [18] that every fuzzy mapping is a particular kind of soft set-valued map. Since every fuzzy mapping has its corresponding multifunction analogue (see [10, Theorem 2]), hence, the idea of e-soft fixed point theorems is a generalization of the concept of fuzzy fixed points and fixed points of multi-valued maps.

The main focus of this article is twofold. First, the existence of common fixed points of soft set-valued maps is investigated under new generalized contractive inequalities. The second direction deals with applying some of the key results obtained herein to deduce their analogues in the setting of fuzzy set-valued and crisp multi-valued mappings. Consequently, it is pointed out that our results unify, generalize and complement the results established in [1, 2, 5, 11, 18, 26] and some references therein.

The paper is organized as follows: Section 2 gathers basic notions, definitions and results needed to establish the main results. In Section 3, the main results of the manuscript are presented. Section 4 applies the results of Section 3 to derive their fuzzy and classical set-valued versions. Finally, Section 5 contains the summary of the paper.

2. Preliminaries

In this section, we collect some important notations, useful definitions and basic results coherent with the literature. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ and $\mathbb{X}^*$, the sets of all positive integers, non-negative reals, real numbers, nonempty closed and bounded subsets of Ψ, respectively. These preliminary concepts are recorded from [18, 26, 20, 21]. Let E be the parameter set, $\nabla\subseteq E$ and $P(\Psi)$ represents the power set of an initial universe of discourse Ψ. Molodstov [21] introduced the notion of soft sets with the following definition.

Definition 1. [21] A pair (F,∇) is called a soft set over Ψ under E, where $\nabla \subseteq E$ and F *is* a mapping given by $F:\nabla \longrightarrow P(\Psi)$.

In other words, a soft set over Ψ is a parameterized family of subsets of Ψ. For each $e \in E$, $F(e)$ is considered as the set of e-approximate elements of (F,∇).

Example 1. [20] Suppose the following:

Ψ- is the universal set of all students at a certain university,

E- is the set of parameters, given as:

$E = \{\text{intelligent, hardworking, dull, hardworking and intelligent}\}.$

Assume that they are one hundred students at the university Ψ given as

$\Psi = \{\jmath_1, \jmath_2, \jmath_3, \jmath_4, \jmath_5 \cdots \jmath_{100}\}$, and $E = \{e_1, e_2, e_3, e_4\}$,

where

- e_1= intelligent, e_2 = hardworking,
- e_3= dull, e_4= hardworking and intelligent.

Then $F: E \longrightarrow P(\Psi)$ defined by $F(e_1) = \{\jmath_1, \jmath_2, \cdots \jmath_{10}\}$ means that $\jmath_1, \jmath_2, \cdots \jmath_{10}$ are intelligent , $F(e_2) = \{\jmath_{11}, \jmath_{12}, \cdots \jmath_{30}\}$ means that $\jmath_{11}, \jmath_{12}, \cdots \jmath_{30}$ are hardworking, $F(e_3) = \emptyset$ means that there is no dull student in the university in question, $F(e_4) = \{\jmath_{15}, \jmath_{81}\}$ means that the students $\jmath_{15}$ and $\jmath_{81}$ are both intelligent and hardworking. Then we can view the soft set (F,E) describing the "kind of students" as the following approximations:

$$(F,E) = \{(\text{intelligent students}, \{\jmath_1, \jmath_2, \cdots \jmath_{10}\}), (\text{hardworking students}, \{\jmath_{11}, \jmath_{12}, \cdots \jmath_{30}\}). (\text{dull}, \emptyset), (\text{intelligent and hardworking students}, \{\jmath_{15}, \jmath_{81}\})\}.$$

Mohammed and Azam [18] initiated the concepts of soft-valued maps and e-soft fixed points through the following preliminaries.

Let (Ψ, ϱ) be a metric space and $\mathbb{X}^*$ be the set of all nonempty closed and bounded subsets of Ψ. Denote by $[P(\Psi)]^E$, the family of soft sets over Ψ. Then consider two soft sets (F,∇) and (G,Δ), $(a,b) \in \nabla \times \Delta$. Assume that $F(a), G(b) \in \mathbb{X}^*$. For $\epsilon > 0$, define $N^\varrho(\epsilon, F(a))$, $S_{EX}^{(a,b)}(F,G)$ and $E^\varrho_{(F_a,G_b)}$, respectively, as follows:

$$N^\varrho(\epsilon, F(a)) = \{\jmath \in \Psi : \varrho(\jmath, \ell) < \epsilon, \quad \text{for} \quad \text{some} \quad \ell \in F(a)\}$$

$$E^\varrho_{(F_a,G_b)} = \{\epsilon > 0 : F(a) \subseteq N^\varrho(\epsilon, G(b)), \quad G(b) \subseteq N^\varrho(\epsilon, F(a))\},$$

and

$$S_{EX}^{(a,b)}(F,G) = \inf E^\varrho_{(F_a,G_b)},$$

Define a distance function $S_{EX}^\infty : [P(\Psi)]^E \times [P(\Psi)]^E \longrightarrow \mathbb{R}_+$ by

$$S_{EX}^\infty(F,G) = \sup_{(a,b)\in \overline{\nabla} \times \overline{\Delta}} S_{EX}^{(a,b)}(F,G), \text{where}$$

$$\overline{\nabla} \times \overline{\Delta} = \{(a,b) \in \nabla \times \Delta : F(a), G(b) \in \mathbb{X}^*\}.$$

Remark 1. Note that in terms of the Hausdorff metric $\aleph$, the distance function $S_{EX}^{(a,b)}(F,G)$ reduces to:

$$S_{EX}^{(a,b)}(F,G) = \aleph(F(a), G(b)) = \max\left\{\sup_{\jmath \in F(a)} \varrho(\jmath, G(b)), \sup_{\ell \in G(b)} \varrho(\ell, F(a))\right\}.$$

Similarly, $S_{EX}^\infty(F,G)$ corresponds to the notion of ϱ_∞-metric for fuzzy sets.

Definition 2. [18] A mapping $T: \Psi \longrightarrow [P(\Psi)]^E$ is called a soft set-valued map. A point $u \in \Psi$ is called an e-soft fixed point of T if $u \in (Tu)(e)$, for some $e \in E$. This is also written as $u \in Tu$, for short. If $\mathrm{DomT}_\jmath = \mathrm{E}$

and $u \in (Tu)(e)$ for all $e \in E$, then u is said to be an E-soft fixed point of T.

We shall denote the set of all E-soft fixed points of a soft set-valued map T by $E_{Fix(T)}$. Here, the domain of T, denoted as $DomT$, is given by

$$DomT = \{\jmath \in \Psi : (T\jmath)(e) \subseteq \Psi, e \in E\}.$$

Analogously, we define the image of T, imT as $imT = \{\ell | \exists \jmath \in \Psi : \ell \in (T\jmath)(e), e \in E\}$.

Notice that if $T : \Psi \longrightarrow [P(\Psi)]^E$ is a soft set-valued map, then $(T\jmath, E)$ is a soft set over Ψ, for all $\jmath \in \Psi$. Throughout this paper, if $T : \Psi \longrightarrow [P(\Psi)]^E$ is a soft set-valued map, then the set $(T\jmath)(e)$ shall also be written as $(T_e\jmath)$. For simplicity, a soft set (F, E) in $[P(\Psi)]^E$ shall be indicated as $F \in [P(\Psi)]^E$(to mean $F : E \longrightarrow P(\Psi)$).

Example 2. Let $\Psi = \{6,7,8\}$ and $E = \{1,2\}$. Define $T : \Psi \longrightarrow [P(\Psi)]^E$ as follows:

$$(T_e\jmath) = \begin{cases} \{6,8\}, & if \quad e = 1 \\ \{7,8\}, & if \quad e = 2. \end{cases}$$

Then T is a soft set-valued map.

Notice that $6 \in (T_e 6)$ for $e = 1$ and $7 \in (T_e 7)$ for $e = 2$; hence, 6 and 7 are e-soft fixed points of T. But, $7 \notin (T_e 7)$ and $6 \notin (T_e 6)$ for $e = 1$ and $e = 2$, respectively. If follows that 6 and 7 are not E-soft fixed points of T. On the other hand, $8 \in (T_e 8)$ for all $e \in E$; thus, the set of all E-soft fixed points of T is given by $E_{Fix(T)} = \{8\}$. The map T can be represented as in Figure 1. Notice that in Figure 1, the dots represent other subsets of Ψ.

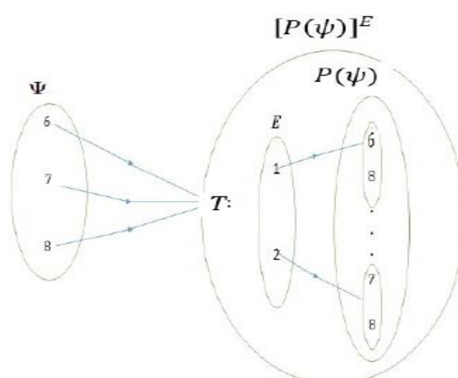

Figure 1. Graphical representation of the soft set-valued map in Example 2.

3. Main Results

We start this section by presenting some auxiliary results as follows.

Lemma 1. Let (Ψ, ϱ) be a metric space and $F \in [P(\Psi)]^E$. Then, for any $\jmath, \ell \in \Psi$ and $a(\jmath), a(\ell) \in E$,

$$p_{EX}^{a(\jmath)}(\jmath, F) \leq \varrho(\jmath, \ell) + p_{EX}^{a(\ell)}(\ell, F).$$

Proof. By definition of $p_{EX}^{a(\jmath)}(,)$, we have

$$\begin{aligned} p_{EX}^{a(\jmath)}(\jmath, F) &= \inf_{r \in F(a)} \varrho(\jmath, r) \\ &\leq \inf_{r \in F(a)} [\varrho(\jmath, \ell) + \varrho(\ell, r)] \\ &\leq \inf_{r \in F(a)} \varrho(\jmath, \ell) + \inf_{r \in F(a)} \varrho(\ell, r) \\ &= \varrho(\jmath, \ell) + p_{EX}^{a(\ell)}(\ell, F). \square \end{aligned}$$

Lemma 2. Let (Ψ,ϱ) be a metric space, $\jmath,\ell\in\Psi$ and $F\in[P(\Psi)]^E$. If $\{\jmath\}$ is a subset *of* $F(a(\jmath))$ for any $a(\jmath)\in E$, then for each $G\in[P(\Psi)]^E$, there exists $a(\ell)\in E$ such that

$$p_{EX}^{a(\jmath)}(\jmath,G)\leq \inf E^{\varrho}_{(F_{a(\jmath)},G_{a(\ell)})}.$$

Proof. By definition of $p_{EX}^{a(\jmath)}$, we have

$$p_{EX}^{a(\jmath)}(\jmath,G)=\inf_{\ell\in G(b)}\varrho(\jmath,\ell)$$
$$\leq \sup_{\jmath\in F(a),\ell\in G(b)}\inf\varrho(\jmath,\ell)$$
$$\leq \inf E^{\varrho}_{(F_{a(\jmath)},G_{a(\ell)})}.\square$$

Lemma 3. Let (Ψ,ϱ) be a metric space and $F\in[P(\Psi)]^E$. Then $p_{EX}^{a(\jmath)}(\jmath,F)=0$ if and only if $\{\jmath\}\subseteq F(a(\jmath))$, for any $a(\jmath)\in E$.

Lemma 4. Let (Ψ,ϱ) be a metric space and $F\in[P(\Psi)]^E$. For any $\jmath\in\Psi$ and $a(\jmath)\in E$, if there exists $\ell\in\Psi$ such that $\ell\in F(a(\jmath))$, then $p_{EX}^{a(\jmath)}(\jmath,F)\leq\varrho(\jmath,\ell)$.

Proof. By Lemma 2, for any $\jmath,\ell\in\Psi$ and $a(\jmath),a(\ell)\in E$, we have $p_{EX}^{a(\jmath)}(\jmath,F)\leq\varrho(\jmath,\ell)+p_{EX}^{a(\ell)}(\ell,F)$. If $\ell\in F(a)$, then by Lemma 3, $p_{EX}^{a(\ell)}(\ell,F)=0$. Hence, $p_{EX}^{a(\jmath)}(\jmath,F)\leq\varrho(\jmath,\ell)$.$\square$

Lemma 5. Let (Ψ,ϱ) be a metric space and $T:\Psi\longrightarrow[P(\Psi)]^E$ be a soft set-valued map. Assume that for some $\jmath_0\in\Psi$ and $a(\jmath_0)=e\in E$, $(T_e\jmath_0)$ is a nonempty compact subset of Ψ. Then, there exists $\jmath_1\in\Psi$ such that $\jmath_1\in(T_e\jmath_0)$.

The following Lemma is a direct consequence of the definition of the distance function $S_{EX}^{(a,b)}(,)$.

Lemma 6. Let (Ψ,ϱ) be a metric space and $F,G\in[P(\Psi)]^E$. Assume that $F(a)$ and $G(b)$ are nonempty closed and bounded subsets of Ψ, for some $a,b\in E$. Then, for each $\jmath\in F(a)$, and all $\ell\in G(b)$,

$$\varrho(\jmath,G)\leq S_{EX}^{(a,b)}(F,G)\text{ and }\varrho(\jmath,G)\leq\varrho(\jmath,\ell).$$

In what follows, we present the first main result of this section.

Theorem 1. Let (Ψ,ϱ) be a complete metric space and $M,T:\Psi\longrightarrow[P(\Psi)]^E$ be soft set-valued maps. Assume that for each $\jmath,\ell\in\Psi$, there exist $a(\jmath),a(\ell)\in E$ with $a(\jmath)\in DomM\jmath$ and $a(\ell)\in DomT\ell$ such that $M\jmath,T\ell\in\mathbb{X}^*$. If

$$\inf E^{\varrho}_{(M_{a(\jmath)},T_{a(\ell)})}\leq l_1p_{EX}^{a(\jmath)}(\jmath,M\jmath)+l_2p_{EX}^{a(\ell)}(\ell,T\ell)$$
$$+l_3p_{EX}^{a(\ell)}(\ell,M\jmath)+l_4p_{EX}^{a(\jmath)}(\jmath,T\ell)+l_5\varrho(\jmath,\ell)\qquad(1)$$

where $\sum_{i=i}^{5}l_i<1$ and $l_3=l_4$. Then, there exists $u\in\Psi$ such that $u\in Mu:=(Mu)(a(u))$ and $u\in Tu:=(Tu)(a(u))$, for some $a(u)\in E$.

Proof. Let $\jmath_0\in\Psi$; then, by hypothesis, there exists $a(\jmath_0)\in E$ with $a(\jmath_0)\in DomM\jmath_0$ such that $M\jmath_0$ is a nonempty closed and bounded subset of Ψ. Choose $\jmath_1\in M\jmath_0$; then for this $\jmath_1\in\Psi$, we can find $a(\jmath_1)\in E$ with $a(\jmath_1)\in DomT\jmath_1$ such that $T\jmath_1\in\mathbb{X}^*$. Then, there exists $\jmath_2\in\Psi$ such that $\jmath_2\in T\jmath_1$ and $\varrho(\jmath_1,\jmath_2)\leq\inf E^{\varrho}_{(M_{a(\jmath_0)},T_{a(\jmath_1)})}$. On same steps, there exist $\jmath_3\in\Psi$ and $a(\jmath_2)\in DomM\jmath_2$ such that $\jmath_3\in M\jmath_2$ and $\varrho(\jmath_2,\jmath_3)\leq\inf E^{\varrho}_{(T_{a(\jmath_1)},M_{a(\jmath_2)})}$. Proceeding recursively, one generates a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ in Ψ such that

$$\jmath_{2n+1}\in M\jmath_{2n},\ \jmath_{2n+2}\in T\jmath_{2n+1},$$

and

$$\varrho(\jmath_{2n+1},\jmath_{2n+2})\leq\inf E^{\varrho}_{(M_{a(\jmath_{2n})},T_{a(\jmath_{2n+1})})},\qquad(2)$$

$$\varrho(J_{2n+2},J_{2n+3}) \leq \inf E^{\varrho}_{(M_{a(J_{2n+1})},T_{a(J_{2n+2})})}. \quad (3)$$

Taking $n=0$ in (2), and using (10) together with Lemma 6, in that order, we have

$$\begin{aligned}\varrho(J_1,J_2) \leq\ & \inf E^{\varrho}_{(M_{a(J_0)},T_{a(J_1)})}\\ \leq\ & l_1 p_{EX}^{a(J_0)}(J_0,Mx_0)+l_2 p_{EX}^{a(J_1)}(J_1,TJ_1)+l_2 p_{EX}^{a(J_1)}(J_1,MJ_0)\\ & +l_4 p_{EX}^{a(J_0)}(J_0,TJ_1)+l_5\varrho(J_0,J_1)\\ \leq\ & l_1\varrho(J_0,J_1)+l_2\varrho(J_1,J_2)+l_3\varrho(J_1,J_1)\\ & +l_4\varrho(J_0,J_2)+l_5\varrho(J_0,J_1)\\ \leq\ & l_1\varrho(J_0,J_1)+l_2\varrho(J_1,J_2)\\ & +l_4[\varrho(J_0,J_1)+\varrho(J_1,J_2)]+l_4\varrho(J_0,J_1)\\ \leq\ & \left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)\varrho(J_0,J_1)=\kappa\varrho(J_0,J_1)\end{aligned} \quad (4)$$

where $\kappa=\left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)$. Notice that $\sum_{i=1}^{5} l_i<1$ implies $l_1+l_4+l_5<1-l_2-l_3$. So, for $l_3=l_4$, we have $0<\kappa<1$.

Again, take $n=0$ in (3), and using Lemma 6, accordingly, we get

$$\begin{aligned}\varrho(J_2,J_3) \leq\ & \inf E^{\varrho}_{(M_{a(J_1)},T_{a(J_2)})}\\ \leq\ & l_1 p_{EX}^{a(J_1)}(J_1,MJ_1)+l_2 p_{EX}^{a(J_2)}(J_2,TJ_2)+l_2 p_{EX}^{a(J_2)}(J_2,MJ_1)\\ & +l_4 p_{EX}^{a(J_1)}(J_1,TJ_2)+l_5\varrho(J_1,J_2)\\ \leq\ & l_1\varrho(J_1,J_2)+l_2\varrho(J_2,J_3)+l_3\varrho(J_2,J_2)\\ & +l_4\varrho(J_1,J_3)+l_5\varrho(J_1,J_2)\\ \leq\ & l_1\varrho(J_1,J_2)+l_2\varrho(J_1,J_3)\\ & +l_4[\varrho(J_1,J_2)+\varrho(J_2,J_3)]+l_4\varrho(J_1,J_2)\\ \leq\ & \left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)\varrho(J_1,J_2)\\ \leq\ & \left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)^2\varrho(J_0,J_1)=\kappa^2\varrho(J_0,J_1).\end{aligned} \quad (5)$$

Letting $n=1$ in (4), and using Lemma 6, we have

$$\begin{aligned}\varrho(J_3,J_4) \leq\ & \inf E^{\varrho}_{(M_{a(J_2)},T_{a(J_3)})}\\ \leq\ & l_1 p_{EX}^{a(J_2)}(J_2,MJ_2)+l_2 p_{EX}^{a(J_3)}(J_3,TJ_3)+l_2 p_{EX}^{a(J_3)}(J_3,MJ_2)\\ & +l_4 p_{EX}^{a(J_2)}(J_2,TJ_3)+l_5\varrho(J_2,J_3)\\ \leq\ & l_1\varrho(J_2,J_3)+l_2\varrho(J_3,J_4)+l_3\varrho(J_3,J_3)\\ & +l_4\varrho(J_1,J_3)+l_5\varrho(J_1,J_2)\\ \leq\ & l_1\varrho(J_2,J_3)+l_2\varrho(J_2,J_4)\\ & +l_4[\varrho(J_2,J_3)+\varrho(J_3,J_4)]+l_4\varrho(J_2,J_3)\\ \leq\ & \left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)\varrho(J_2,J_3)\\ \leq\ & \left(\frac{l_1+l_4+l_5}{1-l_2-l_4}\right)^3\varrho(J_0,J_1)=\kappa^3\varrho(J_0,J_1).\end{aligned} \quad (6)$$

By continuing in this fashion for all $n\in\mathbb{N}$, we have

$$\varrho(J_n,n_{n+1}) \leq \kappa^n\varrho(J_0,J_1). \quad (7)$$

On similar arguments as above, one can see that $0<\kappa^n<1$. Furthermore, for $m,n\in\mathbb{N}$, with $m>n$, by triangle inequality,

$$\begin{aligned}&\varrho(J_n,J_m)\leq\varrho(J_n,J_{n+1})+\varrho(J_{n+1},J_{n+2})+\cdots+\varrho(J_{m-1},J_m)\\ &\leq(\kappa^n+\kappa^{n+1}+\cdots+\kappa^{m-1})\varrho(J_0,J_1)\end{aligned}$$

$$\leq \frac{\kappa^n}{1-\kappa}\varrho(\jmath_0,\jmath_1) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This proves that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Ψ. By completeness of Ψ, there exists $u \in \Psi$ such that $\jmath_n \longrightarrow u$ as $n \longrightarrow \infty$.

Now, applying lemmas 4, 5 and 6, in that order, we obtain

$$\begin{aligned}
& p_{EX}^{a(u)}(u,Mu) \leq \varrho(u,\jmath_{2n+1}) + p_{EX}^{a(\jmath_{2n+1})}(\jmath_{2n+1},Mu) \\
& \leq \varrho(u,\jmath_{2n+1}) + \inf E^{\varrho}_{(M_{a(u)},T_{a(\jmath_{2n})})} \\
& \leq \varrho(u,\jmath_{2n+1}) + l_1 p_{EX}^{a(u)}(u,Mu) + l_2 p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Tx_{2n}) \\
& + l_3 p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Mu) + l_4 p_{EX}^{a(u)}(u,T\jmath_{2n}) + l_5\varrho(u,\jmath_{2n}) \\
& \leq \varrho(u,\jmath_{2n}) + l_1 p_{EX}^{a(u)}(u,Mu) + l_2\varrho(\jmath_{2n},\jmath_{2n+1}) \\
& + l_3 p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Mu) + l_4\varrho(u,\jmath_{2n+1}) + l_5\varrho(u,\jmath_{2n}).
\end{aligned} \tag{8}$$

Notice that by Lemma 6, for $u \in \Psi$, we get

$$p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Mu) \leq \varrho(\jmath_{2n},u) + p_{EX}^{a(u)}(u,Mu).$$

Therefore, becomes

$$\begin{aligned}
p_{EX}^{a(u)}(u,Mu) \leq\ & \varrho(u,\jmath_{2n}) + l_1 p_{EX}^{a(u)}(u,Mu) + l_2\varrho(\jmath_{2n},\jmath_{2n+1}) \\
& + l_3[\varrho(\jmath_{2n},u) + p_{EX}^{a(u)}(u,Mu)] \\
& + l_4\varrho(u,\jmath_{2n}) + l_5\varrho(u,\jmath_{2n}).
\end{aligned} \tag{9}$$

Letting $n \longrightarrow \infty$ in (9), gives

$$p_{EX}^{a(u)}(u,Mu) \leq (l_1+l_3)p_{EX}^{a(u)}(u,Mu).$$

The above expression implies that $(1-l_1-l_3)p_{EX}^{a}(u,Mu) \leq 0$. Hence, by Lemma 6, $u \in Mu$. On similar steps, one can show that $p_{EX}^{a(u)}(u,Tu) = 0$. Consequently, $u \in Tu$. □

Example 3. Let $\Psi = \{0,1,2,3,4,5\}$ and for all $\jmath,\ell \in \Psi$, defined $\varrho: \Psi\times\Psi \longrightarrow \mathbb{R}_+$ by

$$\varrho(\jmath,\ell) = \begin{cases} 0, & if\ \ \jmath = \ell \\ \frac{1}{16}, & if\ \ \jmath \neq \ell \text{ and } \jmath,\ell \in \{1,2\} \\ \frac{1}{8}, & if\ \ \jmath \neq \ell \text{ and } \jmath,\ell \in \{4,5\} \\ \frac{1}{14}, & if\ \ \jmath \neq \ell \text{ and } \jmath,\ell \in \{1,5\} \\ \frac{1}{24}, & if\ \ \jmath \neq \ell \text{ and } \jmath,\ell \in \{1,3\} \\ \frac{1}{18}, & if\ \ \jmath \neq \ell \text{ and } \jmath,\ell \in \{1,4\} \\ \frac{1}{4}, & if\ \jmath \neq \ell \text{ and } \jmath,\ell \in \{2,3,5\}. \end{cases}$$

Then (Ψ,ϱ) is a complete metric space. Let $E = [0,1]$ and $a(\jmath) = e \in E$ for $\jmath \in \Psi$. Then, consider two soft set-valued maps $M,T: \Psi \longrightarrow [P(\Psi)]^E$ defined by

$$(M_e\jmath) := M\jmath = \begin{cases} \{0\}, & if\ \ \jmath = 0 \text{ and } 0 \leq e < \frac{1}{10} \\ \{1\}, & if\ \ \jmath \in \{1,2,3,4\} \text{ and } \frac{1}{10} \leq e < \frac{1}{5} \\ \{1,2,3,4\}, & if\ \ \jmath = 5 \text{ and } \frac{1}{5} \leq e \leq 1, \end{cases}$$

and

$$(T_e\jmath) := T\jmath = \begin{cases} \{0\}, & if\ \ \jmath = 0 \text{ and } 0 \leq e < \frac{1}{10} \\ \{1\}, & if\ \ \jmath \in \{1,2,3,4\} \text{ and } \frac{1}{10} \leq e < \frac{1}{5} \\ \{2,3,4\}, & if\ \ \jmath = 5 \text{ and } \frac{1}{5} \leq e \leq 1. \end{cases}$$

Then, consider the following cases:

- For $\jmath = 1$ and $\ell = 5$, we have $\inf E^{\varrho}_{(M_{\alpha(1)},T_{\alpha(5)})} = \frac{1}{16}$, $P^{\alpha(1)}_{EX}(1,M1) = 0$, $P^{\alpha(5)}_{EX}(5,T5) = \frac{1}{8}$, $P^{\alpha(5)}_{EX}(5,M1) = \frac{1}{14}$, $P^{\alpha(1)}_{EX}(1,T5) = \frac{1}{16}$ and $\varrho(1,5) = \frac{1}{14}$. Therefore, by taking $l_1 = l_3 = l_4 = l_5 = 0$ and $l_2 = \frac{1}{2}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(1)},T_{\alpha(5)})} = \frac{1}{16} \leq \frac{1}{2} p^{\alpha(5)}_{EX}(5,T5)$$
$$\leq l_1 P^{\alpha(1)}_{EX}(1,M1) + l_2 P^{\alpha(5)}_{EX}(5,T5) + l_3 P^{\alpha(5)}_{EX}(5,M1)$$
$$+ l_4 P^{\alpha(1)}_{EX}(1,T5) + l_5 \varrho(1,5).$$

- For $\jmath = 5$ and $\ell = 1$, we have $\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(1)})} = \frac{1}{16}$, $P^{\alpha(5)}_{EX}(5,M5) = \frac{1}{8}$, $P^{\alpha(1)}_{EX}(1,T1) = 0$, $P^{\alpha(1)}_{EX}(1,M5) = 0$, $P^{\alpha(5)}_{EX}(5,T1) = \frac{1}{14}$ and $\varrho(5,1) = \frac{1}{14}$. Therefore, by putting $l_2 = l_3 = l_4 = l_5 = 0$ and $l_1 = \frac{1}{2}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(1)})} = \frac{1}{16} \leq \frac{1}{2} p^{\alpha(5)}_{EX}(5,M5)$$
$$\leq l_1 P^{\alpha(5)}_{EX}(5,M5) + l_2 P^{\alpha(1)}_{EX}(1,T1) + l_3 P^{\alpha(1)}_{EX}(1,M5)$$
$$+ l_4 P^{\alpha(5)}_{EX}(5,T1) + l_5 \varrho(5,1).$$

- For $\jmath = 2$ and $\ell = 5$, we have $\inf E^{\varrho}_{(M_{\alpha(2)},T_{\alpha(5)})} = \frac{1}{16}$, $P^{\alpha(2)}_{EX}(2,M2) = \frac{1}{16}$, $P^{\alpha(5)}_{EX}(5,T5) = \frac{1}{18}$, $P^{\alpha(5)}_{EX}(5,M2) = \frac{1}{14}$, $P^{\alpha(2)}_{EX}(2,T5) = 0$ and $\varrho(2,5) = \frac{1}{4}$. Therefore, by taking $l_1 = l_2 = l_3 = l_4 = 0$ and $l_5 = \frac{1}{4}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(2)},T_{\alpha(5)})} = \frac{1}{16} \leq \frac{1}{4} \varrho(2,5)$$
$$\leq l_1 P^{\alpha(2)}_{EX}(2,M2) + l_2 P^{\alpha(5)}_{EX}(5,T5) + l_3 P^{\alpha(5)}_{EX}(5,M2)$$
$$+ l_4 P^{\alpha(2)}_{EX}(2,T5) + l_5 \varrho(2,5).$$

- For $\jmath = 5$ and $\ell = 2$, we have $\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(2)})} = \frac{1}{16}$, $P^{\alpha(5)}_{EX}(5,M5) = \frac{1}{8}$, $P^{\alpha(2)}_{EX}(2,T2) = \frac{1}{16}$, $P^{\alpha(2)}_{EX}(2,M5) = 0$, $P^{\alpha(5)}_{EX}(5,T2) = \frac{1}{14}$ and $\varrho(5,2) = \frac{1}{4}$. Therefore, by putting $l_1 = l_2 = l_3 = l_4 = 0$ and $l_5 = \frac{1}{4}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(2)})} = \frac{1}{16} \leq \frac{1}{4} \varrho(5,2)$$
$$\leq l_1 P^{\alpha(5)}_{EX}(5,M5) + l_2 P^{\alpha(2)}_{EX}(2,T2) + l_3 P^{\alpha(2)}_{EX}(2,M5)$$
$$+ l_4 P^{\alpha(5)}_{EX}(5,T2) + l_5 \varrho(5,2).$$

- For $\jmath = 3$ and $\ell = 5$, we have $\inf E^{\varrho}_{(M_{\alpha(2)},T_{\alpha(5)})} = \frac{1}{16}$, $P^{\alpha(3)}_{EX}(3,M3) = \frac{1}{24}$, $P^{\alpha(5)}_{EX}(5,T5) = \frac{1}{8}$, $P^{\alpha(5)}_{EX}(5,M3) = \frac{1}{14}$, $P^{\alpha(3)}_{EX}(3,T5) = 0$ and $\varrho(3,5) = \frac{1}{4}$. Therefore, by taking $l_1 = l_2 = l_3 = l_4 = 0$ and $l_5 = \frac{1}{4}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(3)},T_{\alpha(5)})} = \frac{1}{16} \leq \frac{1}{4} \varrho(3,5)$$
$$\leq l_1 P^{\alpha(3)}_{EX}(3,M3) + l_2 P^{\alpha(5)}_{EX}(5,T5) + l_3 P^{\alpha(5)}_{EX}(5,M3)$$
$$+ l_4 P^{\alpha(3)}_{EX}(3,T5) + l_5 \varrho(3,5).$$

- For $\jmath = 5$ and $\ell = 3$, we have $\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(3)})} = \frac{1}{16}$, $P^{\alpha(5)}_{EX}(5,M5) = \frac{1}{8}$, $P^{\alpha(3)}_{EX}(3,T3) = \frac{1}{24}$, $P^{\alpha(3)}_{EX}(3,M5) = 0$, $P^{\alpha(5)}_{EX}(5,T3) = \frac{1}{14}$ and $\varrho(5,3) = \frac{1}{4}$. Therefore, by putting $l_1 = l_2 = l_3 = l_5 = 0$ and $l_1 = \frac{1}{2}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(5)},T_{\alpha(2)})} = \frac{1}{16} \leq \frac{1}{2} P^{\alpha(5)}_{EX}(5,M5)$$

$$\leq l_1 P_{EX}^{\alpha(5)}(5, M5) + l_2 P_{EX}^{\alpha(3)}(3, T3) + l_3 P_{EX}^{\alpha(3)}(3, M5)$$
$$+ l_4 P_{EX}^{\alpha(5)}(5, T3) + l_5 \varrho(5,3).$$

- For $\jmath = 4$ and $\ell = 5$, we have $\inf E^{\varrho}_{(M_{\alpha(4)}, T_{\alpha(5)})} = \frac{1}{16}$, $P_{EX}^{\alpha(4)}(4, M4) = \frac{1}{18}$, $P_{EX}^{\alpha(5)}(5, T5) = \frac{1}{8}$, $P_{EX}^{\alpha(5)}(5, M4) = \frac{1}{14}$, $P_{EX}^{\alpha(4)}(4, T5) = 0$ and $\varrho(4,5) = \frac{1}{8}$. Therefore, by taking $l_1 = l_2 = l_3 = l_4 = 0$ and $l_5 = \frac{1}{2}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(4)}, T_{\alpha(5)})} = \frac{1}{16} \leq \frac{1}{2}\varrho(4,5)$$
$$\leq l_1 P_{EX}^{\alpha(4)}(4, M4) + l_2 P_{EX}^{\alpha(5)}(5, T5) + l_3 P_{EX}^{\alpha(5)}(5, M4)$$
$$+ l_4 P_{EX}^{\alpha(4)}(4, T5) + l_5 \varrho(4,5).$$

- For $\jmath = 5$ and $\ell = 4$, we have $\inf E^{\varrho}_{(M_{\alpha(5)}, T_{\alpha(4)})} = \frac{1}{16}$, $P_{EX}^{\alpha(5)}(5, M5) = \frac{1}{8}$, $P_{EX}^{\alpha(4)}(4, T4) = \frac{1}{18}$, $P_{EX}^{\alpha(4)}(4, M5) = 0$, $P_{EX}^{\alpha(5)}(5, T4) = \frac{1}{14}$ and $\varrho(5,4) = \frac{1}{4}$. Therefore, by setting $l_2 = l_3 = l_4 = l_5 = 0$ and $l_1 = \frac{1}{2}$, we have

$$\inf E^{\varrho}_{(M_{\alpha(5)}, T_{\alpha(4)})} = \frac{1}{16} \leq \frac{1}{2} P_{EX}^{\alpha(5)}(5, M5)$$
$$\leq l_1 P_{EX}^{\alpha(5)}(5, M5) + l_2 P_{EX}^{\alpha(4)}(4, T4) + l_3 P_{EX}^{\alpha(4)}(4, M5)$$
$$+ l_4 P_{EX}^{\alpha(5)}(5, T4) + l_5 \varrho(5,4).$$

Thus, all the conditions of Theorem 1 are satisfied. Consequently, one can see that 0 and 1 are the common e-soft fixed points of M and T in Ψ.

Corollary 1. Let (Ψ, ϱ) be a complete metric space and $M, T: \Psi \longrightarrow [P(\Psi)]^E$ be soft set-valued maps. Assume that for each $\jmath, \ell \in \Psi$, there exist $\alpha(\jmath), \alpha(\ell) \in E$ with $\alpha(\jmath) \in DomM\jmath$ and $\alpha(\ell) \in DomT\ell$ such that $M\jmath, T\ell \in \mathbb{X}^*$. If

$$S_{EX}^{\infty}(M\jmath, T\ell) \leq l_1 p_{EX}^{\alpha(\jmath)}(\jmath, M\jmath) + l_2 p_{EX}^{\alpha(\ell)}(\ell, T\ell) + l_3 p_{EX}^{\alpha(\ell)}(\ell, M\jmath) + l_4 p_{EX}^{\alpha(\jmath)}(\jmath, T\ell) + l_5 \varrho(\jmath, \ell) \tag{10}$$

for all $\jmath, \ell \in \Psi$, where $\sum_{i=i}^{5} l_i < 1$ and $l_3 = l_4$. Then, there exists $u \in \Psi$ such that $u \in Mu \cap Tu$.

Proof. Since $S_{EX}^{\infty}(M\jmath, T\ell) = \sup_{(\alpha(\jmath), \alpha(\ell)) \in E \times E} \inf E^{\varrho}_{(M_{\alpha(\jmath)}, T_{\alpha(\ell)})}$, therefore, Theorem 1 can be applied to obtain $u \in \Psi$ such that $u \in Mu \cap Tu$. □

Corollary 2. Let (Ψ, ϱ) be a complete metric space and $T: \Psi \longrightarrow [P(\Psi)]^E$ be a soft set-valued map. Assume that for each $\jmath \in \Psi$, there exists $\alpha(\jmath) \in E$ with $\alpha(\jmath) \in DomM\jmath$ such that $T\jmath \in \mathbb{X}^*$. If there exists $\gamma \in (0,1)$ such that $S_{EX}^{\infty}(M\jmath, T\ell) \leq \gamma \varrho(\jmath, \ell)$ for all $\jmath, \ell \in \Psi$, then, there exists $u \in \Psi$ such that $u \in Mu \cap Tu$.

Proof. Put $M = T$, $l_1 = l_2 = l_3 = l_4$ and $l_5 = \gamma$ in Corollary 1. □

Corollary 3. Let (Ψ, ϱ) be a complete metric space and $T: \Psi \longrightarrow [P(\Psi)]^E$ be a soft set-valued map. Assume that for each $\jmath \in \Psi$, there exists $\alpha(\jmath) \in DomT\jmath$ such that $T\jmath \in \mathbb{X}^*$. If for all $\jmath, \ell \in \Psi$,

$$\inf E^{\varrho}_{(T_{\alpha(\jmath)}, T_{\alpha(\ell)})} \leq l_1 p_{EX}^{\alpha(\jmath)}(\jmath, T\jmath) + l_2 p_{EX}^{\alpha(\ell)}(\ell, T\ell) + l_3 p_{EX}^{\alpha(\ell)}(\ell, T\jmath) + l_4 p_{EX}^{\alpha(\jmath)}(\jmath, T\ell) + l_5 \varrho(\jmath, \ell)$$

where $\sum_{i=1}^{5} l_i < 1$ and $l_3 = l_4$ $(l_i \geq 0)$, then there exists $u \in \Psi$ such that $u \in Tu$.

Proof. Put $M = T$ in Theorem 1. □

The following is the main result of Mohammed and Azam [18, Theorem 12] with $f = I_{\Psi}$, the identity mapping on Ψ.

Corollary 3. [18] Let (Ψ, ϱ) be a complete metric space and $T: \Psi \longrightarrow [P(\Psi)]^E$ be a soft set-valued map. Assume that for each $\jmath \in \Psi$, there exists $\alpha(\jmath) \in DomT\jmath$ such that $T\jmath \in \mathbb{X}^*$. If there exists $l \in (0,1)$ such that

$$S_{EX}^{(\alpha(\jmath), \alpha(\ell))}(T\jmath, T\ell) \leq l d(\jmath, \ell)$$

for all $\jmath, \ell \in \Psi$, then there exists $u \in \Psi$ such that $u \in Tu$.

Proof. Take $l_1 = l_2 = l_3 = l_4 = 0$ and $l_5 = l$ in Corollary 2. ☐

Theorem 2. Let (Ψ, ϱ) be a complete metric space and $M, T: \Psi \longrightarrow [P(\Psi)]^E$ be soft set-valued maps. Assume that for each $\jmath, \ell \in \Psi$, there exists $a(\jmath) \in DomM\jmath$ *and* $a(\ell) \in DomTell$ such that $M\jmath, Tell \in \mathbb{X}^*$. If there exist $\gamma \in (0,1)$ such that

$$\inf E^{\varrho}_{(M_{a(\jmath)}, T_{a(\ell)})} \leq \gamma \max\{\varrho(\jmath,\ell), p^{a(\jmath)}_{EX}(\jmath, M\jmath), p^{a(\ell)}_{EX}(\ell, T\ell), \frac{p^{a(\jmath)}_{EX}(\jmath, Tell) + p^{a(\ell)}_{EX}(\ell, M\jmath)}{2}\} \tag{11}$$

Then, there exists $u \in \Psi$ such that $u \in Mu \cap Tu$.

Proof. For $\jmath_0 \in \Psi$, by hypothesis, there exists $a(\jmath_0) \in Dom\jmath_0$ such that $M\jmath_0 \in \Psi^*$. Let $\jmath_1 \in M\jmath_0$ and on same steps, $\jmath_2 \in T\jmath_1$; then, $\varrho(\jmath_1, \jmath_2) \leq \inf E^{\varrho}_{(M_{a(\jmath_0)}, T_{a(\jmath_1)})}$. Hence, using (16) and Lemma 6 in that order, we have

$$\begin{aligned}
\varrho(\jmath_1,\jmath_2) &\leq \inf E^{\varrho}_{(M_{a(\jmath_0)}, T_{a(\jmath_1)})} \\
&\leq \gamma \max\{\varrho(\jmath_0,\jmath_1), p^{a(\jmath_0)}_{EX}(\jmath_0, M\jmath_0), p^{a(\jmath_1)}_{EX}(\jmath_1, T\jmath_1), \\
&\frac{p^{a(\jmath_0)}_{EX}(\jmath_0, T\jmath_1) + p^{a(\jmath_1)}_{EX}(\jmath_1, M\jmath_0)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_0,\jmath_1), \varrho(\jmath_0,\jmath_1), \varrho(\jmath_1,\jmath_2), \\
&\frac{\varrho(\jmath_0,\jmath_2) + \varrho(\jmath_1,\jmath_1)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_0,\jmath_1), \varrho(\jmath_1,\jmath_2), \frac{\varrho(\jmath_0,\jmath_1) + \varrho(\jmath_1,\jmath_2)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_0,\jmath_1), \varrho(\jmath_1,\jmath_2)\}.
\end{aligned}$$

If $\max\{\varrho(\jmath_0,\jmath_1), \varrho(\jmath_1,\jmath_2)\} = \varrho(\jmath_1,\jmath_2)$, then

$$\varrho(\jmath_1,\jmath_2) \leq \gamma \varrho(\jmath_1,\jmath_2) < \varrho(\jmath_1,\jmath_2)$$

is a contradiction. It follows that

$$\varrho(\jmath_1,\jmath_2) \leq \gamma \varrho(\jmath_0,\jmath_1). \tag{12}$$

Similarly, for $\jmath_1 \in \Psi$, by assumption, there exists $a(\jmath_1) \in DomM\jmath_1$ such that $M\jmath_1 \in \Psi^*$. Take $\jmath_2 \in M\jmath_1$, for this $\jmath_2$, there exists $a(\jmath_2) \in T\jmath_2$ such that $T\jmath_2 \in \Psi^*$. Choose $\jmath_3 \in T\jmath_2$ so that $\varrho(\jmath_2,\jmath_3) \leq \inf E^{\varrho}_{(M_{a(\jmath_1)}, T_{a(\jmath_2)})}$. Therefore, (16) and Lemma 6 yield:

$$\begin{aligned}
\varrho(\jmath_2,\jmath_3) &\leq \inf E^{\varrho}_{(M_{a(\jmath_1)}, T_{a(\jmath_2)})} \\
&\leq \gamma \max\{\varrho(\jmath_1,\jmath_2), p^{a(\jmath_1)}_{EX}(\jmath_1, M\jmath_1), p^{a(\jmath_2)}_{EX}(\jmath_2, T\jmath_2), \\
&\leq \frac{p^{a(\jmath_1)}_{EX}(\jmath_1, T\jmath_2) + p^{a(\jmath_2)}_{EX}(\jmath_2, M\jmath_1)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_1,\jmath_2), \varrho(\jmath_1,\jmath_2), \varrho(\jmath_2,\jmath_3), \\
&\frac{\varrho(\jmath_1,\jmath_3) + \varrho(\jmath_2,\jmath_2)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_1,\jmath_2), \varrho(\jmath_2,\jmath_3), \frac{\varrho(\jmath_1,\jmath_2) + \varrho(\jmath_2,\jmath_3)}{2}\} \\
&\leq \gamma \max\{\varrho(\jmath_1,\jmath_2), \varrho(\jmath_2,\jmath_3)\}.
\end{aligned}$$

If $\max\{\varrho(\jmath_1,\jmath_2), \varrho(\jmath_2,\jmath_3)\} = \varrho(\jmath_2,\jmath_3)$, then $\varrho(\jmath_2,\jmath_3) \leq \gamma \varrho(\jmath_2,\jmath_3) < \varrho(\jmath_2,\jmath_3)$yields a contradiction. Hence,

$$\varrho(\jmath_2,\jmath_3) \leq \gamma \varrho(\jmath_1,\jmath_2) \leq \gamma^2 \varrho(\jmath_0,\jmath_1).$$

By continuing recursively, we generate a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ such that $\jmath_{2n+1} \in M\jmath_{2n}$ and $\jmath_{2n+2} \in T\jmath_{2n+1}$ such that

$$\begin{aligned}\varrho(\jmath_n,\jmath_{n+1}) \leq\ & \inf E^{\varrho}_{(M_{a(\jmath_{n-1})},T_{a(\jmath_n)})}\\ \leq\ & \gamma\max\{\varrho(\jmath_{n-1},\jmath_n), p_{EX}^{a(\jmath_{n-1})}(\jmath_{n-1},M\jmath_{n-1}), p_{EX}^{a(\jmath_n)}(\jmath_n,T\jmath_n),\\ & \frac{p_{EX}^{a(\jmath_{n-1})}(\jmath_{n-1},T\jmath_n)+p_{EX}^{a(\jmath_n)}(\jmath_n,M\jmath_{n-1})}{2}\}.\end{aligned} \tag{13}$$

By Lemma 6, the inequality (13) reduces to

$$\begin{aligned}\varrho(\jmath_n,\jmath_{n+1}) \leq\ & \gamma\max\{\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_n,\jmath_{n+1}),\\ & \frac{\varrho(\jmath_{n-1},\jmath_{n+1})+\varrho(\jmath_n,\jmath_n)}{2}\}\\ \leq\ & \gamma\max\{\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_n,\jmath_{n+1}),\\ & \frac{\varrho(\jmath_{n-1},\jmath_n)+\varrho(\jmath_n,\jmath_{n+1})}{2}\}\\ \leq\ & \gamma\max\{\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_n,\jmath_{n+1})\}.\end{aligned}$$

If $\max\{\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_n,\jmath_{n+1})\}=\varrho(\jmath_n,\jmath_{n+1})$, then $\varrho(\jmath_n,\jmath_{n+1})\leq\gamma\varrho(\jmath_n,\jmath_{n+1})<\varrho(\jmath_n,\jmath_{n+1})$ gives a contradiction. Therefore,

$$\begin{aligned}\varrho(\jmath_n,\jmath_{n+1}) &\leq \gamma\varrho(\jmath_{n-1},\jmath_n)\\ &\leq \gamma^2\varrho(\jmath_{n-2},\jmath_{n-1})\\ &\leq \gamma^3\varrho(\jmath_{n-3},\jmath_{n-2})\\ &\ \ \vdots\\ &\leq \gamma^n\varrho(\jmath_0,\jmath_1).\end{aligned}$$

From here, one can follow the steps in Theorem 2 to conclude that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Ψ; and the completeness of Ψ implies that there exists $u\in\Psi$ such that $\jmath_n\longrightarrow u$ as $n\longrightarrow\infty$.

Now, assume that $u\notin Mu$. Then, by applying lemmas 4, 5 and 6, accordingly, one gets

$$\begin{aligned}p_{EX}^{a(u)}(u,Mu) \leq\ & \varrho(u,\jmath_{2n+1}) + p_{EX}^{a(\jmath_{2n+1})}(\jmath_{2n+1},Mu)\\ \leq\ & \varrho(u,\jmath_{2n+1}) + \inf E^{\varrho}_{(M_{a(u)},T_{a(\jmath_{2n})})}\\ \leq\ & \varrho(u,\jmath_{2n+1}) + \gamma\max\{\varrho(u,\jmath_{2n}), p_{EX}^{a(u)}(u,Mu), p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},T\jmath_{2n}),\\ & \frac{p_{EX}^{a(u)}(u,T\jmath_{2n})+p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Mu)}{2}\}\\ \leq\ & \varrho(u,\jmath_{2n+1}) + \gamma\max\{\varrho(u,\jmath_{2n}), p_{EX}^{a(u)}(u,Mu), \varrho(\jmath_{2n},\jmath_{2n+1}),\\ & \frac{\varrho(u,\jmath_{2n+1})+p_{EX}^{a(\jmath_{2n})}(\jmath_{2n},Mu)}{2}\}\\ \leq\ & \varrho(u,\jmath_{2n+1}) + \gamma\max\{\varrho(u,\jmath_{2n}), p_{EX}^{a(u)}(u,Mu), \varrho(\jmath_{2n},\jmath_{2n+1}),\\ & \frac{\varrho(u,\jmath_{2n+1})+\varrho(\jmath_{2n},u)+p_{EX}^{a(u)}(u,Mu)}{2}\}.\end{aligned} \tag{14}$$

Letting $n\longrightarrow\infty$ in (14), yields

$$\begin{aligned}p_{EX}^{a(u)}(u,Mu) &\leq \gamma\max\{p_{EX}^{a(u)}(u,Mu), p_{EX}^{a(u)}(u,Mu)\}\\ &\leq \gamma p_{EX}^{a(u)}(u,Mu) < p_{EX}^{a(u)}(u,Mu),\end{aligned}$$

This is a contradiction. Therefore, $u\in Mu$. On similar arguments, one can show that $u\in Tu$. Consequently, $u\in Mu\cap Tu$. □

Corollary 4. Let (Ψ,ϱ) be a complete metric space and $M,T:\Psi\longrightarrow[P(\Psi)]^E$ be soft set-valued maps. Assume that for each $\jmath,\ell\in\Psi$, there exist $a(\jmath)\in DomM\jmath$ *and* $a(\ell)\in DomTell$ such that $M\jmath, T\ell\in\mathbb{X}^*$. If there exists $\gamma\in(0,1)$ such that

$$S_{EX}^{\infty}(M\jmath, T\ell) \leq \gamma\max\{\varrho(\jmath,\ell), p_{EX}^{a(\jmath)}(\jmath, M\jmath), p_{EX}^{a(\ell)}(\ell, T\ell), \frac{p_{EX}^{a(\jmath)}(\jmath,T\ell)+p_{EX}^{a(\ell)}(\ell,M\jmath)}{2}\} \quad (15)$$

then, there exists $u \in \Psi$ such that $u \in Mu \cap Tu$.

Proof. The proof follows as in Corollary 3. □

Corollary 5. Let (Ψ, ϱ) be a complete metric space and $T: \Psi \longrightarrow [P(\Psi)]^E$ be a soft set-valued mapping. Assume that for each $\jmath \in \Psi$, there exists $a(\jmath) \in DomT\jmath$ *such that* $T\jmath \in \mathbb{X}^*$. If there exist $\gamma \in (0,1)$ such that

$$\inf E^{\varrho}_{(T_{a(\jmath)}, T_{a(\ell)})} \leq \gamma\max\{\varrho(\jmath,\ell), p_{EX}^{a(\jmath)}(\jmath, T\jmath), p_{EX}^{a(\ell)}(\ell, T\ell), \frac{p_{EX}^{a(\jmath)}(\jmath,T\ell)+p_{EX}^{a(\ell)}(\ell,T\jmath)}{2}\} \quad (16)$$

Then, there exists $u \in \Psi$ such that $u \in Tu$.

Proof. Put $M = T$ in Theorem 2. □

4. Consequences in fuzzy set-valued and multivalued maps

In this section, we apply the e-soft fixed point results of the previous section to derive some fixed point theorems in the framework of fuzzy set-valued and multivalued mappings. To this end, we recall a few preliminaries that will be used hereafter.

Let (Ψ, ϱ) be a metric space and $\mathbb{X}^*$ be the family of nonempty closed and bounded subsets of Ψ. For $\nabla, \Delta \in \mathbb{X}^*$, the Hausdorff -Pompeiu metric $\aleph$ on $\mathbb{X}^*$ induced by ϱ is defined as

$$\aleph(\nabla,\Delta) = \max\{\sup_{\jmath\in\nabla}\varrho(\jmath,\Delta), \sup_{\ell\in\Delta}\varrho(\ell,\nabla)\}$$

where $\varrho(\jmath,\nabla) = \inf_{\jmath\in\nabla}\varrho(\jmath,\ell)$.

A fuzzy set in Ψ is a function with domain Ψ and values in $[0,1] = I$. If ∇ is a fuzzy set in Ψ and $\jmath \in \Psi$, then the function value $\nabla(\jmath)$ is called the degree of membership of $\jmath$ in ∇. The α-level set of ∇, denoted by $[\nabla]_\alpha$, is defined as follows:

$$[\nabla]_\alpha = \{\jmath \in \Psi: \nabla(\jmath) \geq \alpha\}, \text{if } \alpha \in (0,1],$$

$$[\nabla]_0 = \overline{\{\jmath \in \Psi: \nabla(\jmath) > 0\}}.$$

Here, $\overline{M}$ represents the closure of a nonfuzzy set M. We shall denote the collection of all fuzzy sets in Ψ by I^{Ψ}. If there exists an $\alpha \in [0,1]$ such that $[\nabla]_\alpha, [\Delta]_\alpha \in \Psi^*$, then define

$$p_\alpha(\nabla,\Delta) = \inf\{\varrho(\jmath,\ell): \jmath \in [\nabla]_\alpha, \ell \in [\Delta]_\alpha\},$$

$$D_\alpha(\nabla,\Delta) = \aleph([\nabla]_\alpha, [\Delta]_\alpha),$$

$$p(\nabla,\Delta) = \sup_\alpha p_\alpha(\nabla,\Delta),$$

$$\varrho_\infty(\nabla,\Delta) = \sup_\alpha D_\alpha(\nabla,\Delta).$$

Definition 3. Let Ψ be an arbitrary set and Y a metric space. A mapping $\nabla: \Psi \longrightarrow I^Y$ is called a fuzzy set-valued map. A fuzzy set-valued map ∇ is a fuzzy subset of $\Psi \times Y$ with membership function $\nabla(\jmath)(\ell)$. The function value $\nabla(\jmath)(\ell)$ is the grade of membership of ℓ in $\nabla(\jmath)$.

Definition 4. Let $\nabla, \Delta: \Psi \longrightarrow I^{\Psi}$ be fuzzy set-valued maps. A point $u \in \Psi$ is called fuzzy fixed point of ∇ if $u \in [\nabla u]_\alpha$. The point u is called a common fuzzy fixed point of ∇ and Δ if $u \in [\nabla u]_\alpha \cap [\Delta\, u]_\alpha$.

Now, we deduce some consequences of our results.

Corollary 6. [2, Theorem 5] Let (Ψ, ϱ) be a complete metric space and $\nabla, \triangle : \Psi \longrightarrow I^{\Psi}$ be fuzzy set-valued maps. Assume that for each $\jmath \in \Psi$, there exists $\alpha(\jmath) \in (0,1]$ such that $[\nabla \jmath]_{\alpha(\jmath)}, [\triangle \jmath]_{\alpha(\jmath)} \in \mathbb{X}^*$, and

$$\aleph([\nabla \jmath]_{\alpha(\jmath)}, [\triangle \ell]_{\alpha(\ell)}) \leq l_1 \varrho(\jmath, [\nabla \jmath]_{\alpha(\jmath)}) + l_2 \varrho(\ell, [\triangle \ell]_{\alpha(\ell)}) + l_3 \varrho(\jmath, [\triangle \ell]_{\alpha(\ell)}) + l_4 \varrho(\ell, [\nabla \jmath]_{\alpha(\jmath)}) + l_5 \varrho(\jmath, \ell)$$

for all $\jmath, \ell \in \Psi$, with $\sum_{i=1}^{5} l_i < 1$ $(l_i \geq 0)$ and $l_1 = l_2$ or $l_3 = l_4$. Then there exists $u \in \Psi$ such that $u \in [\nabla u]_{\alpha(u)} \cap [\triangle u]_{\alpha(u)}$.

Proof. Let $a^*(\jmath), a^*(\ell) \in E$ with $a^*(\jmath) = \alpha(\jmath)$ and $a^*(\ell) = \alpha(\ell)$, for all $\jmath, \ell \in \Psi$. Consider two soft set-valued maps $\Theta_\nabla, \Lambda_\triangle : \Psi \longrightarrow [P(\Psi)]^E$, respectively defined by

$$\Theta_\nabla(\jmath(a^*(\jmath))) = \{t \in \Psi : (\nabla \jmath)(t) \geq a^*(\jmath)\} = [\nabla \jmath]_{\alpha(\jmath)}$$

and

$$\Lambda_\triangle(\ell(a^*(\ell))) = \{t \in \Psi : (\triangle \ell)(t) \geq a^*(\ell)\} = [\triangle \ell]_{\alpha(\ell)}.$$

Then,

$$\begin{aligned}\varrho(\jmath, [\nabla \jmath]_{\alpha(\jmath)}) &= \inf\{\varrho(\jmath, q_1) : q_1 \in [\nabla \jmath]_{\alpha(\jmath)}\} \\ &= \inf\{\varrho(\jmath, q_1) : q_1 \in \Theta_\nabla(\jmath(a^*(\jmath)))\} \\ &= p_{EX}^{a^*(\jmath)}(\jmath, \Theta_\nabla(\jmath(a^*(\jmath)))),\end{aligned}$$

$$\begin{aligned}\varrho(\ell, [\triangle \ell]_{\alpha(\ell)}) &= \inf\{\varrho(\ell, q_2) : q_2 \in [\triangle \ell]_{\alpha(\ell)}\} \\ &= \inf\{\varrho(\ell, q_2) : q_2 \in \Lambda_\triangle(\ell(a^*(\ell)))\} \\ &= p_{EX}^{a^*(\ell)}(\ell, \Lambda_\triangle(\ell(a^*(\ell)))).\end{aligned}$$

Similarly,

$$\varrho(\jmath, [\triangle \ell]_{\alpha(\ell)}) = p_{EX}^{a^*(\ell)}(\jmath, \Lambda_\triangle(\ell(a^*(\ell)))).$$

$$\varrho(\ell, [\nabla \jmath]_{\alpha(\jmath)}) = p_{EX}^{a^*(\ell)}(\ell, \Theta_\nabla(\ell(a^*(\ell)))).$$

Therefore,

$$\begin{aligned}\inf E^{\varrho}_{(\Theta_\nabla(a^*(\jmath)), \Lambda_\triangle(a^*(\ell)))} &= \aleph([\nabla \jmath]_{\alpha(\jmath)}, [\triangle \ell]_{\alpha(\ell)}) \\ &\leq l_1 \varrho(\jmath, [\nabla \jmath]_{\alpha(\jmath)}) + l_2 \varrho(\ell, [\triangle \ell]_{\alpha(\ell)}) + l_3 \varrho(\jmath, [\triangle \ell]_{\alpha(\ell)}) \\ &\quad + l_4 \varrho(\ell, [\nabla \jmath]_{\alpha(\jmath)}) + l_5 \varrho(\jmath, \ell) \\ &= l_1 p_{EX}^{a^*(\jmath)}(\jmath, \Theta_\nabla(\jmath(a^*(\jmath)))) + l_2 p_{EX}^{a^*(\ell)}(\ell, \Lambda_\triangle(\ell(a^*(\ell)))) \\ &\quad + l_3 p_{EX}^{a^*(\jmath)}(\jmath, \Lambda_\triangle(\jmath(a^*(\jmath)))) \\ &\quad + l_4 p_{EX}^{a^*(\ell)}(\ell, \Theta_\nabla(\jmath(a^*(\jmath)))) + l_5 \varrho(\jmath, \ell).\end{aligned}$$

Consequently, Theorem 12 can be applied to find $u \in \Psi$ such that $u \in \Theta_{\nabla u} \cap \Lambda_{\triangle u} = [\nabla u]_{\alpha(u)} \cap [\triangle u]_{\alpha(u)}$.□

Definition 5. [11] Let Ψ be a reference set and ∇ be a fuzzy set in Ψ. Then, ∇ is called an approximate quantity if and only if its α-level set is a compact convex subset of Ψ for each $\alpha \in [0,1]$ and $sup_{\jmath \in \Psi} \nabla(\jmath) = 1$. The set of all approximate *quantities in* Ψ is denoted by $\widetilde{W}$.

Corollary 7. [5] Let (Ψ, ϱ) be a complete metric space and $\nabla, \triangle : \Psi \longrightarrow \widetilde{W}$ be fuzzy set-valued maps. Assume that for all $\jmath, \ell \in \Psi$,

$$\varrho_\infty(\nabla(\jmath), \triangle(\ell)) \leq l_1 p(\jmath, \nabla(\jmath)) + l_2 p(\ell, \triangle(\ell)) + l_3 p(\jmath, \triangle(\ell)) + l_4 p(\ell, \nabla(\jmath)) + l_5 \varrho(\jmath, \ell)$$

where $\sum_{i=1}^{5} l_i < 1$ $(l_i \geq 0)$ and either $l_1 = l_2$ or $l_3 = l_4$. Then, there exists $u \in \Psi$ such that $\{u\} \subset \nabla(u)$ and $\{u\} \subset \triangle(u)$.

Proof. For $\jmath \in \Psi$, let $\Theta_\nabla \jmath, \Lambda_\triangle \jmath \in [P(\Psi)]^E$. Then, following the proof of Corollary 6, one deduces that

$[\nabla \jmath]_1 = \Theta_{\nabla}\jmath(1) \subseteq \widetilde{W}$ and $[\Delta\, \jmath]_1 = \Lambda_{\Delta}\jmath(1) \subseteq \widetilde{W}$,
for some $a^*(\jmath) = 1 \in E$. Now, by Lemma 5, there exists $\jmath_1 \in \Psi$ such that $\{\jmath_1\} \subset \Theta_{\nabla_\jmath}(1) = [\nabla \jmath]_1$ and $\jmath_2 \in \Psi$ such that $\{\jmath_2\} \subset \Lambda_{\Delta_\jmath}(1) = [\Delta\, \jmath]_1$. By definition of $\inf E^{\varrho}_{(\nabla_a, \Delta_b)}$ for soft sets and ϱ_∞-metric for fuzzy sets, for all $\jmath, \ell \in \Psi$, we have

$$\inf E^{\varrho}_{(\nabla_{\jmath(1)}, \Delta_{\ell(1)})} = \aleph([\nabla \jmath]_{\alpha(\jmath)}, [\Delta\, \ell]_{\alpha(\ell)})$$
$$\leq \varrho_\infty(\nabla(\jmath), \Delta\, (\ell)).$$

Hence,

$$\inf E^{\varrho}_{(\nabla_{\jmath(1)}, \Delta_{\ell(1)})} \leq l_1 p(\jmath, \nabla(\jmath)) + l_2 p(\ell, \Delta\, (\ell)) + l_3 p(\jmath, \Delta\, (\ell)) \quad (17)$$
$$+ l_4 p(\ell, \nabla(\jmath)) + l_5 \varrho(\jmath, \ell). \quad (18)$$

Since $[\nabla \jmath]_1 \subseteq [\nabla \jmath]_{\alpha(\jmath)}$ for each $\alpha \in [0,1]$, thus

$$\varrho(\jmath, [\nabla \jmath]_{\alpha(\jmath)}) \leq \varrho(\jmath, [\nabla \jmath]_1) = p^1_{EX}(\jmath, \Theta_\nabla(\jmath(1))).$$

That is, $p(\jmath, \nabla(\jmath)) \leq p^1_{EX}(\jmath, \Theta_\nabla(\jmath(1)))$. Similarly, $p(\jmath, \Delta\, (\jmath)) \leq p^1_{EX}(\jmath, \Lambda_\Delta(\jmath(1)))$. Therefore,

$$\inf E^{\varrho}_{(\nabla_{\jmath(1)}, \Delta_{\ell(1)})} \leq l_1 p^1_{EX}(\jmath, \Theta_{\nabla_\jmath}(1)) + l_2 p^1_{EX}(\ell, \Lambda_{\Delta\ell}(1)) + l_3 p^1_{EX}(\jmath, \Lambda_\Delta(\ell(1)))$$
$$+ l_4 p^1_{EX}(\ell, \Theta_\nabla(\jmath(1))) + l_5 \varrho(\jmath, \ell).$$

Consequently, Theorem 2 can be applied to obtain $u \in \Psi$ such that $\{u\} \subset \{\nabla u\}$ and $\{u\} \subset \Delta\, (u)$. ☐

Corollary 8. [2] Let (Ψ, ϱ) be a complete metric space and $\nabla, \Delta : \Psi \longrightarrow \mathbb{X}^*$ be multivalued mappings. Assume that for all $\jmath, \ell \in \Psi$,

$$\aleph(\nabla \jmath, \Delta\, \ell) \leq l_1 \varrho(\jmath, \nabla \jmath) + l_2 \varrho(\ell, \Delta\, \ell) + l_3 \varrho(\jmath, \Delta\, \ell)$$
$$+ l_4 \varrho(\ell, \nabla \jmath) + l_5 \varrho(\jmath, \ell)$$

where $\sum_{i=1}^{5} l_1 < 1$ and either $l_1 = l_2$ or $l_3 = l_4$. Then there exists $u \in \Psi$ such that $u \in \nabla u \cap \Delta\, u$.

Proof. For $\jmath, \ell \in \Psi$, let $E = \{a^*(\jmath), a^*(\ell)\}$ and consider two soft set-valued maps $\Theta, \Lambda : \Psi \longrightarrow [P(\Psi)]^E$, defined by

$$\Theta_e(\jmath) = \begin{cases} \nabla \jmath, & if \quad e = a^*(\jmath) \\ \Psi, & if \quad e = a^*(\ell). \end{cases}$$

$$\Lambda_e(\jmath) = \begin{cases} \Psi, & if \quad e = a^*(\jmath) \\ \Delta\, \jmath, & if \quad e = a^*(\ell). \end{cases}$$

Then,

$$\varrho(\jmath, \nabla \jmath) = \inf\{\varrho(\jmath, r_1) : r_1 \in \nabla \jmath\}$$
$$= \inf\{\varrho(\jmath, r_1) : r_1 \in \Theta_e(\jmath)\}$$
$$= p^{a^*(\jmath)}_{EX}(\jmath, \Theta(\jmath)).$$

$$\varrho(\ell, \Delta\, \ell) = \inf\{\varrho(\ell, r_2) : r_2 \in \Delta\, \ell\}$$
$$= \inf\{\varrho(\ell, r_2) : r_2 \in \Lambda_e(\jmath)\}$$
$$= p^{a^*(\ell)}_{EX}(\ell, \Lambda(\jmath)).$$

Similarly, $\varrho(\jmath, \Delta\, \ell) = p^{a^*(\jmath)}_{EX}(\jmath, \Lambda\ell)$, and $\varrho(\ell, \nabla \jmath) = p^{a^*(\ell)}_{EX}(\ell, \Theta\jmath)$. Therefore, for all $\jmath, \ell \in \Psi$,

$$\inf E^{\varrho}_{(\Theta_{a^*(\jmath)}, \Lambda_{a^*(\ell)})} = \aleph(\nabla \jmath, \Delta\, \ell)$$
$$\leq l_1 \varrho(\jmath, \nabla \jmath) + l_2 \varrho(\ell, \Delta\, \ell) + l_3 \varrho(\jmath, \Delta\, \ell)$$
$$+ l_4 \varrho(\ell, \nabla \jmath) + l_5 \varrho(\jmath, \ell)$$
$$= l_a p^{a^*(\jmath)}_{EX}(\jmath, \Theta\jmath) + l_2 p^{a^*(\ell)}_{EX}(\ell, \Lambda\ell) + l_3 p^{a^*(\jmath)}_{EX}(\jmath, \Lambda\ell)$$
$$+ l_4 p^{a^*(\ell)}_{EX}(\ell, \Theta\jmath) + l_5 \varrho(\jmath, \ell).$$

Consequently, Theorem 2 can be applied to obtain $u \in \Psi$ such that $u \in \Theta u \cap \Lambda u = \nabla u \cap \Delta\, u$. ☐

Corollary 9. [10, Theorem 12] Let (Ψ, ϱ) be a complete metric space and $T: \Psi \longrightarrow \mathbb{X}^*$ be a multivalued map. Assume that there exists $\gamma \in (0,1)$ such that for all $\jmath, \ell \in \Psi$, we have

$$\aleph(T\jmath, T\ell) \leq \gamma \max\{\varrho(\jmath,\ell), \varrho(\jmath, T\jmath), \varrho(\ell, T\ell), \frac{1}{2}[\varrho(\jmath, T\ell) + \varrho(\ell, T\jmath)].$$

Then, there exists $u \in \Psi$ such that $u \in Tu$.

Proof. By using Corollary 8, the proof follows the idea of Corollary 9. ☐

5. Conclusion

In this research, a novel type of multi-valued map whose range set is a family of soft sets has studied. Specifically, a few fixed point theorems which are generalizations of some known fixed point results of single-valued and set-valued mappings in the corresponding literature have presented. Some of these special cases have highlighted and discussed. Moreover, nontrivial examples have constructed to validate the assumptions of the obtained results. It is known that the notion of cut sets in the study of fuzzy fixed point theorems is one of the elegant ways of connecting fixed point results of contractive multi-valued mappings with fuzzy set-valued maps. Hence, the missing of this concept in fixed point theory of soft set-valued maps is an enormous limitation. Along this line, it is important to point out that the ideas of this paper, being established in the framework of metric space is fundamental. Hence, it can be improved upon when examined in the setting of quasi or pseudo metric spaces. In addition, the soft set-valued component can be examined in other hybrid models such as fuzzy soft sets, N-soft sets, intuitionistic fuzzy soft sets, rough sets, and so on. From application viewpoint, the new contractions in this work can be employed to analyses solvability criteria of some classes of differential and integral inclusions of either integer or non-integer type.

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