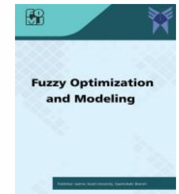
FUZZY OPTIMIZATION AND MODELLING 3(2) (2022) 23-39

E-ISNN: 2676-7007

Contents lists available at FOMJ

Fuzzy Optimization and Modelling

Journal homepage: http://fomj.qaemshahr.iau.ir/



Paper Type: Research Paper

The Interval Rational- Interpolation Method

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A R T I C L E I N F O

Article history:
Received 6 November 2021
Revised 27 December 2021
Accepted 11 January 2022
Available online 1 April 2022

Keywords:
Interval Value
Rational Interpolation
Interval Arithmetic
Rational Function

A B S T R A C T

The rational interpolation sometimes gives better approximations than polynomial interpolation particularly for large sequence of points. In this paper, for the first time we provide a combination of rational interpolation and interval data. we present applied interval arithmetic in rational interpolation, when support points are interval-valued. First, we introduce the basic concepts of the algebraic theories to apply the interval methods to uncertainty analysis. Then the interpolation of interval coefficient is obtained and also the error of the proposed method is analysed and is proved by a theorem for different cases. Some different numerical examples are given to illustrate the proposed method and the results are recorded.

1. Introduction

Interval analysis plays an important role in many fields such as fuzzy theory, statistics and probability theory, approximation theory, and computer science. Many people worked on bounding rounding using Moore's book interval analysis [12] in the 1950's, Hansen [4] provided an overview of the early publication by moor from its introduction in the late 1960's. In 1991 the interval computations journal published the first article. In the last few decades there was an increase in the applications of interval analysis in a wide range of domains, from robotics to neural networks. Despite all these improvements in the number of these applications, a typical problem in biological applications interpolation involving interval data often arises. The main goal of this paper is to present interpolation by Rational Functions under Interval data. We summarize a rational interpolation: as Rational functions are sometimes superior to polynomials, roughly speaking, owing to their ability of modeling functions with poles, which is, zeroes of the denominate of equation. These poles might occur in real values of x. If the function is interpolated, it if has some poles. Such poles can themselves ruin a polynomial

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DOI: 10.30495/fomj.2022.1944092.1045

approximation, even one restricted to real value of x, just as they can ruin the convergence of an infinite power series in x. A rational function approximation, by contrast, will stay 'good' as long as it has enough powers of x in its denominator to account for canceling any nearby poles. Gutknecht [7] presented a general homogeneous treatment of rational interpolation problem in the extended complex domain. Markov et al. [16] designed the linear interpolation under interval data. Hosseini and Jafari [10] presented an extended rational interpolation method. A typical with rational interpolation (unattainable data) was provided by Salazar Ceils et.al in 2008. In 2014, application of interval, algebra operation in interpolation when the support points are intervals were computed interpolation polynomial by newton' divided difference. In 2015, Markov studied some interpolation problems, involving interval data [11].

Rational interpolant can produce satisfying shape-preserving interpolation. There are many effective methods for the construction of shape-preserving interpolant. In [1, 6, 15], rational cubic interpolation splines were constructed to visualize positive data. A united form of the classical Hermite interpolation and shape-preserving interpolation is presented in [5]. In addition, in 2018, Arandiga [2] described an improvement on this nonlinear adaptive rational interpolator. There was a second order non-linear interpolation technique where the convex combination is given by the end-points of the interval. In Yun [8] proposed a new constructive piecewise interpolation method. Tyada et al. [17] introduced shape preserving rational cubic trigonometric fractal interpolation functions.

Our paper is organized as follow. In Section 2, at first, we introduce a basic of interval arithmetic, then, we discuss its rational interpolation. In Section 3, we present our proposed method. In Section 4, some numerical examples are given to present a better illustration for proposed method and the conclusion is explained Section 5.

2. Preliminaries

2.1: Interval number and interval arithmetic

Interval analysis is the theory dealing with interval numbers and arithmetic operations.

Definitions 1. An interval number $[x]$ is defined as a set of all real number x which holds the conditions:

$$[x] = [\underline{x}, \overline{x}] = \{x \in R\, , \underline{x} \leq x \leq \overline{x}\}$$

where $\underline{x}$ *and* $\overline{x}$ denote the lower and upper bounds of interval $[x]$, respectively. Also, $\underline{x} \leq \overline{x}$, $\underline{x}$ *and* $\overline{x}$ are called infimum and supremum. These numbers are also called proper numbers.

The basic computational operations of addition, subtraction, multiplication, and division are as follows:

Definition 2. If $[x]$ and $[y]$ are interval then

$$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$
$$[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$$
$$[x].[y] = [min(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y})\, , max\,(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y})]$$
$$[x]/[y] = [x].[1/\overline{y}, 1/\underline{y}] \qquad \text{if } \ 0 \notin y$$

In the case that $0 \in [y]$, Hansen [4] has defined a set of extended rules for intervals [14]:

$$
[x]/[y] = \begin{cases}
[\overline{x}/\underline{y}, \infty) & if \quad \overline{x} \leq 0 \; and \; \overline{y} = 0 \\
(-\infty, \overline{x}/\overline{y}] \cup [\overline{x}/\underline{y}, \infty) & if \quad \overline{x} \leq 0 \; and \; \underline{y} < 0 < \overline{y} \\
(-\infty, \overline{x}/\overline{y}] & if \quad \overline{x} \leq 0 \; and \; \underline{y} = 0 \\
(-\infty, +\infty) & if \quad \underline{x} < 0 < \overline{x} \\
(-\infty, +\infty) & if \quad \underline{x} \geq 0 \; and \; \overline{x} = 0 \\
(-\infty, \underline{x}/\overline{y}] \cup [\underline{x}/\underline{y}, \infty) & if \quad \underline{x} \geq 0 \; and \; \underline{y} < 0 < \overline{y} \\
[\underline{x}/\overline{y}, \infty) & if \quad \underline{x} \geq 0 \; and \; \overline{y} = 0
\end{cases}
$$

Some properties of interval arithmetic operations:
If $[x]$, $[y]$, and $[z]$ are interval numbers then:

(i) Associative properly is:

1. $([x] + [y]) + [z] = [x] + ([y] + [z])$
2. $([x] * [y]) + [z] = [x] * ([y] * [z])$

(ii) The sub distributive Laws are

1. $[x] * ([y] \pm [z] \subseteq [x] * [y] \pm [x] * [z])$
2. $[x] - [y] \subseteq ([x] + [y]) - ([y] + [z])$
3. $\frac{[x]}{[y]} \subseteq \frac{[xz]}{[yz]}$

All interval arithmetic operations are based on the inclusion principle, which has sometimes referred to the fundamental theorem of interval analysis, which states at the outcome of the operation on a subset of the input interval arguments is included in the outcome of the operation performed on the complete input interval the subsets can be smaller intervals or crisp numbers:

$$x_1^* \in [x_1], x_2^* \in [x_2], \ldots, x_n^* \in [x_n] \Rightarrow f(x_1^*, x_2^*, \ldots, x_n^*) \subset [f([x_1], [x_2], \ldots, [x_n])].$$

2.2 Rational function interpolation

Consider a given set of support points (x_i, f_i) , $i = 0,1,2, \ldots, \mu + v$.

We can interpolate these points by using the following rational function:

$$\emptyset^{\mu,v}(x) = \frac{p^{\mu}(x)}{q^{v}(x)} = \frac{a_0 + a_1 x + \cdots + a_{\mu} x^{\mu}}{b_0 + b_1 x + \cdots + b_v x^{v}}$$

The function $\emptyset^{\mu,v}(x)$is called rational interpolating function to support points (x_i, f_i) , $i = 0,1,2, \ldots, \mu + v$, and the integers μ and v are called the Maximum degrees of the polynomials in the numerator and de monitor respectively, such that satisfies the interpolation conditions:

$$\emptyset^{\mu,v}(x) = \frac{p^{\mu}(x)}{q^{v}(x)} = \frac{a_0 + a_1 x + \cdots + a_{\mu} x^{\mu}}{b_0 + b_1 x + \cdots + b_v x^{v}} = f_i \;, \quad i = 0,1,2, \ldots, \mu + v \qquad (1)$$

We denote the problem of calculating the rational function $\emptyset^{\mu,v}(x)$ from (1) by $A^{\mu,v}$, since solving the above equations $A^{\mu,v}$ is difficult, we can solve the homogeneous system of linear equations

$$p^{\mu,v}(x_i) - f_i \ \emptyset^{\mu,v}(x_i) = 0 \rightarrow (a_0 + a_1 x_i + \cdots + a_\mu {x_i}^{\mu}) - f_i \ (b_0 + b_1 x_i + \cdots + b_v {x_i}^{v}) = 0 \ ,$$

$$i = 0,1,2, \dots, \mu + v$$

Denote this system by $S^{\mu,v}$.

The system, $S^{\mu,v}$, has always nontrivial solutions because it is a system with $\mu + v + 1$ equations and $\mu + v + 2$ unknowns .

Theorem 1. The homogeneous linear system, $S^{\mu,v}$, has always non trivial solutions. For each of such solutions, $\emptyset^{\mu,v}(x) = \frac{P^{\mu,v}(x)}{Q^{\mu,v}(x)}$, $\phi^{\mu,v} \neq 0$ holds, i.e., all nontrivial define rational expressions [16].

Theorem 2. if φ_1 and φ_2 are both (nontrivial) solutions of the homogeneous linear system $S^{\mu,v}$ then they a are $\varphi_1 \sim \varphi_2$, that is, they determine the same rational function [16].

It should be noted that the answer derived from $S^{\mu,v}$ is not always correct in $A^{\mu,v}$, of course, by satisfying each of the following conditions we can be sure that the result from $S^{\mu,v}$ is correct in $A^{\mu,v}$($\emptyset^{\mu,v}$ solves the system $A^{\mu,v}$) [16]:

(i) If $S^{\mu,v}$ has a solution as $\emptyset^{\mu,v}(x)$, which is prime too, then $\emptyset^{\mu,v}(x)$ will also solve $A^{\mu,v}$.

(ii) If $\emptyset^{\mu,v}(x)$ is a solution from $S^{\mu,v}$, $\emptyset^{\mu,v}(x) \equiv \widetilde{\emptyset}^{\mu,v}(x)$, and $\widetilde{\emptyset}^{\mu,v}(x)$ is prime, then $A^{\mu,v}$ is solvable if and only if $\widetilde{\emptyset}^{\mu,v}(x)$ solves the system $S^{\mu,v}$.

(iii) If $S^{\mu,v}$ has the full rank, then $A^{\mu,v}$ is solvable if and only if the solution $\emptyset^{\mu,v}(x)$ from $S^{\mu,v}$ is prime.

3. The interval rational interpolation

Given finite interval-valued data

$$(X_i, [\underline{f_i}, \overline{f_i}]) \qquad i = 0,1, \dots, n$$

We consider the problem of interval rational interpolation

$$\emptyset(x\,;\ \tilde{C}_0, \dots, \tilde{C}_n) = \sum_{j=0}^{n} \tilde{C}_j \emptyset_j\,(x)$$

such that satisfies the interpolation conditions:

$$\emptyset(x\,;\ \tilde{C}_0, \dots, \tilde{C}_n) = [\underline{f_i}, \overline{f_i}] \qquad i = 0, \dots, n \tag{2}$$

$$\frac{\tilde{P}^{\mu}(x_i)}{\tilde{Q}^{v}(x_i)} = [\underline{f_i}, \overline{f_i}]$$

where:

$$\begin{cases} \tilde{P}^{\mu}(x_i; \widetilde{a_0}, \dots, \widetilde{a_n}) = \sum_{j=0}^{n} \tilde{a}_j P_j(x_i) \\ \tilde{Q}^{\mu}(x_i; \widetilde{b_0}, \dots, \widetilde{b_n}) = \sum_{j=0}^{n} \tilde{b}_j Q_j(x_i) \end{cases}$$

Based on the right side of equation it is clear that:

$$\left[\overline{\emptyset_i}, \underline{\emptyset_i}\right] = [\overline{f_i}, \underline{f_i}]$$

$$\emptyset(x_i; \widetilde{C_0}, \dots, \widetilde{C_n}) = \left[\overline{\emptyset_i}, \underline{\emptyset_i}\right]$$

where,

$$\begin{cases} \underline{\emptyset_i} = \min\{u \mid \underline{\emptyset_i} \le u \le \overline{\emptyset_i}\} \\ \overline{\emptyset_i} = \max\{u \mid \underline{\emptyset_i} \le u \le \overline{\emptyset_i}\} \end{cases}$$

According to the equality of intervals:

$$L: \ \underline{\emptyset}\left(x_i; \widetilde{C_0}, \dots, \widetilde{C_n}\right) = \underline{f_i} \quad (3)$$

$$U: \ \overline{\emptyset}\left(x_i; \widetilde{C_0}, \dots, \widetilde{C_n}\right) = \overline{f_i} \qquad i = 0, \dots, n \quad (4)$$

Considering the problem of interpolation:

$$\begin{cases} P^{\mu,v}(x; \widetilde{a_0}, \dots, \widetilde{a_n}) = \sum_{j=0}^{n} \tilde{a}_j P_j(x) \\ Q^{\mu,v}\left(x; \widetilde{b_0}, \dots, \widetilde{b_n}\right) = \sum_{j=0}^{n} \tilde{b}_j Q_j(x) \end{cases}$$

It can be written as follow:

$$[\underline{\emptyset}, \overline{\emptyset}] = \emptyset\left(x; \widetilde{C_0}, \dots, \widetilde{C_n}\right) = \frac{P^{\mu,v}(x; \widetilde{a_0}, \dots, \widetilde{a_n})}{Q^{\mu,v}(x; \widetilde{b_0}, \dots, \widetilde{b_n})} \quad (5)$$

For any i, the problems (IRI) (3) and (4) are considered, thus two systems with (n+1)-order will be obtained. We rewrite (2) as follows:

$$\underline{\emptyset}\left(x; \widetilde{C_0}, \dots, \widetilde{C_n}\right) = \underline{\sum_{j=0}^{n} C_j \emptyset_j(x)}$$

From (4) we have:

$$\underline{P}\,(x; a_0, \dots, a_n) = \underline{\sum_{j=0}^{n} a_j P_j(x)} = \sum_{j=0}^{n} \underline{a_j P_j(x)}$$

where:

$$\underline{a_j P_j(x)} = \begin{cases} \underline{a_j} P_j(x) & P_j(x) \ge 0 \\ \overline{a_j} P_j(x) & P_j(x) < 0 \end{cases},$$

and

$$\underline{Q}\,(x; b_0, \dots, b_n) = \underline{\sum_{j=0}^{n} b_j Q_j(x)} = \sum_{j=0}^{n} \underline{b_j Q_j(x)}$$

where:

$$\underline{b_j Q_j(x)} = \begin{cases} \underline{b_j} Q_j(x) & Q_j(x) \ge 0 \\ \overline{b_j} Q_j(x) & Q_j(x) < 0 \end{cases},$$

So it can be written as follow:

$$\underline{P}\ (x\,;a_0,\dots,a_n)=\sum_{P_j(x)\geq 0}\underline{a_j}P_j(x)+\sum_{P_j(x)<0}\overline{a_j}P_j(x)$$

$$\underline{Q}\ (x\,;b_0,\dots,b_n)=\sum_{Q_j(x)\geq 0}\underline{b_j}Q_j(x)+\sum_{Q_j(x)<0}\overline{b_j}Q_j(x)$$

Remark 1. The coefficients $a_0,\dots,a_\mu,b_0,\dots,b_v$ are interval values.

On the other hand,

$$\overline{\emptyset}(x\,;\widetilde{C_0},\dots,\widetilde{C_n})=\overline{\sum_{j=0}^{n}C_j\emptyset_j(x)}$$

From (4) we get:

$$\overline{P}\ (x\,;a_0,\dots,a_n)=\overline{\sum_{j=0}^{n}a_jp_j(x)}=\sum_{j=0}^{n}\overline{a_jp_j(x)}$$

where:

$$\overline{a_jP_j(x)}=\begin{cases}\overline{a_j}P_j(x) & P_j(x)\geq 0\\ \underline{a_j}P_j(x)\underline{a_j} & P_j(x)<0\end{cases},$$

and,

$$\overline{Q}\ (x\,;b_0,\dots,b_n)=\overline{\sum_{j=0}^{n}b_jQ_j(x)}=\sum_{j=0}^{n}\overline{b_jQ(x)}$$

where:

$$\overline{b_jQ_j(x)}=\begin{cases}\overline{b_j}Q_j(x) & Q_j(x)\geq 0\\ \underline{b_j}Q_j(x)\underline{a_j} & Q_j(x)<0\end{cases},$$

So we get:

$$\overline{P}(x\,;a_0,\dots,a_n)=\sum_{P_j(x)\geq 0}\overline{a_j}P_j(x)+\sum_{P_j(x)<0}\underline{a_j}P_j(x)$$

$$\overline{Q}\ (x\,;b_0,\dots,b_n)=\sum_{Q_j(x)\geq 0}\overline{b_j}Q_j(x)+\sum_{Q_j(x)<0}\underline{b_j}Q_j(x)$$

We introduce the problem of the interval interpolation (L) as follow:

$$\underline{\emptyset}^{\mu,v}(x_i)=\frac{\underline{P}^{\mu,v}(x_i)}{\overline{Q}^{\mu,v}(x_i)}=\underline{f_i}\quad,\qquad \underline{\emptyset}(x_i\,;c_0,\dots,c_n)=\frac{\underline{P}(x_i\,;a_0,\dots,a_n)}{\overline{Q}(x_i\,;b_0,\dots,b_n)}=\underline{f_i}$$

Therefore, the system $A^{\mu,v}$ is as below:

$$A^{\mu,v}:\underline{\emptyset}\ (x_i\,;c_0,\dots,c_n)=\underline{f_i}\Rightarrow$$

$$\underline{\emptyset}\ (x_i\,;c_0,\dots,c_n)=\frac{\sum_{P_j(x)\geq 0}\underline{a_j}P_j(x)+\sum_{P_j(x)<0}\overline{a_j}P_j(x)}{\sum_{Q_j(x)\geq 0}\overline{b_j}Q_j(x)+\sum_{Q_j(x)<0}\underline{b_j}Q_j(x)}=\underline{f_i}$$

We denote the following homogeneous system by $S^{\mu,v}$:

$$\underline{S}^{\mu,v}:\underline{\emptyset}\ (x_i\,;c_0,\dots,c_n)=$$

$$= \sum_{P_j(x)\geq 0} \underline{a_j} P_j(x) + \sum_{P_j(x)<0} \overline{a_j} P_j(x) - \underline{f_i}\left(\sum_{Q_j(x)\geq 0} \overline{b_j} Q_j(x) + \sum_{Q_j(x)<0} \underline{b_j} Q_j(x)\right) = \underline{0} \quad (6)$$

Here, $\tilde{0} = (\underline{0}, \overline{0})$.

Similarly, the problem (U) will be as follows:

$$\overline{\emptyset}^{\mu,\nu}(x_i) = \frac{\overline{P}^{\mu,\nu}(x_i)}{\underline{Q}^{\mu,\nu}(x_i)} = \overline{f_i} \quad , \qquad \overline{\emptyset}(x_i\,; c_0, \dots, c_n\,) = \frac{\overline{P}(x_i\,; a_0, \dots, a_n)}{\underline{Q}(x_i\,; b_0, \dots, b_n)} = \overline{f_i}$$

So, we get the systems $A^{\mu,\nu}$, and $S^{\mu,\nu}$, respectively:

$$\overline{\emptyset}(x_i\,; c_0, \dots, c_n\,) = \frac{\sum_{P_j(x)\geq 0} \overline{a_j} P_j(x) + \sum_{P_j(x)<0} \underline{a_j} P_j(x)}{\sum_{Q_j(x)\geq 0} \underline{b_j} Q_j(x) + \sum_{Q_j(x)<0} \overline{b_j} Q_j(x)} = \overline{f_i}$$

And,

$$\overline{S}^{\mu,\nu} : \overline{\emptyset}(x_i\,; c_0, \dots, c_n\,) =$$

$$= \sum_{P_j(x)\geq 0} \overline{a_j} P_j(x) + \sum_{P_j(x)<0} \underline{a_j} P_j(x) - \underline{f_i}\left(\sum_{Q_j(x)\geq 0} \underline{b_j} Q_j(x) + \sum_{Q_j(x)<0} \overline{b_j} Q_j(x)\right) = \overline{0} \quad (7)$$

As a result:

$$L : \underline{\emptyset}\,(x_i\,; c_0, \dots, c_n\,) = \sum_{P_j(x)\geq 0} \underline{a_j} P_j(x) + \sum_{P_j(x)<0} \overline{a_j} P_j(x) - \underline{f_i}\left(\sum_{Q_j(x)\geq 0} \overline{b_j} Q_j(x) + \sum_{Q_j(x)<0} \underline{b_j} Q_j(x)\right) = \underline{0}$$

$$U : \overline{\emptyset}(x_i\,; c_0, \dots, c_n\,) = \sum_{P_j(x)\geq 0} \overline{a_j} P_j(x) + \sum_{P_j(x)<0} \underline{a_j} P_j(x) - \underline{f_i}\left(\sum_{Q_j(x)\geq 0} \underline{b_j} Q_j(x) + \sum_{Q_j(x)<0} \overline{b_j} Q_j(x)\right) = \overline{0}$$

By applying (6), and (7), we can say that $\underline{\emptyset}, \overline{\emptyset}$ are interval values constantly, in this case, we obtain two types of the interval rational interpolation (L), (U), respectively. Which should be unique either, and (L) interpolates support points $(x_i, \underline{f_i})$, in a similar manner, $(x_i, \overline{f_i})$ is interpolated by (U). Hence, here instead of an interpolation function, a strip of the function of rational interpolation is obtained.

Therefore, it is figured that each set of points $(x_i, \underline{f_i})$, and $(x_i, \overline{f_i})$make a function for any i=0,…,n , and the corresponding interpolation problem of them will be introduced, and defined the problem interpolation correspond to them.

Now, if we define $P_j(x) = x^j$, $j = 0, \dots, n$, $Q_s(x) = x^s$, $s = 0, \dots, \nu$, the interval rational interpolation (L), (U), is defined as:

$$L : \underline{\psi}(x) = \frac{\sum_{x^j\geq 0} \underline{a_j} x^j + \sum_{x^j<0} \overline{a_j} x^j}{\sum_{x^s\geq 0} \overline{b_s} x^s + \sum_{x^s<0} \underline{b_s} x^s}$$

(8)

$$U : \overline{\psi}(x) = \frac{\sum_{x^j\geq 0} \overline{a_j} x^j + \sum_{x^j<0} \underline{a_j} x^j}{\sum_{x^s\geq 0} \underline{b_s} x^s + \sum_{x^s<0} \overline{b_s} x^s}$$

It is obvious that considering the *i*-th power of *x* both as odd and even can be written:

$$L : \underline{\psi}(x) = \frac{\sum_{j=2k} \underline{a_j} x^j + \sum_{j=2k+1} \overline{a_j} x^j}{\sum_{s=2k} \overline{b_s} x^s + \sum_{s=2k+1} \underline{b_s} x^s} \qquad x < 0$$

$$\underline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \underline{a_j} x^j}{\sum_{s=0}^{v} \overline{b_s} x^s} \qquad k = 0, \dots, \left[\frac{n}{2}\right] \qquad x \geq 0$$

and:

$$U:\ \overline{\psi}(x) = \frac{\sum_{j=2k} \overline{a_j} x^j + \sum_{j=2k+1} \underline{a_j} x^j}{\sum_{s=2k} \underline{b_s} x^s + \sum_{s=2k+1} \overline{b_s} x^s} \qquad x < 0$$

$$\overline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \overline{a_j} x^j}{\sum_{s=0}^{v} \underline{b_s} x^s} \qquad k = 0, \dots, \left[\frac{n}{2}\right] \qquad x \geq 0$$

The following states can be considered for the system (8):

Case (1): If for all $x, j, and\ s, x^j and\ x^s$ are non- negative.

In this case:

$$L:\ \underline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \underline{a_j} x^j}{\sum_{s=0}^{v} \overline{b_s} x^s}$$

$$U:\ \overline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \overline{a_j} x^j}{\sum_{s=0}^{v} \underline{b_s} x^s}$$

Case (2): If for all $x, j, and\ s, x^j and\ x^s$ are negative, (however, this will never occur due to the constant coefficient). Thus, we have:

$$L:\ \underline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \overline{a_j} x^j}{\sum_{s=0}^{v} \underline{b_s} x^s}$$

$$U:\ \overline{\psi}(x) = \frac{\sum_{j=0}^{\mu} \underline{a_j} x^j}{\sum_{s=0}^{v} \overline{b_s} x^s}$$

Case (3): if for some $x, j, and\ x^j$ are non-negative, and for some $x, j, and\ x^j$ are negative (for some $x, j, and\ x^j$ are negative) for some $x, s, and\ x^s$ are non-negative (for some $x, s, and\ x^s$ are negative) hence, the system (8) yields the same.

Definition 3. Let $\left\{\left(\underline{a_i}, \overline{a_i}\right),\ 1 \leq i \leq \mu\right\}$ and $\{\left(\underline{b_j}, \overline{b_j}\right),\ 1 \leq j \leq v\}$ denote the coefficients of polynomial $(\underline{P(x)}, \overline{P(x)})$, $(\underline{Q(x)}, \overline{Q(x)})$, respectively. The interval-valued coefficient $V = \{\left(\underline{V_i}, \overline{V_i}\right), 1 \leq i \leq \mu\}$ corresponding to $\underline{P(x)}, \overline{P(x)}$ is defined by:

$$\underline{V_i} = \min\left\{\underline{a_0}, \dots, \underline{a_\mu}, \overline{a_0}, \dots, \overline{a_\mu}\right\}$$

$$\overline{V_i} = \max\left\{\underline{a_0}, \dots, \underline{a_\mu}, \overline{a_0}, \dots, \overline{a_\mu}\right\}$$

It is called interval coefficients of the interval polynomial, $\underline{P(x)}, \overline{P(x)}$.

In a similar manner, $U\{\left(\underline{U_j}, \overline{U_j}\right), 1 \leq j \leq v\}$ corresponds to $\underline{Q(x)}, \overline{Q(x)}$,which are denoted by:

$$\underline{U_j} = \min\left\{\underline{b_0}, \dots, \underline{b_v}, \overline{b_0}, \dots, \overline{b_v}\right\}$$

$$\overline{U_j} = \max\left\{\underline{b_0}, \dots, \underline{b_v}, \overline{b_0}, \dots, \overline{b_v}\right\}$$

It is called interval coefficients of the interval polynomial, $\underline{Q(x)}, \overline{Q(x)}$.

If $\left\{\left(\underline{a_i}, \overline{a_i}\right),\ 1 \leq i \leq \mu\right\}$, $\{\left(\underline{b_j}, \overline{b_j}\right),\ 1 \leq j \leq v\}$ are all interval coefficients which $\underline{V_i} = \underline{a_i}$, $\overline{V_i} = \overline{a_i}$. Similarly, $\underline{U_j} = \underline{b_j}, \overline{U_j} = \overline{b_j}$, $1 \leq i \leq \mu$, $1 \leq j \leq v$ then U, V are called a robust interval coefficient as a result, rational interpolator corresponds to the robust. Otherwise, U and V are called weak interval rational interpolator corresponding to the weak.

3.2. The Error of the interval interpolation

Since the error in the numerical method is inevitable, we are going to analyze the error in this method of the rational interpolation under interval data.

Theorem 3. Consider $\left(x_i, \left[\underline{f_i}, \overline{f_i}\right]\right)$, $i = 0, \dots, n$ the support points (x_i, f_i) for function $[\underline{f_i}, \overline{f_i}]$, and $[\underline{\psi_i(x)}, \overline{\psi_i(x)}]$, rational interpolation, be the corresponding to $[\underline{f_i}, \overline{f_i}]$. We suppose that the derivatives of functions, $\underline{f(x)}$, $\overline{f(x)}$, up to (n+1)-th order in the domain of definition are continuous and they exist, respectively then:

$$\underline{f(x)} = \frac{p(x)}{q(x)} + \underline{R(x)}$$

where:

$$\underline{R(x)} = \frac{(x - x_1) \dots (x - x_n)}{\overline{Q(x)} \cdot \overline{q(x)} \cdot n!} \cdot \left[\frac{d^n}{dx^n}\left(\overline{Q(x)} \cdot \underline{f(x)} \cdot \overline{q(x)}\right)\right]_\eta$$

Similarly, for $\overline{R(x)}$ we have:

$$\overline{R(x)} = \frac{(x - x_1) \dots (x - x_n)}{\underline{Q(x)} \cdot \underline{q(x)} \cdot n!} \cdot \left[\frac{d^n}{dx^n}\left(\underline{Q(x)} \cdot \overline{f(x)} \cdot \underline{q(x)}\right)\right]_\eta.$$

Proof: Suppose $\alpha_1, \dots, \alpha_t$e is inaccessible points in$[a, b]$ with iterated order $r_1, \dots, r_t$ where:

$$\underline{\varphi(x)} = (x - \alpha_1)^{r_1} \dots (x - \alpha_t)^{r_t} \qquad \deg\left(\overline{q(x)}\right) \geq m, \ \sum_{i=1}^{m} r_i = m.$$

So, $\underline{f(x)} \cdot \underline{\varphi(x)}$, $\underline{f(x)} \cdot \overline{\varphi(x)}$, $\overline{f(x)} \cdot \underline{\varphi(x)}$ and $\overline{f(x)} \cdot \overline{\varphi(x)}$ are defined on $[a, b]$. $\overline{T(x)}$ is determined as following :

$$\overline{Q(x)} = \begin{cases} \overline{T(x)} \cdot \overline{\varphi(x)} \\ \overline{T(x)} \cdot \underline{\varphi(x)} \end{cases} or \qquad , \deg\left(\overline{Q(x)}\right) = \deg\left(\overline{q(x)}\right)$$

On the other hand, $\alpha_1, \dots, \alpha_s$ are inaccessible points in $[a, b]$ with iterated order $r_1, \dots, r_s$ whereas,

$$\overline{\varphi(x)} = (x - \alpha_1)^{r_1} \dots (x - \alpha_s)^{r_s} \qquad \deg\left(\underline{q(x)}\right) \geq n, \ \sum_{i=1}^{m} r_i = n..$$

Therefore, $\underline{f(x)} \cdot \underline{\varphi(x)}$, $\underline{f(x)} \cdot \overline{\varphi(x)}$, $\overline{f(x)} \cdot \underline{\varphi(x)}$ and $\overline{f(x)} \cdot \overline{\varphi(x)}$ are defined on $[a, b]$. $\underline{T(x)}$ determine such that:

$$\underline{Q(x)} = \begin{cases} \underline{T(x)} \cdot \underline{\varphi(x)} \\ \underline{T(x)} \cdot \overline{\varphi(x)} \end{cases} or \qquad , \deg\left(\underline{Q(x)}\right) = \deg\left(\underline{q(x)}\right)$$

We consider the following two cases:

Case (1): $n = 2h \Rightarrow \mu + v + 1 = 2h \Rightarrow \mu + v = 2h - 1.$

Hence, in the rational interpolation, $\frac{p(x)}{q(x)}$, $degree\ of\ \underline{p(x)}\ and\ \overline{q(x)}\ are\ h\ and\ h - 1$, respectively. So we can say that, $\deg\left(\underline{p(x)} \cdot \overline{Q(x)}\right) = 2h - 1 = n - 1.$

Case (2): $n = 2h + 1 \Rightarrow \mu + v + 1 = 2h + 1 \Rightarrow \mu + v = 2h.$

In the case, degree of $\underline{p(x)}$and $\overline{q(x)}$ are h. In addition, we will have:

$$\deg\left(\underline{p(x)} \cdot \overline{Q(x)}\right) = 2h = n - 1$$

Now, we define as follows:

$$\underline{R(x)} = \underline{k}\,.\frac{(x-x_1)\dots(x-x_n)}{\overline{Q(x)}\,.\,\overline{q(x)}}$$

$$\underline{f(t)} = \frac{\underline{\underline{p(t)}}}{\overline{f(t)}} + \underline{k}\,.\frac{(t-x_1)\dots(t-x_n)}{\overline{Q(t)}\,.\overline{q(t)}} \quad (9)$$

By using (1), we get:

$$w(t) = \underline{f(t)}.\overline{Q(t)}.\overline{q(t)} - \underline{p(t)}.\overline{Q(t)} - \underline{k}.(t-x_1)\dots(t-x_n)$$

We can find a constant k such that $w(t)$has zeros, $x_0,\dots,x_n,\overline{x}$, by Rolle's theorem, repeatedly, $w'(t)$ has at least n zeros, $w''(t)$ at least $n-1$ zeros and finally $w^n(t)$ at least one zeros. If the same zero is denoted by η. Since $w^n(\eta)$=0 .

$$w^n(\eta) = [\frac{d^n}{dt^n}(\underline{f(t)}.\overline{Q(t)}.\overline{q(t)})]_\eta - 0 - \underline{k}.n! = 0 \quad (10)$$

Thus, from (9), (10), we have:

$$\underline{R(x)} = \frac{(x-x_1)\dots(x-x_n)}{\overline{Q(x)}\,.\,\overline{q(x)}\,.\,n!}.[\frac{d^n}{dx^n}(\overline{Q(x)}.\underline{f(x)}.\overline{q(x)})]_\eta$$

Furthermore, similar $\underline{R(x)}$, we obtain:

$$\overline{R(x)} = \frac{(x-x_1)\dots(x-x_n)}{\underline{Q(x)}\,.\,\underline{q(x)}\,.\,n!}.[\frac{d^n}{dx^n}(\underline{Q(x)}.\overline{f(x)}.\underline{q(x)})]_\eta$$

The proof is complete. ☐

4. Numerical example

In this section we solve examples to illustrate our approach.

Example 1. Consider the points given in Table 1.

Table 1. Data of Example 1

x_i	0	1	2	3
$[\underline{f_i},\overline{f_i}]$	[-0.5,1]	[-1.5]	[-1,0.5]	[8,9.5]

Here,

$$\mu + v = 3 \quad \mu = 2\ ,\ v = 2$$

By using case (1) the interval rational expression

$$\underline{\emptyset^{2,1}}(x_i) = \frac{\underline{a_0} + \underline{a_1}x_i + \underline{a_2}x_i^2}{\overline{b_0} + \overline{b_1}x_i} = \underline{f_i}$$

From (3), we have the homogeneous system $\underline{S^{2,1}}$

$$\underline{S^{2,1}}:\ \underline{a_0} + \underline{a_1}x_i + \underline{a_2}x_i^2 - \underline{f_i}(\overline{b_0} + \overline{b_1}x_i) = 0$$

$$\underline{a_0} + 0.5\overline{b_0} = 0$$

$$\underline{a_0} + \underline{a_1} + \underline{a_2} + 1.5\overline{b_0} + 1.5\overline{b_1} = 0$$

$$\underline{a_0} + 2\underline{a_1} + 4\underline{a_2} + \overline{b_0} + 2\overline{b_1} = 0$$

$$\underline{a_0} + 3\underline{a_1} + 9\underline{a_2} - 8\overline{b_0} + \overline{24b_1} = 0$$

$$\overline{\emptyset^{2,1}}(x_i) = \frac{\overline{a_0} + \overline{a_1}x_i + \overline{a_2}x_i^2}{\underline{b_0} + \underline{b_1}x_i} = \overline{f_i}$$

Similarly:

$$\overline{S^{2,1}} : \overline{a_0} + \overline{a_1} x_i + \overline{a_2} x_i^2 - \overline{f_i}\left(\underline{b_0} + \underline{b_1} x_i\right) = 0$$

$$\overline{a_0} - \underline{b_0} = 0$$
$$\overline{a_0} + \overline{a_1} + \overline{a_2} = 0$$
$$\overline{a_0} + 2\overline{a_1} + 4\overline{a_2} - 0.5\underline{b_0} - \underline{b_1} = 0$$
$$\overline{a_0} + 3\overline{a_1} + 9\overline{a_2} - 9.5\underline{b_0} - 28.5\underline{b_1} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1.5 \\ 1 & 2 & 4 & 0 & 0 & 0 & 0 & 2 \\ 1 & 3 & 9 & 0 & 0 & 0 & 0 & -24 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & -28.5 & 1 & 3 & 9 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{a_0} \\ \underline{a_1} \\ \underline{a_2} \\ \underline{b_1} \\ \overline{a_0} \\ \overline{a_1} \\ \overline{a_2} \\ \overline{b_1} \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 16 \\ 0.5 \\ 0 \\ 0.25 \\ 4.75 \end{bmatrix}$$

$$b_0 = [0.5\ ,2]$$

Now, solving the above systems $\underline{S^{2,1}}$, $\overline{S^{2,1}}$yields the coefficients:

$$\underline{a} = \begin{bmatrix} -1 \\ -2.40196 \\ 1.22549 \end{bmatrix} \quad , \quad \overline{a} = \begin{bmatrix} 0.5 \\ -0.806373 \\ 0.306373 \end{bmatrix}$$
$$\underline{b} = \begin{bmatrix} 0.5 \\ -0.137255 \end{bmatrix} \quad , \quad \overline{b} = \begin{bmatrix} 2 \\ -0.54902 \end{bmatrix}$$

The fact thata_2 , b_2 are interval coefficients and in this case are weak interval coefficients given by

$$\underline{V} = \begin{bmatrix} -1 \\ -2.40196 \\ 0.306373 \end{bmatrix} \quad , \quad \overline{V} = \begin{bmatrix} 0.5 \\ -0.806373 \\ 1.22549 \end{bmatrix}$$
$$\underline{U} = \begin{bmatrix} 0.5 \\ -0.54902 \end{bmatrix} \quad , \quad \overline{U} = \begin{bmatrix} 2 \\ -0.137255 \end{bmatrix}.$$

Finally, we obtain the interval rational interpolation as belows:

$$\underline{\psi}(x) = \frac{-1 - 2.402\,x + 0.306x^2}{2 - 0.137\,x}$$

$$\overline{\psi}(x) = \frac{-0.5 - 0.806\,x + 1.225x^2}{0.5 - 0.549\,x}\ .$$

Example 2. Consider the following points given in Table 2.

Table 2. Data of Example 2

x_i	0	1	2	3
$[\underline{f_i}, \overline{f_i}]$	[0.5,1.5]	[2.3]	[6.5,7.5]	[19.5,20.5]

Here, $\mu + v = 3$, $\mu = 2$, $v = 1$.

By using case (1) the interval rational expression

$$\underline{\emptyset^{2,1}}(x_i) = \frac{\underline{a_0} + \underline{a_1}x_i + \underline{a_2}x_i^2}{\overline{b_0} + \overline{b_1}x_i} = \underline{f_i}$$

From (3), we have the homogeneous system $\underline{S^{2,1}}$

$$\underline{S^{2,1}}:\ \underline{a_0} + \underline{a_1}x_i + \underline{a_2}x_i^2 - \underline{f_i}\left(\overline{b_0} + \overline{b_1}x_i\right) = 0$$

$$\underline{a_0} - 0.5\overline{b_0} = 0$$

$$\underline{a_0} + \underline{a_2} - 2\overline{b_1} = 1.5$$

$$2\underline{a_1} + 4\underline{a_2} - 13\overline{b_1} = 6$$

$$3\underline{a_1} + 9\underline{a_2} - 58.5\overline{b_1} = 19$$

$$\overline{\emptyset^{2,1}}(x_i) = \frac{\overline{a_0} + \overline{a_1}x_i + \overline{a_2}x_i^2}{\underline{b_0} + \underline{b_1}x_i} = \overline{f_i}$$

Similarly:

$$\overline{S^{2,1}}:\ \overline{a_0} + \overline{a_1}x_i + \overline{a_2}x_i^2 - \overline{f_i}\left(\underline{b_0} + \underline{b_1}x_i\right) = 0$$

$$\overline{a_0} - 1.5\underline{b_0} = 0$$

$$\overline{a_1} + \overline{a_2} - 3\underline{b_1} = 1.35$$

$$2\overline{a_1} + 4\overline{a_2} - 15\underline{b_1} = 5.4$$

$$3\overline{a_1} + 9\overline{a_2} - 61.5\underline{b_1} = 17.1$$

$$b_0 = [0.9,1]$$

Solving the above systems $\underline{S^{2,1}}$, $\overline{S^{2,1}}$ yields the coefficients:

$$\underline{a} = \begin{bmatrix} 0.5 \\ 0.539216 \\ 0.529412 \end{bmatrix}, \qquad \overline{a} = \begin{bmatrix} 1.35 \\ 0.291176 \\ 0.476471 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0.9 \\ -0.194118 \end{bmatrix}, \qquad \overline{b} = \begin{bmatrix} 1 \\ -0.215686 \end{bmatrix}$$

The fact that a_1, a_2, b_2 are not interval coefficients and we can modify the following weak interval coefficients:2

$$\underline{V} = \begin{bmatrix} 0.5 \\ 0.291176 \\ 0.476471 \end{bmatrix}, \qquad \overline{V} = \begin{bmatrix} 1.35 \\ 0.539216 \\ 0.529412 \end{bmatrix}$$

$$\underline{U} = \begin{bmatrix} 0.9 \\ -0.215686 \end{bmatrix}, \qquad \overline{U} = \begin{bmatrix} 1 \\ -0.194118 \end{bmatrix}.$$

Therefore, we get the following interval rational interpolation

$$\underline{\psi}(x) = \frac{-0.5 + 0.291\,x + 0.476x^2}{1 - 0.194\,x}$$

$$\overline{\psi}(x) = \frac{1.35 - 0.539\,x + 0.529x^2}{0.9 - 0.216\,x}.$$

Example 3. Consider the points given Table 3.

Table 3. Data of Example 3

x_i	-2	-1	0	1	2
$[\underline{f_i}, \overline{f_i}]$	[0.5,1.5]	[1.5,2.5]	[-0.5,0.5]	[-0.5,0.5]	[0.5,1.5]

Here, $\mu + v = 4 \ , \ \mu = 2 \ , v = 2$

$$\begin{cases} \underline{\psi}(x) = \sum_{j=2k} \underline{a_j} x^j + \sum_{j=2k+1} \overline{a_j} x^j - \underline{f_i}\left(\sum_{s=2k} \overline{b_s} x^s + \sum_{s=2k+1} \underline{b_s} x^s\right) = 0 & x < 0 \\ \underline{\psi}(x) = \sum_{j=0}^{\mu} \underline{a_j} x^j - \underline{f_i}\left(\sum_{s=0}^{v} \overline{b_s} x^s\right) = 0 & x \geq 0 \end{cases}$$

$$\begin{cases} \underline{\psi}(x) = \underline{a_0} + \overline{a_1} x^1 + \underline{a_2} x^2 - \underline{f_i}\left(\overline{b_0} + \underline{b_1} x^1 + \overline{b_2} x^2\right) = 0 & x < 0 \\ \underline{\psi}(x) = \underline{a_0} + \underline{a_1} x^1 + \underline{a_2} x^2 - \underline{f_i}\left(\overline{b_0} + \overline{b_1} x^1 + \overline{b_2} x^2\right) = 0 & x \geq 0 \end{cases}$$

$$\underline{a_0} - 2\overline{a_1} + 4\underline{a_2} - 0.5\overline{b_0} + \underline{b_1} - 2\overline{b_2} = 0$$

$$\underline{a_0} - \overline{a_1} + \underline{a_2} - 1.5\overline{b_0} + 1.5\underline{b_1} - 1.5\overline{b_2} = 0$$

$$\underline{a_0} + 0.5\overline{b_0} = 0$$

$$\underline{a_0} + \underline{a_1} + \underline{a_2} + 0.5\overline{b_0} + 0.5\overline{b_1} + 0.5\overline{b_2} = 0$$

$$\underline{a_0} + 2\underline{a_1} + 4\underline{a_2} - 0.5\overline{b_0} - \overline{b_1} - 2\overline{b_2} = 0$$

$$\begin{cases} \overline{\psi}(x) = \sum_{j=2k} \overline{a_j} x^j + \sum_{j=2k+1} \underline{a_j} x^j - \overline{f_i}\left(\sum_{s=2k} \underline{b_s} x^s + \sum_{s=2k+1} \overline{b_s} x^s\right) = 0 & x < 0 \\ \overline{\psi}(x) = \sum_{j=0}^{\mu} \overline{a_j} x^j - \overline{f_i}\left(\sum_{s=0}^{v} \underline{b_s} x^s\right) = 0 \quad k = 0, \dots, [\frac{n}{2}] & x \geq 0 \end{cases}$$

$$\begin{cases} \overline{\psi}(x) = \overline{a_0} + \underline{a_1} x^1 + \overline{a_2} x^2 - \overline{f_i}\left(\underline{b_0} + \overline{b_1} x^1 + \underline{b_2} x^2\right) = 0 & x < 0 \\ \overline{\psi}(x) = \overline{a_0} + \overline{a_1} x^1 + \overline{a_2} x^2 - \overline{f_i}\left(\underline{b_0} + \underline{b_1} x^1 + \underline{b_2} x^2\right) = 0 & x \geq 0 \end{cases}$$

$$\overline{a_0} - 2\underline{a_1} + 4\overline{a_2} - 1.5\underline{b_0} + 3\overline{b_1} - 6\underline{b_2} = 0$$

$$\overline{a_0} - \underline{a_1} + \overline{a_2} - 2.5\underline{b_0} + 2.5\overline{b_1} - 2.5\underline{b_2} = 0$$

$$\overline{a_0} - 0.5\underline{b_0} = 0$$

$$\overline{a_0} + \overline{a_1} + \overline{a_2} - 0.5\underline{b_0} - 0.5\underline{b_1} - 0.5\underline{b_2} = 0$$

$$\underline{a} = \begin{bmatrix} -1 \\ 4.875 \\ -3.875 \end{bmatrix} \quad , \quad \overline{a} = \begin{bmatrix} 0.25 \\ -4.875 \\ 4.625 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0.5 \\ -3.75 \\ 3.25 \end{bmatrix} \quad , \quad \overline{b} = \begin{bmatrix} 2 \\ 3.75 \\ -5.75 \end{bmatrix}$$

The fact that a_1, b_2 are not interval coefficients and we can modify the following weak interval coefficients:

$$\underline{V} = \begin{bmatrix} -1 \\ -4.875 \\ -3.875 \end{bmatrix} \quad , \quad \overline{V} = \begin{bmatrix} 0.25 \\ 4.875 \\ 4.625 \end{bmatrix}$$

$$\underline{U} = \begin{bmatrix} 0.5 \\ -3.75 \\ -5.75 \end{bmatrix} \quad , \quad \overline{U} = \begin{bmatrix} 2 \\ 3.75 \\ 3.25 \end{bmatrix}.$$

Also, we have:

$$\begin{cases} \underline{\psi}(x) = \frac{-1+4.875\,x-3.875x^2}{2-3.75\,x+3.25x^2} & x<0 \\ \underline{\psi}(x) = \frac{-1-4.875\,x-3.875x^2}{2+3.75\,x+3.25x^2} & x\geq 0 \end{cases},$$

$$\begin{cases} \overline{\psi}(x) = \frac{0.25-4.875\,x+4.625x^2}{0.5+3.75\,x-5.75x^2} & x<0 \\ \overline{\psi}(x) = \frac{0.25+4.875\,x+4.625x^2}{0.5-3.75\,x-5.75x^2} & x\geq 0 \end{cases}.$$

Example 4. Consider the following point given in Table 4.

Table 4. Data of Example 4

x_i	-2	-1	1
$[\underline{f_i}, \overline{f_i}]$	[-10,-8]	[-37,-35]	[-37,-35]

Here, $\mu + v = 2$, $\mu = 1$, $v = 1$

$$\begin{cases} \underline{\psi}(x) = \underline{a_0} + \overline{a_1}x^1 - \underline{f_i}\left(\overline{b_0} + \underline{b_1}x^1\right) = 0 & x<0 \\ \underline{\psi}(x) = \underline{a_0} + \underline{a_1}x^1 - \underline{f_i}\left(\overline{b_0} + \overline{b_1}x^1\right) = 0 & x\geq 0 \end{cases}$$

$$\begin{cases} \overline{\psi}(x) = \overline{a_0} + \underline{a_1}x^1 - \overline{f_i}\left(\underline{b_0} + \overline{b_1}x^1\right) = 0 & x<0 \\ \overline{\psi}(x) = \overline{a_0} + \overline{a_1}x^1 - \overline{f_i}\left(\underline{b_0} + \underline{b_1}x^1\right) = 0 & x\geq 0 \end{cases}$$

$$\underline{a_0} - 2\overline{a_1} + 10\overline{b_0} - 20\underline{b_1} = 0$$

$$\underline{a_0} - \overline{a_1} + 37\overline{b_0} - 37\underline{b_1} = 0$$

$$\underline{a_0} + \underline{a_1} + 37\overline{b_0} + 37\overline{b_1} = 0$$

$$\overline{a_0} - 2\underline{a_1} + 8\underline{b_0} - 16\overline{b_1} = 0$$

$$\overline{a_0} - \underline{a_1} + 35\underline{b_0} + 35\overline{b_1} = 0$$

$$\overline{a_0} + \overline{a_1} + 35\underline{b_0} + 35\underline{b_1} = 0$$

$$\underline{a} = \begin{bmatrix} -793 \\ 256.5 \end{bmatrix} \quad , \quad \overline{a} = \begin{bmatrix} 729 \\ -256.5 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ -13.5 \end{bmatrix} \quad , \quad \overline{b} = \begin{bmatrix} 1 \\ 13.5 \end{bmatrix}.$$

The fact that a_1 are not interval coefficients and we can modify the following weak interval coefficients:

$$\underline{V} = \begin{bmatrix} -793 \\ -256.5 \end{bmatrix} \quad , \quad \overline{V} = \begin{bmatrix} 729 \\ 256.5 \end{bmatrix}$$

$$\underline{U} = \begin{bmatrix} 0 \\ -13.5 \end{bmatrix} \quad , \quad \overline{U} = \begin{bmatrix} 1 \\ 13.5 \end{bmatrix}.$$

Now, we get:

$$\begin{cases} \underline{\psi}(x) = \frac{-793+256.5\,x}{1-13.5\,x} & x<0 \\ \underline{\psi}(x) = \frac{-793-256.5\,x}{1+13.5\,x} & x\geq 0 \end{cases},$$

$$\begin{cases} \overline{\psi}(x) = \frac{729-256.5\,x}{13.5\,x} & x<0 \\ \overline{\psi}(x) = \frac{729+256.5\,x}{-13.5\,x} & x\geq 0 \end{cases}.$$

Remark 2: It is important to note that in example 1, x_i are positive and f_i are intervals both positive and negative. x_is are positive in the second example and f_i intervals include positive numbers. There are x_i and f_i intervals both positive and negative numbers in the third example. In example 4, both positive and negative numbers are included in f_i intervals, and x_i are all negative. In all of the above examples, we obtain weak interval coefficients. Therefore, the corresponding rational interpolator will also be weak, so we have to pick the interval carefully and delicately (support interval). Currently, the proposed method has several advantages.

5. Conclusion

The innovation in rational interpolation under interval data makes this study unique. However, this method has some limitations, including weak interval coefficients and weak interpolators as a result. The intervals to be considered must be carefully and delicately chosen. Using this method, it is important that the distance between intervals be small. In order to compensate for this shortcoming, the authors suggest including whether appropriate intervals can be found to implement this method, and the interpolator responds to interval coefficients with robustness. A second suggestion is to consider the interval interpolation of the poles. Therefore, we can find intervals, there were in the corresponding rational interpolator, without poles, so that we can achieve better approximations.

Conflict of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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https://doi.org/10.30495/fomj.2022.1944092.1045

Received: 06 November 2021 Revised: 27 December 2021 Accepted: 11 January 2022