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Construction of New Implicit Milstein-Type Scheme for Solution of the Systems of SODEs

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ABSTRACT

The main aim of this study is to construct a new approximation method for solving stiff Itô stochastic ordinary differential equations based on explicit Milstein scheme. Under the sufficient conditions such as the Lipschitz condition and linear growth bound, we prove that split-step (α, β)-Milstein method strongly convergence to exact solution with order one. The mean-square stability of our method for linear stochastic differential equation with one-dimensional Wiener process is studied. Stability analysis shows that the mean-square stability of our proposed method contains the mean-square stability region of linear scalar test equation. Finally, numerical examples illustrate the effectiveness of the theoretical results.

1. Introduction

In this work, we look at Itô stochastic ordinary differential equations (SODEs) as

$$dZ_t = U(Z_t)dt + \sum_{q=1}^{Q} V(Z_t)d\omega_t^q, \quad t \geq 0, \tag{1}$$

Here $U, V_q : \mathbb{R}^d \to \mathbb{R}^d$, are drift and diffusion functions, respectively. We assume that the initial data is independent of the Wiener measure driving the equations and ω_t^q, $q = 1,\ldots,Q$ is an one-dimensional Wiener process defined on the complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$. We assume the SODE (1) satisfies the required conditions of the existence and uniqueness theorems [5]. This type of equation (1) often appears in many scientific areas of engineering and applied sciences such as mathematical finance, biology, chemistry, medicine, and etc. [1, 2, 5, 9–11]. Hence, the approximate and numerical solutions of SODE (1) have been considered by many authors in recent years, for example see [4, 12–14, 17, 19–21].

This paper is motivated by the work of Higham et al. [3]. We propose a new Milstein method, and

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investigate its strong convergence when SDE (1) hold Lipschitz condition and linear growth bound for the drift and diffusion terms. Fixed any time $T \ge 0$ and given a step-size $h = \frac{T - t_0}{N}$, $t_l = t_0 + lh$, $l = 0,1,\dots,N$ where $N = 1,2,\dots$, we introduce the split-step (α, β)-Milstein (SSABM) approximation

$$\begin{aligned}
\bar{Y}_l &= Y_l + \alpha h U(\bar{Y}_l),\\
\bar{\bar{Y}}_l &= \bar{Y}_l - \frac{h}{2}\beta \sum_{q=1}^{Q} V_q V_q'(\bar{\bar{Y}}_l),\\
Y_{l+1} &= Y_l + hU(\bar{\bar{Y}}_l) + \sum_{q=1}^{Q} V_q(\bar{\bar{Y}}_l)\Delta\omega_t^q + \sum_{q=1}^{Q} V_q V_q'(\bar{\bar{Y}}_l)((\Delta\omega_t^q)^2 - h),
\end{aligned} \tag{2}$$

Here $\alpha,\beta \in [0,1]$ and $\Delta\omega_l^q = \omega_{l+1}^q - \omega_l^q$ is a Gaussian random variable $N(0,h)$. If choose $\alpha = \beta = 0$, we gives Milstein method [7]. For $\alpha = 1, \beta = 0$, we obtain DSSBM method [18]. Obviously, deterministic two first equations in (2) are implicit in $\bar{\bar{Y}}_l$ and $\bar{Y}_l$ when $\alpha,\beta \in (0,1]$ that must be solved in order to obtain the intermediate approximation $\bar{\bar{Y}}_l$ and $\bar{Y}_l$, respectively.

The following proposition can be found in [5, 8] when considering the strong convergence properties of the numerical methods for SODEs.

Proposition 1. For any $z_1, z_2 \in \mathbb{R}^d$, the satisfaction of Lipschitz conditions

$$|U(z_1) - U(z_2)|^2 \vee \sum_{q=1}^{Q} |V_q(z_1) - V_q(z_2)|^2 \vee \sum_{q=1}^{Q} |V_q V_q'(z_1) - V_q V'(z_2)|^2 \le \ell_1 |z_1 - z_2|^2, \tag{3}$$

and linear growth bounds

$$|U(z_1)|^2 \vee \sum_{q=1}^{Q} |V_q(z_1)|^2 \vee \sum_{q=1}^{Q} |V_q V_q'(z_1)|^2 \le \ell_2 (1 + |z_1|^2), \tag{4}$$

hold for positive constants ℓ_1 and ℓ_2, where $\vee$ denote a maximal operator.

The layout of the paper is as follows. In Section 2, we discuss the strong convergence of the numerical method (2). The mean-square stability properties of the SSABM method (2) is analysed in Section 3. Finally, we show numerically the accuracy and efficiency of the proposed method applied to SODEs.

2. Strong convergence properties

In this section, we prove strong convergence of the SSABM method (2) by Proposition 1, following lemma and Milstein approximation step

$$Y_{l+1}^{Mil} = Y_l^{Mil} + hU(Y_l^{Mil}) + \sum_{q=1}^{Q} V_q(Y_l^{Mil})\Delta\omega_t^q + \frac{1}{2}\sum_{q=1}^{Q} V_q V_q'(Y_l^{Mil})((\Delta\omega_t^q)^2 - h), \tag{5}$$

with the local mean error $O(h)$ and mean-square error $O(h^2)$.

Lemma 1. Assume for a one-step discrete time approximation Y that the local mean error and mean-square error for all $N = 1,2,\dots,$ and $\ell = 0,1,2\dots,N-1$ satisfy the estimates

$$|E(Y_{l+1} - Z_{t_{l+1}}) | F_t| = O(h^{\kappa_1}), \tag{6}$$

and

$$|E|\Upsilon_{l+1}-Z_{t_{l+1}}|^2|F_t|^{\frac{1}{2}}=O(h^{\kappa_2}), \tag{7}$$

when $\kappa_2 \geq \frac{1}{2}$ and $\kappa_1 \geq \kappa_2 + \frac{1}{2}$. Then $|E|\Upsilon_r - Z_{t_r}|^2|F_0|^{\frac{1}{2}} = O(h^{\kappa_2-\frac{1}{2}})$ holds for each $r=0,1,\ldots,N$.

Theorem 1. Under the Proposition 1.1, the numerical solution produced by the SSABM method (2) converges to the exact solution of SODE (1) in the mean-square sense with strong order of convergence 1.

Proof. For estimation of local mean error of our method, we can write

$$\begin{aligned}\varepsilon_1 &= |E[(\Upsilon_{l+1}-Z_{t_{l+1}})|F_t]| \\ &\leq |E[(Y_{l+1}^{Mil}-Z_{t_{l+1}})|F_t]| + |E[(\Upsilon_{l+1}-Y_{l+1}^{Mil})|F_t]| \\ &\leq O(h^2)+\varepsilon_2,\end{aligned} \tag{8}$$

where

$$\begin{aligned}\varepsilon_2 &= |E[(\Upsilon_{l+1}-Y_{l+1}^{Mil})|F_t]| \\ &= |E[\Upsilon_l - Y_{l+1}^{Mil} + h(U(\bar{\bar{\Upsilon}}_l)-U(Y_l^{Mil})) + \sum_{q=1}^{Q}(V_q(\bar{\bar{\Upsilon}}_l)-V_q(Y_l^{Mil}))\Delta W_l^q + \frac{1}{2}\sum_{q=1}^{Q}(V_qV_q'(\bar{\bar{\Upsilon}}_l)-V_qV_q'(Y_l^{Mil}))((\Delta W_l^q)^2-h)]| \\ &\leq h\sqrt{\ell_1}\,|\bar{\bar{\Upsilon}}_l-\Upsilon_l|.\end{aligned} \tag{9}$$

Notice that for obtain of above inequality, we use global Lipschitz condition (3) and $E[\Delta W_l^q]=0$. Also, applying (2) and Proposition 1 yields

$$|\bar{\bar{\Upsilon}}_l-\Upsilon_l| = |\bar{\bar{\Upsilon}}_l-\bar{\Upsilon}_l| + |\bar{\Upsilon}_l-\Upsilon_l|, \tag{10}$$

where

$$\begin{aligned}|\bar{\bar{\Upsilon}}_l-\bar{\Upsilon}_l| &\leq |-\frac{\beta h}{2}\sum_{q=1}^{Q}V_qV_q'(\bar{\bar{\Upsilon}}_l)| \\ &\leq \frac{\beta h}{2}\sum_{q=1}^{Q}|V_qV_q'(\bar{\bar{\Upsilon}}_l)-V_qV_q'(\bar{\Upsilon}_l)| + \frac{\beta h}{2}\sum_{q=1}^{Q}|V_qV_q'(\bar{\Upsilon}_l)-V_qV_q'(\Upsilon_l)| + \frac{\beta h}{2}\sum_{q=1}^{Q}|V_qV_q'(\Upsilon_l)| \\ &\leq \frac{\beta h}{2}\sqrt{\ell_1}\,|\bar{\bar{\Upsilon}}_l-\bar{\Upsilon}_l| + \frac{\beta h}{2}\sqrt{\ell_1}\,|\bar{\Upsilon}_l-\Upsilon_l| + \frac{\beta h}{2}\sqrt{\ell_2}(1+|\Upsilon_l|^2)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}|\bar{\Upsilon}_l-\Upsilon_l| &\leq |\alpha h U(\bar{\Upsilon}_l)| \\ &\leq \alpha h|U(\bar{\Upsilon}_l)-U(\Upsilon_l)| + \alpha h|U(\Upsilon_l)| \\ &\leq \alpha h\sqrt{\ell_1}\,|\bar{\Upsilon}_l-\Upsilon_l| + \alpha h\sqrt{\ell_2}(1+|\Upsilon_l|^2)^{\frac{1}{2}}.\end{aligned}$$

Form above inequalities, we conclude that

$$|\bar{\Upsilon}_l-\Upsilon_l|\le h\frac{\alpha\sqrt{\ell_2}}{1-\alpha h\sqrt{\ell_1}}(1+|\Upsilon_l|^2)^{\frac{1}{2}}, \tag{11}$$

and

$$|\bar{\bar{\Upsilon}}_l-\bar{\Upsilon}_l|\le h\frac{\beta}{2}\sqrt{\ell_2}\frac{1+(1-h)\alpha\sqrt{\ell_1}}{(1-\frac{\beta h}{2}\sqrt{\ell_1})(1-h\alpha\sqrt{\ell_1})}(1+|\Upsilon_l|^2)^{\frac{1}{2}}. \tag{12}$$

Now for $(1-\frac{\beta h}{2}\sqrt{\ell_1})(1-h\alpha\sqrt{\ell_1})>0,$ we have from (8)-(12), $\kappa_1=2.$ Similarly by standard arguments, we can prove the following

$$\begin{aligned}\varepsilon_3&=E[|\Upsilon_{l+1}-Z_{t_{l+1}}|^2|F_t]^{\frac{1}{2}}\\&\le E[|Y_{l+1}^{Mil}-Z_{t_{l+1}}|^2|F_t]^{\frac{1}{2}}+E[|\Upsilon_{l+1}-Y_{l+1}^{Mil}|^2|F_t]^{\frac{1}{2}}\\&\le O(h)+\sqrt{\varepsilon_4}.\end{aligned} \tag{13}$$

Similarly to (9), by

$$E[\Delta W_l^{q_1}\Delta W_l^{q_2}]=\begin{cases}h, & q_1=q_2,\\ 0, & q_1\ne q_2,\end{cases}$$

and

$$E[((\Delta W_l^{q_1})^2-h)((\Delta W_l^{q_2})^2-h)]=\begin{cases}\frac{h^2}{2}, & q_1=q_2,\\ 0, & q_1\ne q_2,\end{cases}$$

we can obtain

$$\begin{aligned}\varepsilon_4&=E[|\Upsilon_{l+1}-Y_{l+1}^{Mil}|^2|F_t]\\&=E[|\Upsilon_l-Y_{l+1}^{Mil}+h(U(\bar{\bar{\Upsilon}}_l)-U(Y_l^{Mil}))+\sum_{q=1}^{Q}(V_q(\bar{\bar{\Upsilon}}_l)-V_q(Y_l^{Mil}))\Delta W_l^q\\&\quad+\frac{1}{2}\sum_{q=1}^{Q}(V_qV_q'(\bar{\bar{\Upsilon}}_l)-V_qV_q'(Y_l^{Mil}))((\Delta W_l^q)^2-h)|^2|F_t]\\&\le h^2E[|U(\bar{\bar{\Upsilon}}_l)-U(Y_l^{Mil})|^2+h\sum_{q=1}^{Q}|V_q(\bar{\bar{\Upsilon}}_l)-V_q(Y_l^{Mil})|^2+\frac{h^2}{2}\sum_{q=1}^{Q}|V_qV_q'(\bar{\bar{\Upsilon}}_l)-V_qV_q'(Y_l^{Mil})|^2\\&\le h(1+\frac{3}{2}h)\ell_1|\bar{\bar{\Upsilon}}_l-\Upsilon_l|^2.\end{aligned} \tag{14}$$

Using two first equations in (2), Proposition 1 and

$$(a_1+a_2+\ldots+a_n)^2\le n(a_1^2+a_2^2+\ldots+a_n^2),$$

we can get

$$|\overline{\overline{Y}}_l - Y_l| \le 2|\overline{\overline{Y}}_l - \overline{Y}_l|^2 + 2|\overline{Y}_l - Y_l|^2, \tag{15}$$

where

$$\begin{aligned}
|\overline{\overline{Y}}_l - \overline{Y}_l|^2 &\le \Big|-\frac{\beta h}{2}\sum_{q=1}^{Q} V_q V_q'(\overline{\overline{Y}}_l)\Big|^2 \\
&\le 3\frac{(\beta h)^2}{4}\sum_{q=1}^{Q}|V_q V_q'(\overline{\overline{Y}}_l) - V_q V_q'(\overline{Y}_l)|^2 + 3\frac{(\beta h)^2}{4}\sum_{q=1}^{Q}|V_q V_q'(\overline{Y}_l) - V_q V_q'(Y_l)|^2 + 3\frac{(\beta h)^2}{4}\sum_{q=1}^{Q}|V_q V_q'(Y_l)|^2 \\
&\le 3\frac{(\beta h)^2}{4}\ell_1|\overline{\overline{Y}}_l - \overline{Y}_l|^2 + 3\frac{(\beta h)^2}{4}\ell_1|\overline{Y}_l - Y_l|^2 + 3\frac{(\beta h)^2}{4}\ell_2(1+|Y_l|^2),
\end{aligned}$$

and

$$\begin{aligned}
|\overline{Y}_l - Y_l|^2 &\le |\alpha h U(\overline{Y}_l)|^2 \\
&\le 2(\alpha h)^2|U(\overline{Y}_l) - U(Y_l)|^2 + 2(\alpha h)^2|U(Y_l)|^2 \\
&\le (\alpha h)^2\ell_1|\overline{Y}_l - Y_l|^2 + (\alpha h)^2\ell_2(1+|Y_l|^2).
\end{aligned}$$

Form above inequalities, we obtain

$$|\overline{Y}_l - Y_l|^2 \le h^2\frac{\alpha^2\ell_2}{1-(\alpha h)^2\ell_1}(1+|Y_l|^2), \tag{16}$$

and

$$|\overline{\overline{Y}}_l - \overline{Y}_l|^2 \le 3h^2\frac{\beta^2}{4}\ell_2\frac{1+(1-h^2)\alpha^2\ell_1}{(1-3\frac{(\beta h)^2}{4}\ell_1)(1-(h\alpha)^2\ell_1)}(1+|Y_l|^2). \tag{17}$$

Combining (13)-(17), implies that for $(1-3\frac{(\beta h)^2}{4}\ell_1)(1-(h\alpha)^2\ell_1) > 0$, $\kappa_2 = \frac{3}{2}$. Thus, we can choose in Lemma 1 and $\kappa_1 = 2, \kappa_2 = \frac{3}{2}$ can prove that the strong order of our method is equal one.□

3. Linear mean-square stability properties

Linear mean-square (MS-) stability of SSABM method, studied in this section of paper. For this aim, we choose a scalar linear SODE with an one-dimensional Wiener process as a test equation, which is given by

$$dZ_t = \upsilon Z_t dt + \nu Z_t d\omega_t, \quad t \ge t_0, \quad \upsilon, \nu \in \mathbb{R}. \tag{18}$$

The zero solution of (18) is said to be MS-stable if $\lim_{t\to\infty} E|Z_t|^2 = 0.$ It is well known [6, 15] that the zero solution of (18) is MS–stable if and only if

$$2\upsilon + \nu^2 < 0. \tag{19}$$

Saito and Mitsui [15] introduce the following definition of MS-stability for a numerical scheme.

Definition 1. The numerical method is said to be MS-stable if

$$\overline{R}(\upsilon,\nu,h)=E[R^2(\upsilon,\nu,h,\xi_l)]<1,\quad \xi_l\sim \mathrm{N}(0,1),$$

where $\overline{R}(\upsilon,\nu,h)$ is called MS-stability function.

Applying SSABM method to the linear test equation (18) yields

$$\mathrm{Y}_{l+1}=R(\upsilon,\nu,h,\xi_l)\mathrm{Y}_l,$$

where

$$R(\upsilon,\nu,h,\xi_l)=1+\frac{\upsilon h+\sqrt{h}\nu\xi_l+\frac{1}{2}h\nu^2((\xi_l)^2-1)}{(1-\alpha\upsilon h)(1+\frac{1}{2}h\beta\nu^2)}. \quad (20)$$

Using Definition 1, the SSABM scheme is MS-stable if

$$a_1+a_2+a_3+a_4<1,$$

where

$$a_1=\frac{(\upsilon h+(1-\alpha\upsilon h)(1+\frac{1}{2}h\beta\nu^2)-\frac{1}{2}h\nu^2)^2}{(1-\alpha\upsilon h)^2(1+\frac{1}{2}h\beta\nu^2)^2},$$

$$a_2=\frac{h\nu^2}{(1-\alpha\upsilon h)^2(1+\frac{1}{2}h\beta\nu^2)^2},$$

$$a_3=\frac{\frac{3}{4}h^2\nu^4}{(1-\alpha\upsilon h)^2(1+\frac{1}{2}h\beta\nu^2)^2},$$

$$a_4=\frac{h(\upsilon h+(1-\alpha\upsilon h)(1+\frac{1}{2}h\beta\nu^2)-\frac{1}{2}h\nu^2)\nu^2}{(1-\alpha\upsilon h)^2(1+\frac{1}{2}h\beta\nu^2)^2},$$

This is equivalent to

$$(1-2\alpha)(\upsilon h)^2+2\upsilon h+\upsilon h(1-2\alpha\upsilon h)h\beta\nu^2+h\nu^2+\frac{1}{2}h^2\nu^4<0.$$

Considering $x=h\upsilon$ and $y=h\nu^2$, we have drawn the MS-stability regions of the proposed method (2) and test equation (18) in Figure 1. Results of this figure show that the method can solve stiff SDEs when $\alpha,\beta\to 1$.

4. Numerical results

In this section, we present some numerical results obtained by our new scheme (2) with different values of α,β. Comparison of this method with Milstein and DSSBM [18] schemes on the Matlab platform, confirm our theoretical conclusion.

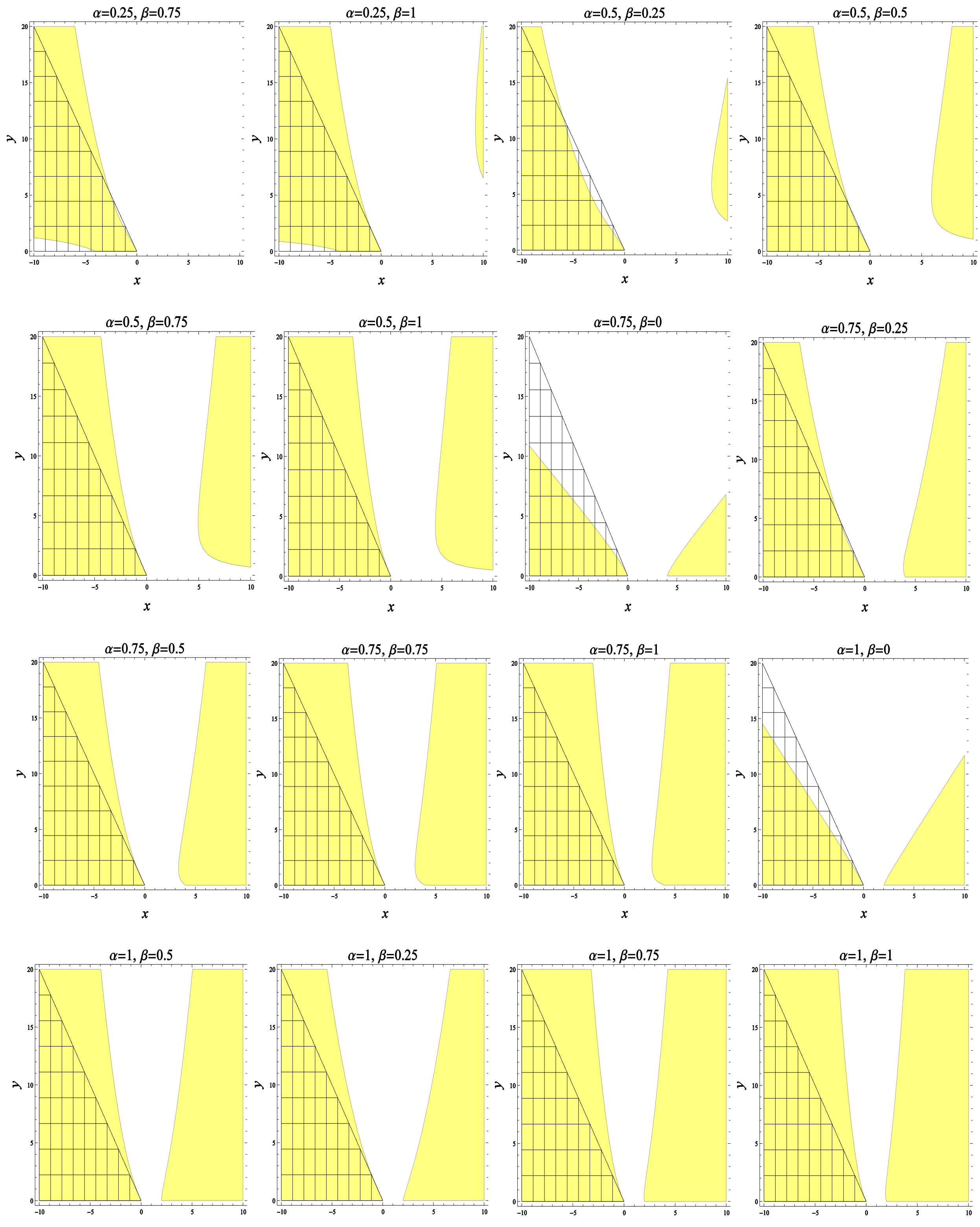

Figure 1. MS-stability regions for the SSABM method (2) applied to scalar linear SODE (18).

Example 1. Consider the following nonlinear stiff SODE,

$$dZ_t = -(a+b^2 Z_t)(1-Z_t^2)dt + (1-Z_t^2)d\omega_t, \quad Z_0 = \frac{1}{2}, \quad t \in [0,T]. \tag{21}$$

The exact solution is given by [5]

$$Z_t = \frac{(1+Z_0)\exp(-2at+2b\omega_t)+Z_0-1}{(1+Z_0)\exp(-2at+2b\omega_t)-Z_0+1}.$$

Figure 2 gives the means square errors (MSE) of the SSABM (2), Milstein and DSSBM [18] methods. From Figure 2, it is clear that the SSABM scheme has better accuracy of Milstein and DSSBM methods.

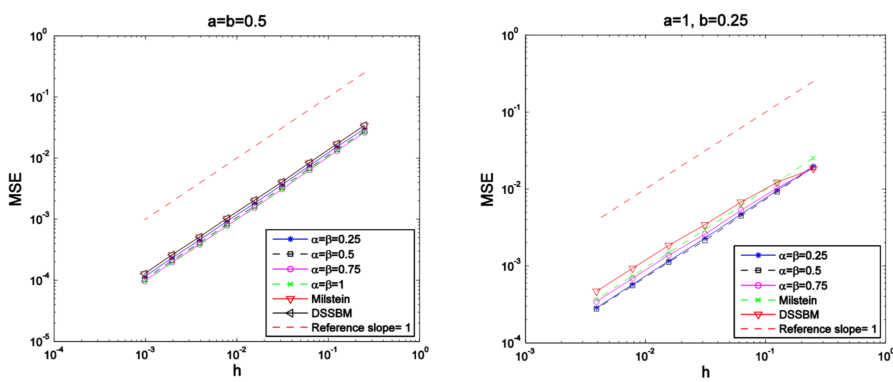

Figure 2. MSE estimation of the SSABM (2), Milstein and DSSBM [18] methods for nonlinear SODE (21).

Example 2. We consider the following stiff stochastic system [16]

$$\begin{aligned} dZ_t^1 &= 10dZ_t^2 dt + 0.2(Z_t^1 + Z_t^2)d\omega_t, \\ dZ_t^1 &= -10dZ_t^1 dt + 0.2(Z_t^1 + Z_t^2)d\omega_t \\ Z_0^1 &= 1, \quad Z_0^2 = 1, \quad t \in [0,20]. \end{aligned} \tag{22}$$

Figure 3 illustrates the numerical simulations of the SSABM and Milstein schemes for step-size h = 0.01. From this figures, we can see that the stability properties of the SSABM scheme are better than the Milstein scheme for stiff stochastic system (22).

5. Conclusion

In this article, we applied the new implicit Milstein method to the stochastic differential systems. The split-step (α,β)-Milstein schemes have strong convergence order of one and are stable in mean-square sense for the scalar linear SODE with an one-dimensional Wiener process when $\alpha,\beta \to 1$. In numerical results, the presented method which is implicit scheme has superior efficiency and accuracy than the Milstein and DSSBM [18] methods.

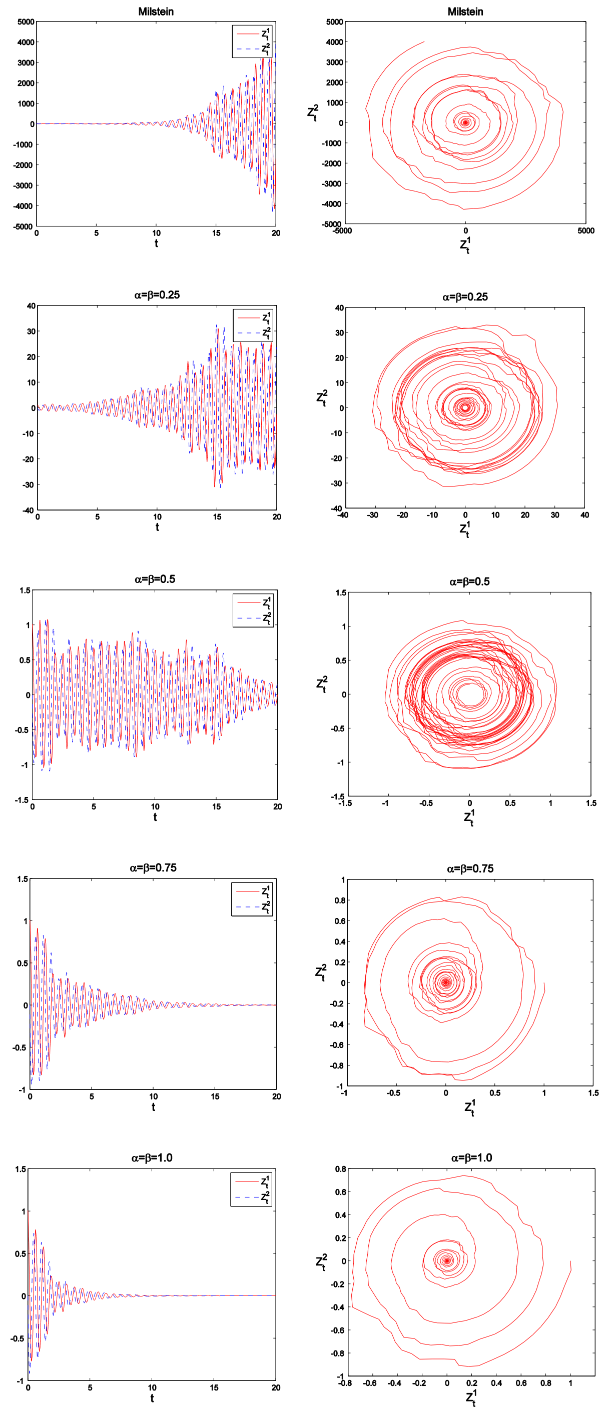

Figure 3. Numerical simulation of the Milstein and SSABM (2) methods for SDE system (22)

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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