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Ishita Approach to Construct an Intuitionistic Fuzzy Linear Regression Model

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A R T I C L E I N F O

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A B S T R A C T

To the best of our knowledge, there is only two approaches for constructing an intuitionistic fuzzy linear regression model (regression model in which all the variables and coefficients are considered as intuitionistic triangular fuzzy numbers). However, after a deep study, some mathematical incorrect assumptions have been considered in these approaches. Therefore, it is scientifically incorrect to use these approaches for general real-life data. Keeping the same in mind, in this paper, a new approach (named as Ishita approach) is proposed to construct an intuitionistic fuzzy linear regression model. The proposed approach overcomes the limitations of the existing approaches. It is fit for positive, negative or mixed of positive and negative datasets represented as symmetric or asymmetric intuitionistic triangular fuzzy numbers. Moreover, the constructed models of the proposed approach guarantee the homogeneity principle such that for symmetric intuitionistic fuzzy data, the constructed model is symmetric, i.e., the estimated model's coefficients are symmetric intuitionistic fuzzy numbers. Furthermore, the proposed approach is illustrated with the help of a numerical example.

1. Introduction

Statistical regression analysis is reliable and powerful to construct the causal relationship between independent and dependent variable and has numerous applications in various fields. The complexity of real life problems where information is frequently uncertain and ambiguous imposed researchers to extend regression models into fuzzy environment. Tremendous approaches and methods have been developed to construct fuzzy regression models with different types of fuzzy numbers [6]. However, as far as the author knows, there is only two existing researches to construct intuitionistic fuzzy regression model [2, 5].

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Arefi and Taheri [2, Section I, pp. 1142] discussed the need of fuzzy sets [18] and intuitionistic fuzzy sets [3] in regression analysis and claimed that there has not been any study on this topic except the existing approach [12]. Arefi and Taheri [2, Section VII, pp. 1152] pointed out that as in existing approach [12], the dependent and independent variables are considered as real numbers whereas the regression coefficients are considered as ATIFNs. Therefore, the existing approach [2] cannot be used to construct such intuitionistic fuzzy linear regression model in which dependent variable(s), independent variables and all the regression coefficients are considered as ATIFNs. To overcome this limitation of the existing approach [12], Arefi and Taheri [2, Section IV, pp. 1145] proposed a least-square based approach to construct an intuitionistic fuzzy linear regression model by considering dependent variable(s), the independent variables and regression coefficients as ATIFNs.

Arefi and Taheri [2] formulated their solution to estimate their model parameters based on that limited multiplication approximation. Consequently, given data of the independent variables and model's parameters needed to be determined as positive symmetric TAIFNs although in their example there is a negative parameter explains the negative relationship between the output variable and one of two input variables.

According to Zadeh's extension principle [17], fuzzy numbers operations including multiplication as well as their applications such as fuzzy regression models must have no conflict with mathematical and physical principles. One fundamental principle must be maintained in any computation and calculations on fuzzy numbers, is the homogeneity principle. The homogeneity is that for full symmetric intuitionistic fuzzy linear regression model where the output and input variables as well as model's coefficients, all of the three model's components must be symmetric intuitionistic fuzzy numbers. Moreover, with dataset of pairs, $(y_1, x_{1j}, \dots, x_{1p}), (y_2, x_{2j}, \dots, x_{2p}), \dots, (y_2, x_{nj}, \dots, x_{np})$, represented as symmetric TAIFNs the estimated coefficients must be symmetric ATIFN. So, symmetric intuitionistic fuzzy linear regression model can fit homogeneous datasets. Keeping in mind the homogeneity in fuzzy regression models and especially for symmetric intuitionistic fuzzy linear regression model, Chen and Nien [5] proposed a mathematical programming approach to formulate symmetric intuitionistic fuzzy linear regression model based on least absolute of discrepancies as a generalized approach avoids the limitations and incontinent multiplication assumption of Arefi and Taheri [2] and to avoid the effect of the sign of the unknown model's parameters, Chen and Nien [5] setup two dummy ATIFNs one is non-positive and another is nonnegative (with the help of an example problems related to this assumption will be discussed in more details in Section (2)). The multiplication concerning assumptions of Arefi and Taheri [2] as well as Chen and Nien [5] with more details will be discussed in Section (2). Chen and Nien [5] mentioned that their approach is general such that it is not limited to symmetric ATIFNs but it constructs fuzzy model fits asymmetric ATIFNs as well. Moreover, on claiming that their proposed approach overcomes the conflict made by Arefi and Taheri [2] with homogeneity principle, however the model proposed by Chen and Nien [5], as result on applying the example [5, Section 4, Equation 35, pp. 205] is not symmetric AIFRM although the data are represented as asymmetric ATIFNs. Moreover, the assumptions of dummy variables set by Chen and Nien [5] to parametrize the unknown model's slopes are not always be satisfied although the related constraints are made in Chen and Nien [5] mathematical programming formulation.

After a critical study of the existing approaches [2, 5], it is noticed that they have used some mathematical incorrect assumptions in their approaches. In this paper, mathematical incorrect assumptions, considered by Arefi and Taheri [2], and Chen and Nien [5], are pointed out. Furthermore, in the literature, it is pointed out that it is better to use the fuzzy least absolute deviation based approach as compared to fuzzy least square based approach [1, 2, 4, 7, 9-12, 14-16, 19]. Therefore, a new approach (named as Ishita approach), based on least absolute deviation, is proposed to construct an intuitionistic fuzzy linear regression model. Therefore, motivated by intuitionistic fuzzy linear regression model and the limitations of the existing approaches [2, 5], the proposed Ishita approach constructs a sound and general to fit any type of intuitionistic fuzzy data and it conserve the homogeneity of data such that the model to fit data represented as symmetric ATIFNs, is symmetric, i.e., its estimated coefficients are symmetric ATIFNs.

This paper is organized as follows: In Section 2, mathematical incorrect assumptions, considered in the existing approaches [2, 5], are pointed out. In Section 3, multiplication of two ATIFNs is proposed. In Section 4, a new approach (named as Ishita approach) is proposed to construct the intuitionistic fuzzy linear regression model. In Section 5, the proposed Ishita approach is illustrated by a numerical example. Section 6 concludes the work.

2. Mathematical incorrect assumptions of the existing approach

2.1 Arefi and Taheri's approach

In this section, the mathematical incorrect assumptions, considered in Arefi and Taheri approach [2], are pointed out as follows:

1. It is obvious from Step 1 and Step 2 of Arefi and Taheri [2, Section IV, pp. 1145] approach, discussed in Section 2, that Arefi and Taheri [2, Section IV, pp. 1145] have used the multiplication $(a;\alpha;\ \alpha^*)(b;\ \beta;\ \beta^*) = (ab;a\beta+b\alpha;a\beta^*+b\alpha^*)$ to transform the intuitionistic fuzzy linear regression model (1) into the intuitionistic fuzzy linear regression model (2).
2. It is pertinent to mention that this multiplication is valid only if both $(a;\alpha;\ \alpha^*)$ and $(b;\ \beta;\ \beta^*)$ are nonnegative ASTIFNs i.e., $a-\alpha^*\geq 0$and $b-\ \beta^*\ \geq 0$. However, if $(a;\alpha;\ \alpha^*)$ and/or $(b;\ \beta;\ \beta^*)$ are not non-negative ASTIFNs then the number, obtained by this multiplication, will not necessarily be an Atanassov's symmetric triangular intuitionistic fuzzy number (ASTIFN). e.g., if $\tilde{a}=(-3;\ 1;\ 2)$ and $\tilde{b}=(2;\ 1;\ 1)$ then according to this multiplication the value of $(-3;\ 1;\ 2)(2;\ 1;\ 1)=(-6;\ -5;\ 2)$ which is not a ASTIFN as in ASTIFN $(a;\alpha;\ \alpha^*)$, α and α^* should always be non-negative real numbers. While, in the number $(-6;\ -5;\ 2)$ the value of α is -5 which is not a non-negative real number.
3. Since, in the intuitionistic fuzzy linear regression model the regression coefficients $(a_j;\omega_j\,;\ \omega_j^*)$ (which are unknown) will not necessarily be positive ASTIFN. So, it is mathematically incorrect to use the multiplication$\left(a_j;\omega_j\,;\ \omega_j^*\right)\otimes\left(x_{ij};\ \sigma_{ij};\ \sigma_{ij}^*\right)=(a_jx_{ij};a_j\sigma_{ij}+x_{ij}\,\omega_j\,;a_j\sigma_{ij}^*+x_{ij}\omega_j^*)$, valid only for nonnegative ASTIFNs, for transforming the intuitionistic fuzzy linear regression model (1) into the intuitionistic fuzzy linear regression model (2).
4. In the ASTIFN $(a;\omega;\ \omega^*)$ the condition $\omega^*\geq\omega$ should always be satisfied. Since, Arefi and Taheri [2, Section IV, pp. 1145] have obtained the value of $\omega_j$and ω_j^* by solving the system of linear equations without considering the restriction $\omega_j^*\geq\ \omega_j$. Therefore, for the values of ω_j and ω_j^*, obtained by Arefi and Taheri [2, Section IV, pp. 1145] approach, the condition $\omega_j^*\geq\ \omega_j$ will not necessarily be satisfied. Hence, the number $(a_j;\omega_j\,;\ \omega_j^*)$, obtained by Arefi and Taheri [2, Section IV, pp. 1145] approach, will not necessarily be a ASTIFN.
5. It is well-known fact that in the ASTIFN $(a_j;\omega_j\,;\ \omega_j^*)$ the value of ω_j and ω_j^* should be non-negative real numbers. Due to the same reason Arefi and Taheri [2, Section IV, pp. 1145] have suggested that if the obtained value of ω_j and/or ω_j^* is a negative real number then again solve the system of linear equations by considering ω_j and/or ω_j^* equal to zero. However, to do the same is not valid e.g., on solving a system of linear equation $3x+y=2$ and $2x+y=1$ the obtained values of x and y are $x=1$and $y=-1$. Since, the value of y is a negative real number. Therefore, according to Arefi and Taheri [2, Section IV, pp. 1145], there is need to solve the system of linear equations $3x+y=2$ and $2x+y=1$ with assumption $y=0$. However, on considering y $=$ 0, there doesn't exist any solution for the system of linear equations $3x+y=2$ and $2x+y=1$.

2.2 Chen and Nien's approach

Chen and Nien [5] tackled the effect of the sign of an unknown model's parameter on their constructed model, they assume two dummy variables such that the first is non-positive and the second is nonnegative in the place of an unknown AATIFN which is multiplied by the known TAIFN. In order to highlight Chen and Nien [5] multiplication strategy, let $\tilde{A}=\left(a_j-\omega_{j1}^*,a_j-\omega_{j1},a_j\,,a_j+\omega_{j2},a_j+\omega_{j2}^*\right)$ and is an unknown ATIFN and $\tilde{X}=\left(x_{ij}-\sigma_{ij1}^*,x_{ij}-\sigma_{ij1},\ x_{ij},x_{ij}+\sigma_{ij2};x_{ij}+\sigma_{ij2}^*\right)$ is a known ATIFN both expressed as left, main, right points, respectively. To find the multiplication, i.e., $\tilde{A}\otimes\tilde{X}$, transform $\tilde{A}\otimes\tilde{X}$ into $(\tilde{A}_1\oplus\ \tilde{A}_2)\otimes\tilde{X}$ assuming that unrestricted ATIFN $\tilde{A}=\left(a_j-\omega_{j1}^*,a_j-\omega_{j1},a_j\,,a_j+\omega_{j2},a_j+\omega_{j2}^*\right)$ is a sum of a non-positive ATIFN $\tilde{A}_1=\left(a_{j1}-\omega_{j11}^*,a_{j1}-\omega_{j11},a_{j1}\,,a_{j1}+\omega_{j21},a_{j1}+\omega_{j21}^*\right)$ and a nonnegative ATIFN $\tilde{A}_2=\big(a_{j2}-\omega_{j12}^*,a_{j2}-$

$\omega_{j12}, a_{j2}, a_{j2}+\omega_{j22}, a_{j2}+\omega^*_{j22})$. So, $(\tilde{A}_1 \otimes \tilde{A}_2) \otimes \tilde{X}$ is transformed into $\tilde{A}_1 \otimes \tilde{X} \oplus \tilde{A}_2 \otimes \tilde{X}$. Hence, $\tilde{A} \otimes \tilde{X} = \tilde{A}_1 \otimes \tilde{X} \oplus \tilde{A}_2 \otimes \tilde{X}$. While $\tilde{A}_1 \leq 0$ and $\tilde{A}_2 \geq 0$, it is clear to notice that, on using the equations (11) and (12) [5, Section 2, Definition 6, pp.196], they have tacitly considered the given input variable as positive ATIFN so their model can fit only positive datasets. Moreover, to build up the mathematical programming problem Chen and Nien [5] guaranteed to get one optimal of the two dummy variables, which is the coefficient of an independent variable by restricting the multiplication of the two dummy ATIFNs to zero. However, the distributive property is not absolutely satisfied for ATIFNs. For example, let $A = (12; 13, 14; 15, 17)$, $B = (-5; -4, -3; -2, -1)$ and $C = (2;\ 3,\ 4;\ 5,\ 7)$ be three ATIFNs. Then, we have $(A \oplus B) \otimes C = (7; 9, 11;\ 13, 16) \otimes (2;\ 3, 4;\ 5, 7)$ whereas,

$$A \otimes C \oplus B \otimes C = (24; 39, 56;\ 75, 119) \oplus (-35; -20, -12;\ -6, -2)$$

It is obvious that $(A \oplus B) \otimes C \neq A \otimes C \oplus B \otimes C$.

Furthermore, the approach proposed by Chen and Nien [5] doesn't maintain the homogeneity, i.e., for given data represented as ASTIFNs, the model is not necessarily symmetric as it is shown in the illustrative example.

3. Proposed multiplication

It is obvious that to find the regression coefficients $\left(\tilde{a}_j, j = 1,2,\ldots,m\right)$ of intuitionistic fuzzy linear regression model $\hat{\tilde{y}} = \tilde{a}_0 \oplus \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{x}_j$, there is need to multiply the unknown ASTIFN $\left(\tilde{a}_j\right)$ with a known ASTIFN $\left(\tilde{x}_j\right)$. However, to the best of our knowledge no such multiplication of ATIFNs is defined in the literature.

Therefore, in this section the multiplication of two Atanassov's asymmetric triangular intuitionistic fuzzy numbers (AATIFNs) $\left(a_j; \omega_{j1}, \omega_{j2};\ \omega^*_{j1}, \omega^*_{j2}\right)$ and $\left(x_{ij}; \sigma_{ij1}, \sigma_{ij2}; \sigma^*_{ij1}, \sigma^*_{ij2}\right)$ is proposed. The proposed multiplication can also be used to find the multiplication of two ASTIFNs $\left(a_j; \omega_j;\ \omega^*_j\right)$ and $\left(x_{ij}; \sigma_{ij},; \sigma^*_{ij}\right)$ by considering $\omega_{j1} = \omega_{j2} = \omega_j$, $\omega^*_{j1} = \omega^*_{j2} = \omega^*_j$, $\sigma_{ij1} = \sigma_{ij2} = \sigma_{ij}$ and $\sigma^*_{ij1} = \sigma^*_{ij2} = \sigma^*_{ij}$ in the proposed multiplication.

Since, $\tilde{x}_j = \left(x_{ij}; \sigma_{ij1}, \sigma_{ij2}; \sigma^*_{ij1}, \sigma^*_{ij2}\right)$ is a known ATIFN i.e., it can be easily checked that the following condition is satisfied:

$$0 \leq x_{ij} - \sigma_{ij1} \leq x_j - \sigma_{ij2} \leq x_{ij} \leq x_{ij} - \sigma^*_{ij2} \leq x_{ij} - \sigma^*_{ij1}$$

or

$$x_{ij} - \sigma_{ij1} \leq 0 \leq x_{ij} - \sigma_{ij2} \leq x_{ij} \leq x_{ij} - \sigma^*_{j2} \leq x_{ij} - \sigma^*_{ij1}$$

or

$$x_{ij} - \sigma_{ij1} \leq x_{ij} - \sigma_{ij2} \leq x_{ij} \leq 0 \leq x_{ij} - \sigma^*_{ij2} \leq x_{ij} - \sigma^*_{ij1}$$

or

$$x_{ij} - \sigma_{ij1} \leq x_{ij} - \sigma_{ij2} \leq 0 \leq x_{ij} \leq x_{ij} - \sigma^*_{ij2} \leq x_{ij} - \sigma^*_{ij1}$$

or

$$x_{ij} - \sigma_{ij1} \leq x_{ij} - \sigma_{ij2} \leq x_{ij} \leq x_{ij} - \sigma^*_{ij2} \leq 0 \leq x_{ij} - \sigma^*_{ij1}$$

or

$$x_{ij} - \sigma_{ij1} \leq x_{ij} - \sigma_{ij2} \leq x_{ij} \leq x_{ij} - \sigma^*_{ij2} \leq x_{ij} - \sigma^*_{ij1} \leq 0$$

So, the following multiplication can be proposed by generalizing the existing multiplication [8].

Case 1: If $0 \leq x_{ij} - \sigma_{ij1} \leq x_j - \sigma_{ij2} \leq x_{ij} \leq x_{ij} - \sigma^*_{j2} \leq x_{ij} - \sigma^*_{j1}$.

Then,

$\left(a_j; \omega_{j1}, \omega_{j2};\ \omega^*_{j1}, \omega^*_{j2}\right) \otimes \left(x_{ij}; \sigma_{ij1}, \sigma_{ij2}; \sigma^*_{ij1}, \sigma^*_{ij2}\right) = \left(b_{ij}; \lambda_{ij1}, \lambda_{ij2};\ \lambda^*_{ij1}, \lambda^*_{ij2}\right)$.

where,

$$b_{ij} = a_j x_{ij}$$

$$\lambda_{ij1} = b_{ij} - minimum\left(\left(a_j - \omega_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right), \left(a_j - \omega_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right)\right)$$

$$= b_{ij} - \left(\frac{\left(a_j - \omega_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right) + \left(a_j - \omega_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right)}{2} - \frac{\left|\left(a_j - \omega_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right) - \left(a_j - \omega_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right)\right|}{2}\right)$$

$$\lambda_{ij2} = b_{ij} - minimum\left(\left(a_j - \omega_{j2}\right)\left(x_{ij} - \sigma_{ij2}\right), \left(a_j - \omega_{j2}\right)\left(x_{ij} + \sigma^*_{ij2}\right)\right)$$
$$= b_{ij} - \left(\frac{(a_j-\omega_{j2})(x_{ij}-\sigma_{ij2}) + (a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)}{2} - \frac{\left|(a_j-\omega_{j2})(x_{ij}-\sigma_{ij2}) - (a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)\right|}{2}\right)$$
$$\lambda^*_{ij1} = maximum\left(\left(a_j + \omega^*_{j2}\right)\left(x_{ij} + \sigma^*_{ij2}\right), \left(a_j + \omega^*_{j2}\right)\left(x_{ij} - \sigma_{ij2}\right)\right) - b_{ij}$$

$$= \frac{\left(a_j+\omega^*_{j2}\right)\left(x_{ij}+\sigma^*_{ij2}\right)+\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})}{2} + \frac{\left|\left(a_j+\omega^*_{j2}\right)\left(x_{ij}+\sigma^*_{ij2}\right)-\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})\right|}{2} - b_{ij}$$
$$\lambda^*_{ij2} = maximum\left(\left(a_j + \omega^*_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right), \left(a_j + \omega^*_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right)\right) - b_{ij}$$
$$= \frac{\left(a_j+\omega^*_{j1}\right)\left(x_{ij}+\sigma^*_{ij1}\right)+\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1})}{2} + \frac{\left|\left(a_j+\omega^*_{j1}\right)\left(x_{ij}+\sigma^*_{ij1}\right)-\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1})\right|}{2} - b_{ij}$$

Case 2: If $x_{ij} - \sigma_{ij1} \le 0 \le x_{ij} - \sigma_{ij2} \le x_{ij} \le x_{ij} - \sigma^*_{j2} \le x_{ij} - \sigma^*_{ij1}$.

Then,

$$\left(a_j; \omega_{j1}, \omega_{j2};\ \omega^*_{j1}, \omega^*_{j2}\right) \otimes \left(x_{ij}; \sigma_{j1}, \sigma_{ij2}; \sigma^*_{ij1}, \sigma^*_{ij2}\right) = \left(b_{ij}; \lambda_{ij1}, \lambda_{ij2};\ \lambda^*_{ij1}, \lambda^*_{ij2}\right).$$

where,

$$b_{ij} = a_j x_{ij}$$
$$\lambda_{ij1} = b_{ij} - minimum\left(\left(a_j + \omega^*_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right), \left(a_j - \omega_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right)\right)$$
$$b_{ij} - \left(\frac{\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1}) + (a_j-\omega_{j1})\left(x_{ij}+\sigma^*_{ij1}\right)}{2} - \frac{\left|\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1}) - (a_j-\omega_{j1})\left(x_{ij}+\sigma^*_{ij1}\right)\right|}{2}\right)$$
$$\lambda_{ij2} = b_{ij} - minimum\left(\left(a_j - \omega_{j2}\right)\left(x_{ij} - \sigma_{ij1}\right), \left(a_j - \omega_{j2}\right)\left(x_{ij} + \sigma^*_{ij2}\right)\right)$$
$$= b_{ij} - \left(\frac{(a_j-\omega_{j2})(x_{ij}-\sigma_{ij1}) - (a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)}{2} - \frac{\left|(a_j-\omega_{j2})(x_{ij}-\sigma_{ij1}) - (a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)\right|}{2}\right)$$
$$\lambda^*_{ij1} = maximum\left(\left(a_j + \omega^*_{j2}\right)\left(x_{ij} + \sigma^*_{ij2}\right), \left(a_j + \omega^*_{j2}\right)\left(x_{ij} - \sigma_{ij2}\right)\right) - b_{ij}$$
$$= \frac{\left(a_j+\omega^*_{j2}\right)\left(x_{ij}+\sigma^*_{ij2}\right)+\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})}{2} + \frac{\left|\left(a_j+\omega^*_{j2}\right)\left(x_{ij}+\sigma^*_{ij2}\right)-\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})\right|}{2} - b_{ij}$$
$$\lambda^*_{ij2} = maximum\left(\left(a_j + \omega^*_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right), \left(a_j - \omega_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right)\right) - b_{ij}$$
$$= \frac{\left(a_j+\omega^*_{j1}\right)\left(x_{ij}+\sigma^*_{ij1}\right)+(a_j-\omega_{j1})(x_{ij}-\sigma_{ij1})}{2} + \frac{\left|\left(a_j+\omega^*_{j1}\right)\left(x_{ij}+\sigma^*_{ij1}\right)-(a_j-\omega_{j1})(x_{ij}-\sigma_{ij1})\right|}{2} - b_{ij}$$

Case 3: If $x_{ij} - \sigma_{ij1} \le x_{ij} - \sigma_{ij2} \le x_{ij} \le 0 \le x_{ij} - \sigma^*_{ij2} \le x_{ij} - \sigma^*_{ij1}$

or

$$x_{ij} - \sigma_{ij1} \le x_{ij} - \sigma_{ij2} \le 0 \le x_{ij} \le x_{ij} - \sigma^*_{ij2} \le x_{ij} - \sigma^*_{ij1}\ .$$

Then,

$$\left(a_j; \omega_{j1}, \omega_{j2};\ \omega^*_{j1}, \omega^*_{j2}\right) \otimes \left(x_{ij};\ \sigma_{ij1}, \sigma_{ij2}; \sigma^*_{ij1}, \sigma^*_{ij2}\right) = \left(b_{ij}; \lambda_{ij1}, \lambda_{ij2};\ \lambda^*_{ij1}, \lambda^*_{ij2}\right).$$

where,

$$b_{ij} = a_j x_{ij}$$
$$\lambda_{ij1} = b_{ij} - minimum\left(\left(a_j + \omega^*_{j1}\right)\left(x_{ij} - \sigma_{ij1}\right), \left(a_j - \omega_{j1}\right)\left(x_{ij} + \sigma^*_{ij1}\right)\right)$$
$$= b_{ij} - \left(\frac{\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1})+(a_j-\omega_{j1})\left(x_{ij}+\sigma^*_{ij1}\right)}{2} - \frac{\left|\left(a_j+\omega^*_{j1}\right)(x_{ij}-\sigma_{ij1}) - (a_j-\omega_{j1})\left(x_{ij}+\sigma^*_{ij1}\right)\right|}{2}\right)$$
$$\lambda_{ij2} = b_{ij} - minimum\left(\left(a_j - \omega_{j2}\right)\left(x_{ij} + \sigma^*_{ij2}\right), \left(a_j + \omega^*_{j2}\right)\left(x_{ij} - \sigma_{ij2}\right)\right)$$
$$= b_{ij} - \left(\frac{(a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)+\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})}{2} - \frac{\left|(a_j-\omega_{j2})\left(x_{ij}+\sigma^*_{ij2}\right)-\left(a_j+\omega^*_{j2}\right)(x_{ij}-\sigma_{ij2})\right|}{2}\right)$$

$$\lambda_{ij1}^{*} = maximum\left(\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right),\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)\right)-b_{ij}$$
$$=\frac{\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)+\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)}{2}+\frac{\left|\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)-\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)\right|}{2}-b_{ij}$$
$$\lambda_{ij2}^{*} = maximum\left(\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right),\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)\right)-b_{ij}$$
$$=\frac{\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)+\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)}{2}+\frac{\left|\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)-\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)\right|}{2}-b_{ij}$$

Case 4: If $x_{ij}-\sigma_{ij1}\leq x_{ij}-\sigma_{ij2}\leq x_{ij}\leq x_{ij}-\sigma_{ij2}^{*}\leq 0\leq x_{ij}-\sigma_{ij1}^{*}$.

Then,

$$\left(a_j;\omega_{j1},\omega_{j2};\omega_{j1}^{*},\omega_{j2}^{*}\right)\otimes\left(x_{ij};\ \sigma_{ij1},\sigma_{ij2};\sigma_{ij1}^{*},\sigma_{ij2}^{*}\right)=\left(b_{ij};\lambda_{ij1},\lambda_{ij2};\ \lambda_{ij1}^{*},\lambda_{ij2}^{*}\right). \quad (1)$$

where,

$$b_{ij}=a_jx_{ij}$$
$$\lambda_{ij1}=b_{ij}-minimum\left(\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right),\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)\right)$$
$$=b_{ij}-\left(\frac{\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right)+\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)}{2}-\frac{\left|\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right)-\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right|}{2}\right)$$
$$\lambda_{ij2}=b_{ij}-minimum\left(\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right),\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right)$$
$$=b_{ij}-\left(\frac{\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right)+\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)}{2}-\frac{\left|\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right)-\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)\right|}{2}\right)$$
$$\lambda_{ij1}^{*}=maximum\left(\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right),\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right)-b_{ij}$$
$$=\frac{\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)+\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)}{2}+\frac{\left|\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)-\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right|}{2}-b_{ij}$$
$$\lambda_{ij2}^{*}=maximum\left(\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right),\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)\right)-b_{ij}$$
$$=\frac{\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)+\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)}{2}+\frac{\left|\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)-\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)\right|}{2}-b_{ij}$$

Case 5: If $x_{ij}-\sigma_{ij1}\leq x_{ij}-\sigma_{ij2}\leq x_{ij}\leq x_{ij}-\sigma_{ij2}^{*}\leq x_{ij}-\sigma_{ij1}^{*}\leq 0$.

Then,

$$\left(a_j;\omega_{j1},\omega_{j2};\ \omega_{j1}^{*},\omega_{j2}^{*}\right)\otimes\left(x_{ij};\sigma_{ij1},\sigma_{ij2};\sigma_{ij1}^{*},\sigma_{ij2}^{*}\right)=\left(b_{ij};\lambda_{ij1},\lambda_{ij2};\lambda_{ij1}^{*},\lambda_{ij2}^{*}\right). \quad (2)$$

where,

$$b_{ij}=a_jx_{ij}$$
$$\lambda_{ij1}=b_{ij}-minimum\left(\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right),\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right)\right)$$
$$b_{ij}-\left(\frac{\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)+\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right)}{2}-\frac{\left|\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)-\left(a_j+\omega_{j1}^{*}\right)\left(x_{ij}-\sigma_{ij1}\right)\right|}{2}\right)$$
$$\lambda_{ij2}=b_{ij}-minimum\left(\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right),\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right)\right)$$
$$=b_{ij}-\left(\frac{\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)+\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right)}{2}-\frac{\left|\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)-\left(a_j+\omega_{j2}^{*}\right)\left(x_{ij}-\sigma_{ij2}\right)\right|}{2}\right)$$
$$\lambda_{ij1}^{*}=maximum\left(\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right),\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right)-b_{ij}$$
$$\frac{\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)+\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)}{2}+\frac{\left|\left(a_j-\omega_{j2}\right)\left(x_{ij}-\sigma_{ij2}\right)-\left(a_j-\omega_{j2}\right)\left(x_{ij}+\sigma_{ij2}^{*}\right)\right|}{2}-b_{ij}$$
$$\lambda_{ij2}^{*}=maximum\left(\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right),\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)\right)-b_{ij}$$
$$\frac{\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)+\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)}{2}+\frac{\left|\left(a_j-\omega_{j1}\right)\left(x_{ij}-\sigma_{ij1}\right)-\left(a_j-\omega_{j1}\right)\left(x_{ij}+\sigma_{ij1}^{*}\right)\right|}{2}-b_{ij}\ .$$

4. Proposed Ishita approach

In this section, with the help of the multiplication, proposed in Section 4, a new approach (named as Ishita approach) is proposed to construct intuitionistic fuzzy linear regression model $\hat{\tilde{y}}_i = \tilde{a}_0 \oplus \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{x}_{ij}, i = 1,2,\ldots,n$ by considering the regression coefficients $\tilde{a}_j$, $j = 0,1,\ldots,m$, the estimated responses $(\hat{\tilde{y}}_i, i = 1,2,\ldots,n)$ and the observed data$(\tilde{x}_{ij}, \tilde{y}_i)$,$i = 1,2,\ldots,n; j = 1,2,\ldots,m$, as AATIFNs. The proposed Ishita approach can also be used to construct the intuitionistic fuzzy linear regression model, considered by Arefi and Taheri [2], by considering $s_{i1} = s_{i2} = s_i, s_{i1}^* = s_{i2}^* = s_i^*, \omega_{j1} = \omega_{j2} = \omega_j, \omega_{j1}^* = \omega_{j2}^* = \omega_j^*, \sigma_{ij1} = \sigma_{ij2} = \sigma_{ij}$and $\sigma_{ij1}^* = \sigma_{ij2}^* = \sigma_{ij}^*$.

The steps of the proposed Ishita approach are as follows.

Step 1: Transform the intuitionistic fuzzy linear regression model $\hat{\tilde{y}}_i = \tilde{a}_0 \oplus \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{x}_{ij}, i = 1,2,\ldots,n$ into its equivalent intuitionistic fuzzy linear regression model (3) by assuming $\tilde{a}_j$, $\tilde{x}_{ij}$ and $\hat{\tilde{y}}_i$as AATIFNs $(a_j;\ \omega_{j1},\omega_{j2};\ \omega_{j1}^*,\omega_{j2}^*)$, $(x_{ij};\sigma_{ij1},\sigma_{ij2};\sigma_{ij1}^*,\sigma_{ij2}^*)$ and $(\hat{y}_i;\ s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*)$ respectively.

$$(\hat{y}_i;\ s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*) = (a_0;\omega_{01},\omega_{02};\ \omega_{01}^*,\omega_{02}^*) \oplus \sum_{j=1}^{m}(a_j;\ \omega_{j1},\omega_{j2};\omega_{j1}^*,\omega_{j2}^*) \otimes (x_{ij};\ \sigma_{ij1},\sigma_{ij2};\sigma_{ij1}^*,\sigma_{ij2}^*),\ \ i = 1,2,\ldots,n. \quad (3)$$

Step 2: Transform the intuitionistic fuzzy linear regression model (4) into its equivalent intuitionistic fuzzy linear regression model (4) by replacing$(a_j;\omega_{j1},\omega_{j2};\ \omega_{j1}^*,\omega_{j2}^*) \otimes (x_{ij};\ \sigma_{ij1},\sigma_{ij2};\sigma_{ij1}^*,\sigma_{ij2}^*) = (b_{ij};\lambda_{ij1},\lambda_{ij2};\ \lambda_{ij1}^*,\lambda_{ij2}^*)$, obtained by using the multiplication proposed in Section (4).

$$(\hat{y}_i;\ s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*) = (a_0;\omega_{01},\omega_{02};\ \omega_{01}^*,\omega_{02}^*) \oplus \sum_{j=1}^{m}(b_{ij};\lambda_{ij1},\lambda_{ij2};\lambda_{ij1}^*,\lambda_{ij2}^*), i = 1, 2, \ldots, n. \quad (4)$$

Step 3: Transform the intuitionistic fuzzy linear regression model (4) into its equivalent intuitionistic fuzzy linear regression model (5).

$$(\hat{y}_i;\ s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*) = \left(a_0 + \sum_{j=1}^{m} b_{ij};\ \omega_{01} + \sum_{j=1}^{m}\lambda_{ij1}, \omega_{02} + \sum_{j=1}^{m}\lambda_{ij2};\ \omega_{01}^* + \sum_{j=1}^{m}\lambda_{ij1}^*, \omega_{02}^* + \sum_{j=1}^{m}\lambda_{ij2}^*\right), i = 1,2,\ldots,n \quad (5)$$

Step 4: Find the absolute deviation (D_i) between the estimated value of $(\hat{y}_i; s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*)$ and observed values $(y_i;\ s_{i1},s_{i2};\ s_{i1}^*,s_{i2}^*)$ using the relation $D_i = \left|y_i - (a_0 + \sum_{j=1}^{m} b_{ij})\right| + \left|s_{i1} - (\omega_{01} + \sum_{j=1}^{m}\lambda_{ij1})\right| + \left|s_{i2} - (\ \omega_{02} + \sum_{j=1}^{m}\lambda_{ij2})\right| + \left|s_{i1}^* - (\ \omega_{01}^* + \sum_{j=1}^{m}\lambda_{ij1}^*)\right| + \left|s_{i2}^* - (\ \omega_{02}^* + \sum_{j=1}^{m}\lambda_{ij2}^*)\right|, i = 1,2,\ldots,n,$

where, the expression of D_i is obtained by generalizing the existing expression [9, 16] for finding the absolute distance between two fuzzy numbers.

Step 5: Find the sum of absolute deviations, $\sum_{i=1}^{n} D_i = \left(\sum_{i=1}^{n}\left|y_i - (a_0 + \sum_{j=1}^{m} b_{ij})\right| + \left|s_{i1} - (\omega_{01} + \sum_{j=1}^{m}\lambda_{ij1})\right| + \left|s_{i2} - (\ \omega_{02} + \sum_{j=1}^{m}\lambda_{ij2})\right| + \left|s_{i1}^* - (\ \omega_{01}^* + \sum_{j=1}^{m}\lambda_{ij1}^*)\right| + \left|s_{i2}^* - (\ \omega_{02}^* + \sum_{j=1}^{m}\lambda_{ij2}^*)\right|\right)$.

Step 6: Find the optimal solution $\{a_j,\omega_{j1},\omega_{j2},\omega_{j1}^*,\omega_{j2}^*: j = 0,1,\ldots,m\}$ of the following mathematical programming problem.

$$\text{Minimize}\left(\sum_{i=1}^{n}\left|y_i - (a_0 + \sum_{j=1}^{m} b_{ij})\right| + \left|s_{i1} - (\omega_{01} + \sum_{j=1}^{m}\lambda_{ij1})\right| + \left|s_{i2} - (\ \omega_{02} + \sum_{j=1}^{m}\lambda_{ij2})\right| + \left|s_{i1}^* - (\ \omega_{01}^* + \sum_{j=1}^{m}\lambda_{ij1}^*)\right| + \left|s_{i2}^* - (\ \omega_{02}^* + \sum_{j=1}^{m}\lambda_{ij2}^*)\right|\right)$$

Subject to

$0 \le \omega_{j2} \le \omega_{j1}$, $\quad 0 \le \omega_{j2}^* \le \omega_{j1}^*$, $j = 0,1,\ldots,m$.

Step 7: Using the values of $a_j,\omega_{j1},\omega_{j2},\omega_{j1}^*,\omega_{j2}^*: j = 0,1,\ldots,m$, obtained in Step 6, find $\tilde{a}_j = (a_j;\omega_{j1},\omega_{j2};\ \omega_{j1}^*,\omega_{j2}^*)$.

Step 8: Put $\tilde{a}_j = \left(a_j; \omega_{j1}, \omega_{j2};\ \omega_{j1}^*, \omega_{j2}^*\right)$, obtained in Step 7, in equation (4) to obtain the intuitionistic fuzzy linear regression model.

The proposed Ishita approach constructs a model to fit any given real-life data such that for dataset of n patterns, the mathematical programming problem of Step 6 can be solved to minimize $5n$ of least absolute deviations since there is 5 ATIFN's component for each pattern. The minimization problems in this paper are solved using Maple Software tool.

5. Illustrative example

Arefi and Tehri [2, Section V, pp. 1149] constructed an intuitionistic fuzzy linear regression model to illustrate their proposed approach by considering the data [2, Section V, Table 1, pp. 1149]. On applying the proposed Ishita approach for the existing data [2, Section V, Table 1, pp. 1149] the obtained values of ω_2 and ω_2^* are negative real numbers. Since, ω_j and ω_j^* represents the distance and hence, negative values of ω_j and ω_j^* does not have any physical significance. Therefore, it may be concluded that no intuitionistic fuzzy linear regression model can be obtained for the existing data [2, Section V, Table 1, pp. 1149].

In this section, to illustrate the proposed Ishita approach, an intuitionistic fuzzy linear regression model is obtained by considering the following observed data.

$$\tilde{x}_{11} = (3\,;1,\ 1;\ 2,2),\ \tilde{x}_{21} = (2;\ 1,1;2,2), \tilde{y}_1 = (10;\ 5,5;9,\ 9), \tilde{y}_2 = (11;\ 4,\ 4;7,7).$$

Using the proposed Ishita approach, an intuitionistic fuzzy linear regression model for the considered observed data can be obtained as follows.

Step 1: Assuming $\tilde{a}_1 = (a_1;\ \omega_{11}, \omega_{12};\ \omega_{11}^*, \omega_{12}^*)$,$\hat{\tilde{y}}_i = (\hat{y}_i;\ s_{i1}, s_{i2};\ s_{i1}^*, s_{i2}^*), i = 1,2$, and using Step 1 of the proposed Ishita approach, we have

$$(\hat{y}_1;\ s_{11}, s_{12};\ s_{11}^*, s_{12}^*) = (a_0; \omega_{01}, \omega_{02};\ \omega_{01}^*, \omega_{02}^*) \oplus (a_1; \omega_{11}, \omega_{12};\ \omega_{11}^*, \omega_{12}^*) \otimes (3\,; 1,1;\ 2,2).$$

$$(\hat{y}_2;\ s_{21}, s_{22};\ s_{21}^*, s_{22}^*) = (a_0;\ \omega_{01}, \omega_{02};\ \omega_{01}^*, \omega_{02}^*) \oplus (a_1;\ \omega_{11}, \omega_{12};\ \omega_{11}^*, \omega_{12}^*) \otimes (2; 1,1; 2,2).$$

Step 2: Since, for the ATIFN $(3;\ 1,\ 1;\ 2,2)$, the condition $3 - 1 = 2 > 0$ is satisfying as well as for the ATIFN $(2;\ 1,1;2,2)$, the condition $2 - 1 = 1 > 0$ is satisfying. So, using the multiplication, proposed in Section 3, $(a_1;\ \omega_{11}, \omega_{12};\ \omega_{11}^*, \omega_{12}^*) \otimes (3;\ 1,\ 1;\ 2,2) = (3a_1; |a_1| + 3\omega_{11}\,, |a_1| + 3\omega_{12}\,; 2|a_1| + 3\omega_{11}^*, 2|a_1| + 3\omega_{12}^*)$ as well as $(a_1; \omega_{11}, \omega_{12};\ \omega_{11}^*, \omega_{12}^*) \otimes (2;\ 1,1;2,2) = (2a_1;\ |a_1| + 2\omega_{11}\,, |a_1| + 2\omega_{12}\,;\ 2|a_1| + 2\omega_{11}^*, 2|a_1| + 2\omega_{12}^*)$.

Therefore, we have

$$(\hat{y}_1;\ s_{11}, s_{12};\ s_{11}^*, s_{12}^*) = (a_0;\ \omega_{01}, \omega_{02};\ \omega_{01}^*, \omega_{02}^*) \oplus (3a_1; |a_1| + 3\omega_{11}\,, |a_1| +; 2|a_1| + 3\omega_{11}^*, 2|a_1| + 3\omega_{12}^*).$$

$$(\hat{y}_2;\ s_{21}, s_{22};\ s_{21}^*, s_{22}^*) = (a_0, \omega_{01}, \omega_{02};\ \omega_{01}^*, \omega_{02}^*) \oplus (2a_1; |a_1| + 2\omega_{11}\,, |a_1| + 2\omega_{12}\,;, 2|a_1| + 2\omega_{12}^*).$$

Step 3: Using Step 3 of the proposed Ishita approach, we have,

$$(\hat{y}_1;\ s_{11}, s_{12};\ s_{11}^*, s_{12}^*) = (a_0 + 3a_1; \omega_{01} + |a_1| + 3\omega_{11}\,, \omega_{02} + |a_1| + 3\omega_{12}\,;\ \omega_{01}^* +, \omega_{02}^* + 2|a_1| + 3\omega_{12}^*).$$

$$(\hat{y}_2;\ s_{21}, s_{22};\ s_{21}^*, s_{22}^*) = (a_0 + 2a_1;\ \omega_{01} + |a_1| + 2\omega_{11}\,, \omega_{02} + |a_1| + 2\omega_{12}\,;\ \ \omega_{02}^* + 2|a_1| + 2\omega_{12}^*).$$

Step 4: Using Step 4 of the proposed Ishita approach, we have,

$D_1 = |10 - (a_0 + 3a_1)| + |5 - (\omega_{01} + |a_1| + 3\omega_{11})| + |5 - (\omega_{02} + |a_1| + 3\omega_{12})| + |9 - (\omega_{01}^* + 2|a_1| + 3\omega_{11}^*)| + |9 - (\omega_{02}^* + 2|a_1| + 3\omega_{12}^*)|$

$D_2 = |11 - (a_0 + 2a_1)| + |4 - (\omega_{01} + |a_1| + 2\omega_{11})| + |4 - (\omega_{02} + |a_1| + 2\omega_{12})| + |7 - (\omega_{01}^* + 2|a_1| + 2\omega_{11}^*)| + |7 - (\omega_{02}^* + 2|a_1| + 2\omega_{12}^*)|$.

Step 5: Using Step 5 of the proposed Ishita approach, we have,

$\sum_{i=1}^{2} D_i = |10 - (a_0 + 3a_1)| + |5 - (\omega_{01} + |a_1| + 3\omega_{11})| + |5 - (\omega_{02} + |a_1| + 3\omega_{12})| + |9 - (\omega_{01}^* + 2|a_1| + 3\omega_{11}^*)| + |9 - (\omega_{02}^* + 2|a_1| + 3\omega_{12}^*)| + |11 - (a_0 + 2a_1)| + |4 - (\omega_{01} + |a_1| + 2\omega_1)| + |4 - (\omega_{02} + |a_1| + 2\omega_{12})| + |7 - (\omega_{01}^* + 2|a_1| + 2\omega_{11}^*)| + |7 - (\omega_{02}^* + 2|a_1| + 2\omega_{12}^*)|$.

Step 6: Using Step 6 of the proposed Ishita approach, the following mathematical programming problem is obtained.

Minimize $(|10 - (a_0 + 3a_1)| + |5 - (\omega_{01} + |a_1| + 3\omega_{11})| + |5 - (\omega_{02} + |a_1| + 3\omega_{12})| + |9 - (\omega_{01}^* + 2|a_1| + 3\omega_{11}^*)| + |9 - (\omega_{02}^* + 2|a_1| + 3\omega_{12}^*)| + |11 - (a_0 + 2a_1)| + |4 - (\omega_{01} + |a_1| + 2\omega_1)| + |4 - (\omega_{02} + |a_1| + 2\omega_{12})| + |7 - (\omega_{01}^* + 2|a_1| + 2\omega_{11}^*)| + |7 - (\omega_{02}^* + 2|a_1| + 2\omega_{12}^*)|)$.

Subject to

$0 \le \omega_{j2} \le \omega_{j1}, \qquad 0 \le \omega_{j2}^* \le \omega_{j1}^*, \;\; j = 0,1.$

On solving this mathematical programming problem the obtained optimal solution is $a_0 = 13, a_1 = -1$, $\omega_{01} = \omega_{02} = \omega_{01}^* = \omega_{02}^* = 1, \omega_{11} = \omega_{12} = 1, \omega_{11}^* = \omega_{12}^* = 2$.

Step 7: Using the values of $a_j, \omega_{j1}, \omega_{j2}, \omega_{j1}^*, \omega_{j2}^*: j = 0,1$, obtained in Step 6,

$\tilde{a}_0 = (13; 1, 1;\; 1, 1)$ and $\tilde{a}_1 = (-1;\; 1, 1;\;\; 2, 2)$.

Step 8: Putting $\tilde{a}_j = (a_j;\; \omega_{j1}, \omega_{j2};\; \omega_{j1}^*, \omega_{j2}^*)$, $j = 0,1$, obtained in Step 7, in $(\hat{y};\; s_1, s_2;\; s_1^*, s_2^*) = (a_0;\; \omega_{01}, \omega_{02};\; \omega_{01}^*, \omega_{02}^*) \oplus ((a_1;\; \omega_1, \omega_2;\; \omega_1^*, \omega_2^*) \otimes (x_1;\; \sigma_1, \sigma_2;\; \sigma_1^*, \sigma_2^*))$, the obtained intuitionistic fuzzy linear regression model is

$(\hat{y};\; s_1, s_2;\; s_1^*, s_2^*) = (13; 1, 1;\; 1, 1) \oplus (-1;\; 1, 1;\;\; 2, 2) \otimes (x_1;\; \sigma_1, \sigma_2;\; \sigma_1^*, \sigma_2^*)$.

$\omega_{j1} = \omega_{j2} = \omega_j, \omega_{j1}^* = \omega_{j2}^* = \omega_j^*, \sigma_{ij1} = \sigma_{ij2} = \sigma_{ij}$ and $\sigma_{ij1}^* = \sigma_{ij2}^* = \sigma_{ij}^*$

Now, we investigate another example to illustrate our approach. The dataset in this example is sized in four patterns of $\{\langle \tilde{Y}_i, \tilde{X}_{i1}\rangle \big| i = \overline{1,4}\}$ where $\tilde{Y}_i$ is the volume (given in Liter) of one mole of methane gas at under a constant pressure of one atom and $\tilde{X}_{i1}$ is the temperature on Celsius scale (℃). Due to uncertainty in expressing temperature and volume, $\tilde{X}_{i1}$ and $\tilde{Y}_i$ are represented as symmetric ATIFNs, i.e, $\sigma_1 = \sigma_2 = \gamma_{x_{1_\mu}}$, $\sigma_1^* = \sigma_2^* = \gamma_{x_{1_v}}$ for every $\tilde{X}_{i1}$ and $s_1 = s_2 = \gamma_{y_\mu}$, $s_1^* = s_2^* = \gamma_{y_v}$ for $\tilde{Y}_i, i = 1, .., 4$. It assumed that there is direct relation between the temperature and the volume of methane gas. And the relation stops at -162 ℃. The four patterns are

$$\{\langle (22; 0.0110, 0.9890), (-3;\; 0.0110, 0.9890)\rangle,$$
$$\langle (21;\; 0.0198, 0.9802), (-23;\; 0.0842, 0.9158), \rangle$$
$$\langle (18;\; 0.0081, 0.9919), (-53; 0.1940, 0.80510)\rangle,$$
$$\langle (\,9;\; 0.0045, 0.9955), (-162;\; 0.5931, 0.4069)\rangle\}$$

The intuitionistic fuzzy regression model of this example has the following general form.

$$\hat{\tilde{Y}} = (\hat{y};\; s, s^*) = (\hat{a}_0;\; \omega_0, \omega_0^*) \oplus ((\hat{a}_1;\; \omega_1, \omega_1^*) \otimes (x_1;\; \sigma_1, \sigma_1^*)) \tag{6}$$

Although Arefi and Taheri [2] considered the only positive symmetric datasets in their least squared deviations method, their method applied in this example to show that their method cannot be generalized. The estimating the model's parameters of Arefi and Taheri [2] method is The model's intercept is $(\hat{a}_0;\; \omega_0, \omega_0^*) = (10.9534;\; 0.0111, 1.0975)$ and the slope is $(\hat{a}_1;\; \omega_1, \omega_1^*) = (-0.1087;\; -0.0004, 0.0004)$.

While $\omega_1 = -0.0004$, the slope is not TAIFN because the spread cannot be negative and consequently, using Arefi and Taheri [2] approach, an intuitionistic fuzzy linear regression model doesn't exist for such dataset.

Although Chen and Nien [12] mentioned that their approach is general and they have set dummy variables to confront the effect of negative sign of slope(s), i.e., $\tilde{A}_1 \otimes \tilde{X}_1 = \tilde{A}_{11} \otimes \tilde{X} \oplus \tilde{A}_{12} \otimes \tilde{X}$ such that $\tilde{A}_{11} \otimes \tilde{A}_{12} = \big(0; (0,0), (0,0)\big)$, the estimated model's parameters using Chen and Nien [12] approach are not suitable for negative data. On applying Chen and Nien [12] approach for this example the obtained models' parameters according to the general model (6) as it is $\hat{a}_0 = 22.3860$, $\hat{a}_0 - \omega_{01} = 22.2522$ $(\omega_{01} = 0.1338)$ $\hat{a}_0 - \omega_{01}^* = 21.3315$ $\omega_{01}^* = 1.0545$, $\hat{a}_0 + \omega_{02} = 22.3860$ $(\omega_2 = 0)$, $\hat{a}_0 + \omega_{02}^* = 23.3191$ $(\omega_{02}^* = 0.9331)$. Therefore, the intercept is $\tilde{A}_0 = (22.3860; 0.1338, 1.0545; 0, 0.9331)$. However, the two dummy variables are both non zero TAIFNs. The estimated values of $\tilde{A}_{11}$ and $\tilde{A}_{12}$ are $(-0.0029;\ 0,\ 0.0012;\ 0,\ 0)$ and $(0.0859;\ 0.0014,\ 0.0014,\ 0,\ 0)$, respectively. It is obvious that neither $\tilde{A}_{11} = 0$ nor $\tilde{A}_{11} = 0$ and the independent variable cannot have two slopes simultaneously. Therefore, there is no model exists to fit the given data of this example using Chen and Nien [12] approach.

However, on applying the Steps of proposed Ishita approach with the help of proposed multiplication and since all of the input data patterns are negative ASTIFNs Case 5 of the proposed multiplication is considered, the intuitionistic fuzzy linear model of this example after obtaining the model's parameters is constructed as follows:

$$\hat{\hat{Y}} = (\hat{y};\ s, s^*) = (23.2910;\ 0.0081, 0.9890) \oplus (0.09910;\ 0, 0) \otimes (x_1;\ \sigma_1, \sigma_1^*) \tag{7}$$

It is clear by this example that the proposed approach is sound and general. For the data given in this example, there is only one model fit such data and predict a volume of methane gas from a given temperature in Celsius scale this model is model (7) which only can be constructed using the proposed approach.

6. Conclusions

This study used mathematical programming problems to construct an intuitionistic fuzzy linear regression models. The least absolutely deviations between the predicted and observed ATIFN are considered as the objective function, makes the constructed model more robust. The mathematical incorrect assumptions, considered in the existing approaches are pointed out. Also, a new approach (named as Ishita approach) is proposed to construct the intuitionistic fuzzy linear regression model. The model used proposed multiplication of an unknown ATIFN and a known ATIFN which is the silent feature of Ishita approach. Ishita approach is sound and general to fit any type of given data represented as ATIFNs. It conserves the homogeneity principle where for ASTIFNs the constructed model is symmetric. Furthermore, to illustrate the proposed Ishita approach a numerical example has been solved. The real-life example shows the advantage of Ishita approach over existing approaches. The proposed multiplication is a bit complicated and nonlinear so, that can be improved as future research work.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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