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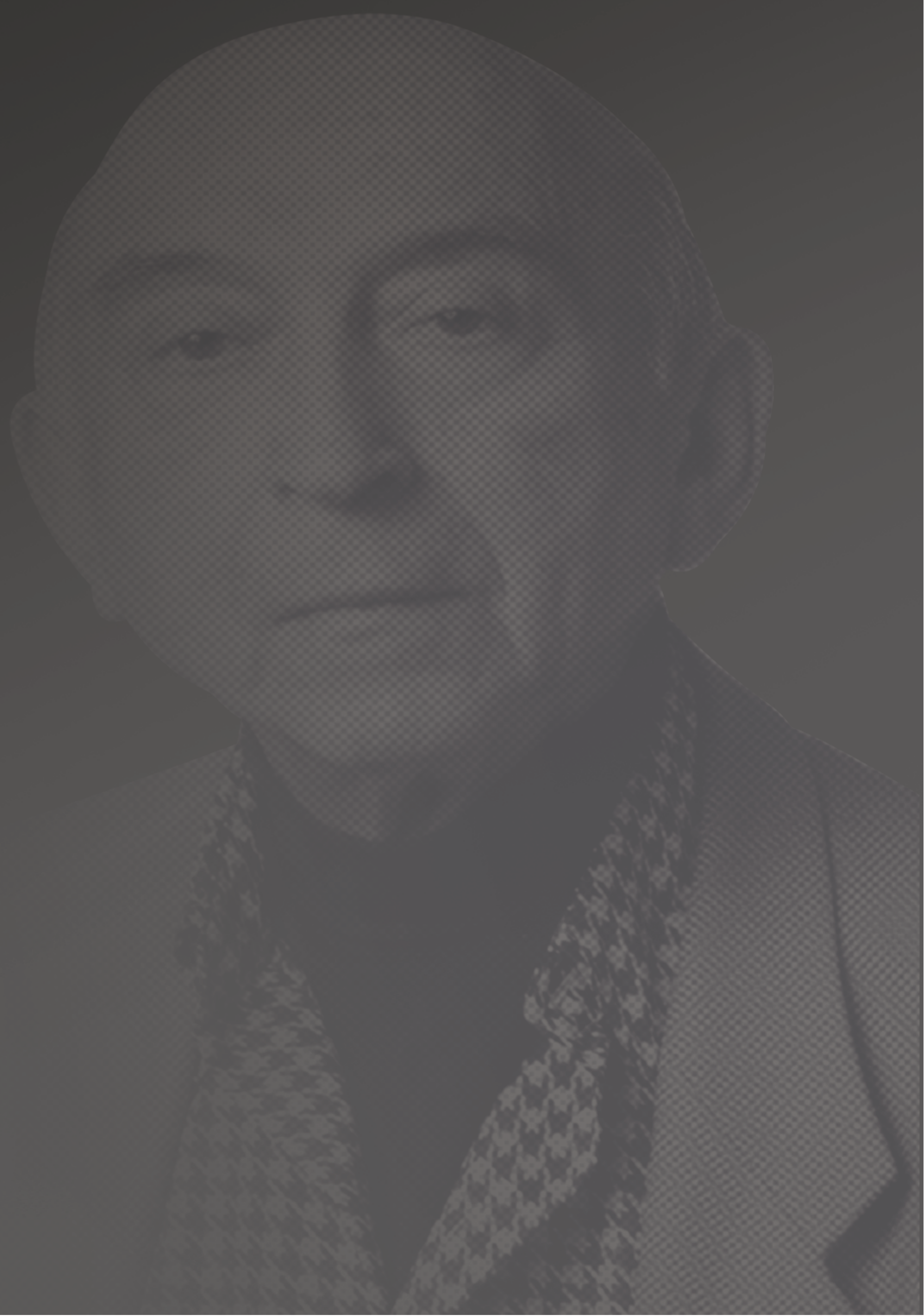

Transactions on
Fuzzy Sets & Systems

A warm welcome to the inaugural Volume 1, Issue 1, of the Journal of Transactions on fuzzy sets and systems (TFSS), a new exciting title for journals for 2022. The launch of this pioneering journal marks a new era for the multidisciplinary field. TFSS aims to publish high-quality original articles that make a significant contribution to the research areas of both theoretical and applied mathematics in the field of fuzzy. Therefore, the journals focus is to provide a new platform for disseminating the latest research and current practices in the emerging fields from fuzzy logic, fuzzy mathematics, intelligent systems theory, fuzzy generalizations, and related areas. Fuzzy sets and systems are an emerging and rapidly growing multidisciplinary field.

Fuzzy sets are also the cornerstone of a non-additive uncertainty theory, namely possibility theory, and of a versatile tool for both linguistic and numerical modeling: fuzzy rule-based systems. Numerous works now combine fuzzy concepts with other scientific disciplines as well as modern technologies. Fuzzy sets have triggered new research topics in connection with category theory, topology, algebra, analysis, optimization, Soft sets, etc. Furthermore, fuzzy sets have strong logical underpinnings in the tradition of many-valued logic.

Our vision for TFSS is to establish a vigorously peer-reviewed, open-access, online platform with the mission to publish the most inspiring and impactful research discoveries in fundamental areas for all cognitive systems and computational studies worldwide. The Journal will publish original research articles that review recent developments in specific subject areas and forward-looking perspectives.

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I have the great honor of being the founding Editor-in-Chief and accept the challenges and responsibilities of developing TFSS into a premier professional journal on fuzzy sets and systems. Moreover, it is our great honor to work with an exceptional team of editorial board members who give generously of their time and expertise to the journal. We will rely heavily on them for their willingness to process submitted papers and make a sound recommendation on each manuscript and work with me to develop this new journal to serve the research community best.

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We wish to close this inaugural editorial by showing heartfelt gratitude to all for your support as editors, authors, reviewers, and information users who have helped ensure the quality of all articles included in this inaugural issue of TFSS. We will welcome and value greatly your suggestions and ideas for making the journal highly impactful. Together, with your support, we can build a journal of excellence to serve the research community.

Arsham Borumand Saeid

Transactions on Fuzzy Sets and Systems

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On Quantum-MV algebras-Part II: Orthomodular Lattices, Softlattices and Widelattices

Afrodita Iorgulescu

Abstract. Orthomodular lattices generalize the Boolean algebras; they have arisen in the study of quantum logic. Quantum-MV algebras were introduced as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices.

In this paper, we continue the research in the "world" of involutive *algebras* of the form $(A, \odot, ^-, 1)$, with $1^- = 0$, 1 *being the last element*. We clarify now some aspects concerning the quantum-MV (QMV) algebras as non-idempotent generalizations of orthomodular lattices. We study in some detail the orthomodular lattices (OMLs) and we introduce and study two generalizations of them, the orthomodular softlattices (OMSLs) and the orthomodular widelattices (OMWLs). We establish systematically connections between OMLs and OMSLs/OMWLs and QMV, pre-MV, metha-MV, orthomodular algebras and ortholattices, orthosoftlattices/orthowidelattices - connections illustrated in 22 Figures. We prove, among others, that the transitive OMLs coincide with the Boolean algebras, that the OMSLs coincide with the OMLs, that the OMLs are included in OMWLs and that the OMWLs are a proper subclass of QMV algebras. The transitive and/or the antisymmetric case is also studied.

AMS Subject Classification 2020: 06D35; 06F99; 03G05
Keywords and Phrases: m-MEL algebra, m-BE algebra, m-pre-BCK algebra, m-BCK algebra, MV algebra, Quantum-MV algebra, Pre-MV algebra, Metha-MV algebra, Orthomodular algebra, Ortholattice, Orthosoftlattice, Orthowidelattice, Boolean algebra

1 Introduction

The *algebraists* work usually with the commutative additive groups and with the positive (right) cone of a partially-ordered commutative group $(G, \leq, +, -, 0)$, where there are essentially a sum $\oplus = +$ and an element 0. Sometimes, the negative (left) cone is needed also, where there are essentially a product $\odot = +$ and an element $1 = 0$. They work with algebras that have associated an (pre-order) order relation, which usually does not appear explicitly in the definitions. The presence of the (pre-order) order relation implies the presence of the (generalized) duality principle. Thus, each algebra has a dual one, the (pre-order) order relation has a dual one. We have given names to the dual algebras [14], [16], [15]: "left" algebra and "right" algebra, names connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. Hence, the algebraists usually work with the commutative *right-unital magmas*.

By the contrary, the *logicians* work with the logic of *truth*, where the *truth* is represented by 1, and there is essentially one implication; we could name this logic "left-logic". One can imagine also a "right-logic", as

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a logic of *false*, where the *false* is represented by 0 and there is a "right-implication". Hence, the logicians usually work with the commutative *left-algebras of logic*.

In this paper, *regarding from (algebras of) logic side, we shall work with left-algebras (left-unital magmas) as principal algebras*, therefore, the unital magmas will be defined multiplicatively.

Thus, the commutative algebraic structures connected directly or indirectly with classical/ nonclassical logics belong to two parallel "worlds":

1. the "world" of *(left) algebras of logic*, where there are essentially one implication, $\to$ (two, in the non-commutative case), and an element 1 (that can be the last element); the algebras $(A, \to, 1)$, verifying the basic property (M): $1 \to x = x$, are called *M algebras* [16], [15]; an internal binary relation can be defined by: $x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$ ($\leq$ can be a pre-order, an order, or even a lattice order); algebras belonging to this "world" are [17], [16], [15]: the bounded MEL, BE and aBE, pre-BCK algebras, BCK algebras, bounded BCK algebras, BCK(P) algebras, Hilbert algebras, Wajsberg algebras, implicative-Boolean algebras, etc. A "Big map" (hierarchy of algebras of logic) is presented in ([15], Figure 1).

2. the "world" of *(left) algebras*, where there are essentially a product, $\odot$, and an element 1 (that can be the last element); the algebras $(A, \odot, 1)$, verifying the corresponding basic properties (PU): $1 \odot x = x$ and (Pcomm): $x \odot y = y \odot x$, are called *commutative unital magmas*; in algebras with an additional operation, $(A, \odot, ^-, 1)$, an internal binary relation can be defined by: $x \leq_m y \Longleftrightarrow x \odot y^- = 0$ ($\leq_m$ can be a pre-order, an order, or even a lattice order), where 'm' comes from 'magma'; algebras belonging to this "world" are [16], [15]: the m-MEL, m-BE and m-aBE, m-pre-BCK algebras, m-BCK algebras, pocrims, (bounded) lattices, residuated lattices, BL algebras, MTL algebras, NM algebras, MV algebras, Boolean algebras, etc. A corresponding "Big map" (hierarchy of algebras) is presented in ([15], Figure 10).

Between the two parallel "worlds" there are some connections, as for example: the equivalence between BCK(P) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$). In [15], Theorems 9.1 and 9.3 connect the two "worlds" in the involutive case.

Beside the classical and non-classical logics, there exist the quantum logics. Examples of algebraic structures connected with quantum logics (= quantum structures/ algebras) are the bounded implicative (implication) lattices, the De Morgan algebras, the ortholattices, the orthomodular lattices, the quantum-MV algebras, etc.

The ortholattice is an important example of *sharp* structure (which satisfies the noncontradiction principle) from sharp quantum theory [4] (Birkhoff, 1967; Kalmbach, 1983).

Orthomodular lattices (particular ortholattices) generalize the Boolean algebras. They have arisen, cf. [25], "in the study of quantum logic, that is, the logic which supports quantum mechanics and which does not conform to classical logic. As noted by Birkhoff and von Neumann in 1936 [2], the calculus of propositions in quantum logic "is formally indistinguishable from the calculus of linear subspaces [of a Hilbert space] with respect to set products, linear sums and orthogonal complements" in the role of *and*, *or* and *not*, respectively. This has led to the study of the closed subspaces of a Hilbert space, which form an orthomodular lattice in contemporary terminology. As often happens in algebraic logic, the study of orthomodular lattices has tremendously developed, both for their interest in logic and for their own sake, see Kalmbach [23]".

Quantum-MV algebras (or QMV algebras) were introduced by Roberto Giuntini in [11] (see also [9], [8], [12], [10], [13], [7], [6]), as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices.

The connections between algebras of logic/ algebras and quantum algebras were not very clear. But, in papers [15], [20], [21], we established important connections, by redefining equivalently the bounded involutive lattices and De Morgan algebras as involutive *m-MEL algebras*, the ortholattices, the MV, the Boolean algebras and the quantum-MV algebras as involutive *m-BE algebras*, verifying some properties, and then putting all of them on the involutive "Big map"; thus, we have proved that the quantum algebras belong, in fact, to the "world" of *algebras* (involutive commutative unital magmas).

In this paper, we continue the research from [21], [18], based on [22], [20], [15], in the "world" of involutive *algebras* of the form $(A, \odot, ^-, 1)$, with $1^- = 0$, 1 *being the last element*. We clarify now some aspects concerning the quantum-MV algebras as non-idempotent generalizations of orthomodular lattices. We study the orthomodular lattices and we introduce and study two generalizations of them, the orthomodular softlattices and the orthomodular widelattices - in connection with the lattices/ ortholattices and their two generalizations, the softlattices/ orthosoftlattices and the widelattices/ orthowidelattices, generalizations introduced in [22]. Many results were obtained by the powerful computer program *Prover*9/*Mace*4 (version DEC. 2007) created by William W. McCune (1953 – 2011) [24]. By lack of space, we shall not present here the examples we have. This paper, like [15], [20], [22], [21], [18], presents the facts in the same *unifying way*, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different algebras and for obtaining results.

The paper is organized as follows.

In Section 2 (**Preliminaries**), we recall the notions and the results necessary for making the paper self-contained as much as possible.

In Section 3 (**Orthomodular lattices**), we study in some detail the orthomodular lattices (OMLs), that are QMV algebras. We establish connections between OMLs and QMV, pre-MV, metha-MV, orthomodular (OM) algebras and ortholattices (OLs), connections illustrated in Figures $3-8$. We prove that the antisymmetric OMLs and the transitive OMLs coincide with the Boolean algebras and that transitive OLs are included in transitive metha-MV algebras. We introduce the new notion of *modular algebra* and we prove that the modular algebras coincide with the modular ortholattices.

In Section 4 (**Orthomodular softlattices, widelattices**), based on the two generalizations of OLs: the orthosoftlattices (OSLs) and the orthowidelattices (OWLs), introduced in [22], we introduce and study, in separate subsections, two corresponding generalizations of OMLs: the orthomodular softlattices (OMSLs) and the orthomodular widelattices (OMWLs). We establish connections between OMSLs/OMWLs and QMV, pre-MV, metha-MV, OM algebras and OLs, OSLs/OWLs, connections illustrated in Figures $9-15/16-22$, respectively. We prove that the OMLs coincide with the OMSLs and that transitive OSLs are included in transitive metha-MV algebras. We also prove that the OMLs are included in OMWLs, which in turn are included in QMV algebras too, and that transitive OWLs are included in transitive metha-MV algebras, hence that transitive OMWLs are included in transitive QMV algebras.

2 Preliminaries

2.1 The "Big map" of algebras

Recall from [15] the following:

Let $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ be an algebra of type $(2, 1, 0)$ and define $0 \overset{def.}{=} 1^-$. Define an *internal* binary relation $\leq_m$ on A^L by: for all $x, y \in A^L$,

(m-dfrelP) $\quad x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$.

Consider the following list **m-A** of basic properties that can be satisfied by $\mathcal{A}^L$ [15]:

(PU) $\quad 1 \odot x = x = x \odot 1$ (unit element of product, the *identity*),
(Pcomm) $\quad x \odot y = y \odot x$ (commutativity of product),
(Pass) $\quad x \odot (y \odot z) = (x \odot y) \odot z$ (associativity of product);

(Neg1-0) $\quad 1^- = 0$,
(Neg0-1) $\quad 0^- = 1$;

(m-An) $(x \odot y^- = 0 \text{ and } y \odot x^- = 0) \Longrightarrow x = y$ (antisymmetry),
(m-B) $[(x \odot y^-)^- \odot (x \odot z)] \odot (y \odot z)^- = 0$,
(m-BB) $[(z \odot x)^- \odot (y \odot x)] \odot (y \odot z^-)^- = 0$,
(m-*) $x \odot y^- = 0 \Longrightarrow (z \odot y^-) \odot (z \odot x^-)^- = 0$,
(m-**) $x \odot y^- = 0 \Longrightarrow (x \odot z) \odot (y \odot z)^- = 0$,
(m-L) $x \odot 0 = 0$ (last element),
(m-Re) $x \odot x^- = 0$ (reflexivity),
(m-Tr) $(x \odot y^- = 0 \text{ and } y \odot z^- = 0) \Longrightarrow x \odot z^- = 0$ (transitivity),
etc.

Dually, let $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ be an algebra of type $(2, 1, 0)$ and define $1 \stackrel{def.}{=} 0^-$. Define an *internal* binary relation $\geq_m$ on A^R by: for all $x, y \in A^R$,

(m-dfrelS) $x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1$.

The list of dual properties is omitted.

Recall from [15] the definitions of the following algebras needed in this paper (the dual ones are omitted):

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an algebra of type $(2, 1, 0)$ through this paper. Define $0 \stackrel{def.}{=} 1^-$ (hence (Neg1-0) holds) and suppose that $0^- = 1$ (hence (Neg0-1) holds too). We say that $\mathcal{A}^L$ is a [15]:

- *left-m-MEL algebra*, if (PU), (Pcomm), (Pass), (m-L) hold;
- *left-m-BE algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) hold;
- *left-m-pre-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) and (m-BB) hold;
- *left-m-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An) and (m-BB) hold.

Denote by **m-MEL**, **m-BE**, **m-pre-BCK**, **m-BCK** these classes of left-algebras, respectively.

In ([15], Figure 10), the "Big map", connecting the commutative unital magmas, including these algebras, was drawn.

We say that $\mathcal{A}^L$ is [15] *reflexive*, if $\leq_m$ is reflexive (i.e. (m-Re) holds); *transitive*, if $\leq_m$ is transitive (i.e. (m-Tr) holds); *antisymmetric*, if $\leq_m$ is antisymmetric (i.e. (m-An) holds). If **X** is a class of algebras, we shall denote by **tX** (**aX**, **atX**=**taX**) the subclass of all transitive (antisymmetric, transitive and antisymmetric, respectively) algebras of **X**.

We say that an algebra is *involutive*, if it verifies (DN) ($(x^-)^- = x$ or $x^= = x$). If **X** is a class of algebras, we shall denote by $\mathbf{X}_{(DN)}$ the subclass of all involutive algebras of **X**. By ([15], Theorem 6.12), in any involutive m-BE algebra we have the equivalences: (m-BB) $\Leftrightarrow$ (m-B) $\Leftrightarrow$ (m-**) $\Leftrightarrow$ (m-*) $\Leftrightarrow$ (m-Tr).

Note that: $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)} = \mathbf{pre\text{-}m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}tBE}_{(DN)})$.

Any left-m-BCK algebra is involutive, by ([15], Theorem 6.13). We write: **m-BCK**= $\mathbf{m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}taBE}_{(DN)})$. Note that a (involutive) m-BCK algebra satisfies all the properties in the list **m-A** of properties and, additionally, (DN) and other properties.

Note that the binary relation $\leq_m$ is only **reflexive** in $\mathbf{m\text{-}BE}_{(DN)}$, it is a **pre-order** in $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)}$ and it is an **order** in **m-BCK**.

2.1.1 Involutive m-MEL algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Because of the axiom (DN), we have introduced in [20] the new operation sum, $\oplus$, the dual of product, $\odot$, by: for all $x, y \in A^L$,

$$x \oplus y \stackrel{def.}{=} (x^- \odot y^-)^-. \tag{1}$$

Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-MEL algebra.

Proposition 2.1. *(See ([6], Proposition 2.1.2), in dual case, [9])* *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:*

$$0 \oplus x = x = x \oplus 0, \quad i.e. \quad (SU) \quad holds, \tag{2}$$

$$x \oplus y = y \oplus x, \quad i.e. \quad (Scomm) \quad holds, \tag{3}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad i.e. \quad (Sass) \quad holds, \tag{4}$$

$$x \oplus 1 = 1, \quad i.e. \quad (m - L^R) \quad holds; \tag{5}$$

$$(x \oplus y)^- = x^- \odot y^- \quad (De\ Morgan\ law\ 1), \tag{6}$$

$$(x \odot y)^- = x^- \oplus y^- \quad (De\ Morgan\ law\ 2),\ and\ hence \tag{7}$$

$$x \odot y = (x^- \oplus y^-)^-. \tag{8}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [20] a new binary relation:
(m-dfP) $x \ \leq_m^P \ y \stackrel{def.}{\Longleftrightarrow} x \odot y = x$ and, dually,
(m-dfS) $x \ \geq_m^S \ y \stackrel{def.}{\Longleftrightarrow} x \oplus y = x$.
By ([20], Proposition 3.11), $\leq_m^P$ is antisymmetric and transitive and $0 \leq_m^P x \leq_m^P 1$, for any x.

Proposition 2.2. *([20], Proposition 3.14)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. If (m-Pimpl) holds, then:
(1) the order relation $\leq_m^P$ is a lattice order (denoted by $\leq_m^O$),
(2) $x \leq_m^P y \Longleftrightarrow y \geq_m^S x$,
(3) $x \leq_m^P y \Longrightarrow y^- \leq_m^P x^-$.

With the notations from this subsection, the definition of MV algebras [3], [5] becomes [15]:

Definition 2.3. *(The dual one is omitted)*
A left-MV algebra is an algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-L}, 1)$ of type $(2, 1, 0)$ verifying (PU), (Pcomm), (Pass), (m-L), (DN) and:
($\wedge_m$-comm) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.

We recall the following important remark, which was the motivation of paper [15]:
The left-MV algebra is just the involutive left-m-MEL algebra verifying ($\wedge_m$-comm).
We have denoted by **MV** the class of all left-MV algebras.

2.1.2 Involutive m-BE algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-BE algebra.

Remark 2.4. *(See ([15], Theorem 6.21) (The dual one is omitted)*
Since ($\wedge_m$-comm) implies (m-Re), by ([15], (mB1)), it follows that **any left-MV algebra is in fact an involutive left-m-BE algebra verifying ($\wedge_m$-comm)***. And since ($\wedge_m$-comm) implies also (m-An) and (m-BB) ($\Longleftrightarrow$ ... (m-Tr)), by ([15], (mB2), (mCBN1)), respectively, it follows that* **any left-MV algebra is in fact a left-m-BCK algebra***, i.e. we have:*

$$\mathbf{MV} \ \subset \ \mathbf{m-BCK} = \mathbf{m-BCK_{(DN)}} \ (= \mathbf{m-taBE_{(DN)}}).$$

We have introduced in [21], in an involutive left-m-MEL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$, the following new operations:

$$x \wedge_m^M y \stackrel{def.}{=} (x^- \odot y)^- \odot y \stackrel{(Pcomm)}{=} y \odot (y \odot x^-)^- \quad and,\ dually, \tag{9}$$

$$x \vee_m^M y \stackrel{def.}{=} (x^- \wedge_m^M y^-)^- = [(x \odot y^-)^- \odot y^-]^- = y \oplus (y \oplus x^-)^- \tag{10}$$

and

$$x \wedge_m^B y \stackrel{def.}{=} (y^- \odot x)^- \odot x \stackrel{(Pcomm)}{=} x \odot (x \odot y^-)^- = y \wedge_m^M x \quad and,\ dually, \tag{11}$$

$$x \vee_m^B y \stackrel{def.}{=} (x^- \wedge_m^B y^-)^- = ((y \odot x^-)^- \odot x^-)^- = x \oplus (x \oplus y^-)^- = y \vee_m^M x. \tag{12}$$

Proposition 2.5. *(See [6], Proposition 2.1.2, in dual case) ([21], Proposition 3.2)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:

$$x \wedge_m^M 1 = x = 1 \wedge_m^M x, \quad x \wedge_m^M 0 = 0, \tag{13}$$

$$x \vee_m^M 0 = x = 0 \vee_m^M x, \quad x \vee_m^M 1 = 1, \tag{14}$$

$$(x \vee_m^M y)^- = x^- \wedge_m^M y^- \quad (De\ Morgan\ law\ 1), \tag{15}$$

$$(x \wedge_m^M y)^- = x^- \vee_m^M y^- \quad (De\ Morgan\ law\ 2),\ and\ hence \tag{16}$$

$$x \wedge_m^M y = (x^- \vee_m^M y^-)^-. \tag{17}$$

Proposition 2.6. *(See ([6], Proposition 2.1.2), in dual case) ([21], Proposition 3.3)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

$$if \quad x \odot y = 1, \quad then \quad x = y = 1; \tag{18}$$

$$if \quad x \wedge_m^M y = 1, \quad then \quad x = y = 1, \tag{19}$$

$$0 \wedge_m^M x = 0, \tag{20}$$

$$1 \vee_m^M x = 1, \tag{21}$$

$$x \wedge_m^M x = x, \quad x \vee_m^M x = x. \tag{22}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [21] two new binary relations: for all $x, y \in A^L$,
(m-dfWM) $x \ \leq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^M y = x$ and, dually,
(m-dfVM) $x \ \geq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^M y = x$,
and
(m-dfWB) $x \ \leq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^B y = x \ (\Longleftrightarrow y \wedge_m^M x = x)$ and, dually,
(m-dfVB) $x \ \geq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^B y = x \ (\Longleftrightarrow y \vee_m^M x = x)$.

Proposition 2.7. *([21], Proposition 3.6)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:
(1) $x \leq_m y \Longleftrightarrow x \leq_m^B y$ *and, dually*
(1') $x \geq_m y \Longleftrightarrow x \geq_m^B y$.
(2) If ($\wedge_m$-comm) holds (i.e. $x \wedge_m^M y = y \wedge_m^M x$), then
$x \leq_m y \ (\Longleftrightarrow x \leq_m^B y) \ \Longleftrightarrow x \leq_m^M y$.
(2') If ($\wedge_m$-comm) holds, then ($\vee_m$-comm) holds (i.e. $x \vee_m^M y = y \vee_m^M x$) and
$x \geq_m y \ (\Longleftrightarrow x \geq_m^B y) \ \Longleftrightarrow x \geq_m^M y$.

Remark 2.8. *([21], Remark 3.7)*

The equivalence $\leq_m \iff \leq_m^B$ implies that $\leq_m$ is an order relation if and only if $\leq_m^B$ is an order relation. But, it does not imply that if $\leq_m$ is a lattice order, then $\leq_m^B$ is a lattice order too with respect to $\wedge_m^B, \vee_m^B$ - see the examples in the last section.

Corollary 2.9. *(See [6], Corollary 2.1.3 and [21], Corollary 3.9)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, the binary relation $\leq_m^M$ is reflexive and antisymmetric and $0 \leq_m^M x \leq_m^M 1$, for all $x \in A^L$, where $0 \stackrel{def.}{=} 1^-$.

2.2 Ortholattices, orthosoftlattices and orthowidelattices

Definition 2.10. *An algebra $\mathcal{A} = (A, \wedge, \vee)$ or, dually, $\mathcal{A} = (A, \vee, \wedge)$, of type $(2,2)$, will be said to be a (Dedekind) lattice, if the following properties hold [1]: for all $x, y, z \in A$,*

(m-Wid)	(idempotency of $\wedge$)	$x \wedge x = x$,
(m-Wcomm)	(commutativity of $\wedge$)	$x \wedge y = y \wedge x$,
(m-Wass)	(associativity of $\wedge$)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(m-Wabs)	(absorption of wedge over vee)	$x \wedge (x \vee y) = x$, and also
(m-Vid)	(idempotency of $\vee$)	$x \vee x = x$,
(m-Vcomm)	(commutativity of $\vee$)	$x \vee y = y \vee x$,
(m-Vass)	(associativity of $\vee$)	$(x \vee y) \vee z = x \vee (y \vee z)$,
(m-Vabs)	(absorption of vee over wedge)	$x \vee (x \wedge y) = x$,

where "W" comes from "wedge" (the LaTeX command for the meet symbol) and "V" comes from "vee" (the LaTeX command for the join symbol).

Moreover, if there exist $0, 1 \in A$ such that: for all $x \in A$,
(m-WU) $1 \wedge x = x$ and, dually,
(m-VU) $0 \vee x = x$,
then $\mathcal{A}$ is said to be a bounded *(Dedekind) lattice (with last element 1 and first element 0) and is denoted by $\mathcal{A} = (A, \wedge, \vee, 0, 1)$ or, dually, by $\mathcal{A} = (A, \vee, \wedge, 0, 1)$.*

Naming convention for the dual lattices: $(A, \wedge, \vee)$ is the *left-lattice* and $(A, \vee, \wedge)$ is the *right-lattice* (names coming from the *left-continuity* of a t-norm and the *right-continuity* of a t-conorm; see more on *left- and right- algebras* in [14]).

We have analysed the ortholattices in [15], [20]. Recall the following definition:

Definition 2.11. *(See [25], [4]) (Definition 1) (The dual one is omitted)*

A left-ortholattice, or a left-OL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded (Dedekind) left-lattice and the unary operation $^-$ satisfies (DN), (DeM1) ($(x \vee y)^- = x^- \wedge y^-$), (DeM2) ($(x \wedge y)^- = x^- \vee y^-$) and the complementation laws:
(m-WRe) $x \wedge x^- = 0$ (noncontradiction principle) and, dually,
(m-VRe) $x \vee x^- = 1$ (excluded middle principle).

We have denoted by **OL** the class of all left-ortholattices.

Since, in a lattice, the absorption laws (m-Wabs) and (m-Vabs) are not independent (they imply the idempotency laws (m-Wid) and (m-Vid)), we have introduced in [22] the following two dual *independent absorption laws*:

(m-Wabs-i) $x \wedge (x \vee x \vee y) = x$ and, dually,
(m-Vabs-i) $x \vee (x \wedge x \wedge y) = x$ (dual laws of independent absorption).

We have proved that the system of eight axioms: **L8-i** = {(m-Wid), (m-Vid), (m-Wcomm), (m-Vcomm), (m-Wass), (M-Vass), (m-Wabs-i), (m-Vabs-i)} is equivalent with the "standard" system **L8** of axioms for lattices from Definition 2.10 ([22], Theorem 3.2).

We have then introduced in [22] the following two generalizations of lattices/ bounded lattices.

Definition 2.12. *(The dual ones are omitted) ([22], Definition 3.3)*
(1) A left-softlattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee)$ of type $(2,2)$ such that the axioms (m-Wid), (m-Vid), (m-Wcomm), (m-Vcomm), (m-Wass), (m-Vass) are satisfied.
(2) A bounded left-softlattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, 0, 1)$ of type $(2,2,0,0)$ such that the reduct $(A^L, \wedge, \vee)$ is a left-softlattice and the elements 0 and 1 verify the axioms: for all $x \in A^L$,
(m-WU) $1 \wedge x = x$, *(m-VU)* $0 \vee x = x$,
(m-WL) $0 \wedge x = 0$, *(m-VL)* $1 \vee x = 1$.

Definition 2.13. *(The dual ones are omitted) ([22], Definition 3.9)*
(1') A left-widelattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee)$ of type $(2,2)$ such that the axioms (m-Wcomm), (m-Vcomm), (m-Wass), (m-Vass), (m-Wabs-i), (m-Vabs-i) are satisfied.
(2') A bounded left-widelattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, 0, 1)$ of type $(2,2,0,0)$ such that the reduct $(A^L, \wedge, \vee)$ is a left-widelattice and the elements 0 and 1 verify the axioms: for all $x \in A^L$,
(m-WU) $1 \wedge x = x$, *(m-VU)* $0 \vee x = x$.

We have introduced in [22] the following two generalizations of OLs.

Definition 2.14. *(Definition 1) (The dual one is omitted) ([22], Definition 5.1)*
A left-orthosoftlattice, or a left-OSL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded left-softlattice (Definition 2.12) and the unary operation $^-$ satisfies (DN), (DeM1), (DeM2) and (m-WRe), (m-VRe).

Definition 2.15. *(Definition 1) (The dual one is omitted) ([22], Definition 5.6)*
A left-orthowidelattice, or a left-OWL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded left-widelattice (Definition 2.13) and the unary operation $^-$ satisfies (DN), (DeM1), (DeM2) and (m-WRe), (m-VRe).

We have denoted by **OSL** the class of all left-OSLs and by **OWL** the class of all left-OWLs.
Consider the following properties (the dual ones are omitted):
(m-Pimpl) $[(x \odot y^-)^- \odot x^-]^- = x$,
(G) $x \odot x = x$,
(m-Pabs-i) $x \odot (x \oplus x \oplus y) = x$.

Proposition 2.16. *([22], Proposition 3.15)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Then,

$$(m-Pimpl) \iff (G) + (m-Pabs-i).$$

We have obtained the following equivalent definitions.

Definition 2.17. *(Definition 2) (The dual ones are omitted)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. $\mathcal{A}^L$ is a:
- left-ortholattice (left-OL), if (m-Pimpl) holds ([20], Definition 4.15),
- left-orthosoftlattice (left-OSL), if (G) holds ([22], Definition 5.3),
- left-orthowidelattice (left-OWL), if (m-Pabs-i) holds ([22], Definition 5.8),
i.e. **OL** = $\mathbf{m\text{-}BE}_{(DN)}$ + *(m-Pimpl)*, **OSL** = $\mathbf{m\text{-}BE}_{(DN)}$ + *(G)*, **OWL** = $\mathbf{m\text{-}BE}_{(DN)}$ + *(m-Pabs-i)*.

Hence, we have:

$$\mathbf{OL} = \mathbf{OSL} \cap \mathbf{OWL}, \tag{23}$$

i.e. we have the representation from Figure 1, useful in the sequel.

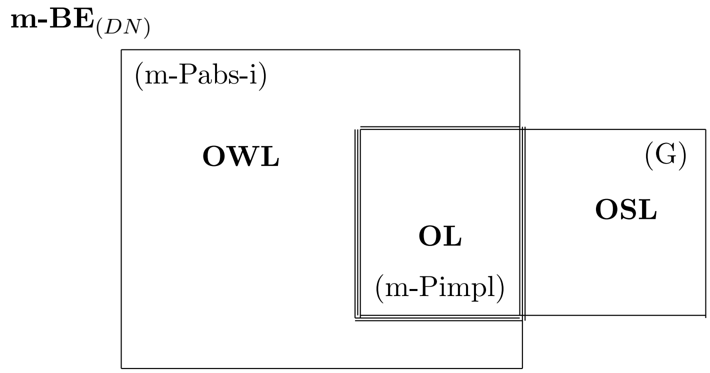

Figure 1: Resuming connection between **OSL**, **OWL** and **OL**

Theorem 2.18. *([20], Theorem 4.16) We have:* **aOL** = **OL** + *(m-An)* = **Boole**.

Finally, recall that [22]: **taOSL** = **Boole**.

2.3 Boolean algebras

Definition 2.19. *(Definition 1) (The dual one is omitted)*

A left-Boolean algebra is a bounded (Dedekind) left-lattice that is distributive and complemented, i.e. is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ *verifying: (m-Wid), (m-Wcomm), (m-Wass), (m-Wabs), (m-WU), (m-Wdis), (m-WRe) and, dually, (m-Vid), (m-Vcomm), (m-Vass), (m-Vabs), (m-VU), (m-Vdis), (m-VRe), where:*

(m-Wdis) $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$,
(m-Vdis) $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y)$.

We have denoted by **Boole** the class of all left-Boolean algebras.

Consider the following properties (the dual ones are omitted):

(m-Pdiv) $x \odot (x \odot y^-)^- = x \odot y$,
(m-Pdis) $z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$.

We have obtained the following equivalent definitions.

Definition 2.20. *(Definitions 2 and 3) (The dual ones are omitted)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra.* $\mathcal{A}^L$ *is a:*
- left-Boolean algebra, if (m-Pdiv) holds ([20], Definition 4.19) or, equivalently,
- left-Boolean algebra, if (m-Pdis) holds ([20], Definition 4.21),
i.e. **Boole** = **m-BE**$_{(DN)}$ + *(m-Pdiv)* = **m-BE**$_{(DN)}$ + *(m-Pdis)*.

2.4 QMV algebras. OM, PreMV, MMV algebras. MV algebras

Consider the following properties (the dual ones are omitted):

(Pqmv) $x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$,
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$,
(Pmv) $x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$ or, equivalently, $x \odot (x^- \vee_m^M y) = x \odot y$,
(Δ_m) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$.

Definition 2.21. *(The dual ones are omitted)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. $\mathcal{A}^L$ is a:

- *left-quantum-MV algebra (left-QMV algebra), if (Pqmv) holds ([21], Definition 3.10),*
- *left-orthomodular algebra (left-OM algebra), if (Pom) holds ([21], Definition 4.1),*
- *left-pre-MV algebra (left-PreMV algebra), if (Pmv) holds ([21], Definition 4.1),*
- *left-metha-MV algebra (left-MMV algebra), if (Δ_m) holds ([21], Definition 4.1).*

We have denoted by **QMV**, **OM**, **PreMV**, **MMV** the corresponding classes of left-algebras. Hence, we have:

QMV = $\textbf{m-BE}_{(DN)}$ + (Pqmv), **OM** = $\textbf{m-BE}_{(DN)}$ + (Pom),
PreMV = $\textbf{m-BE}_{(DN)}$ + (Pmv), **MMV** = $\textbf{m-BE}_{(DN)}$ + (Δ_m).

Theorem 2.22. *[21] Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

(1) (Pqmv) $\Longleftrightarrow$ (Pmv) + (Pom), i.e. **QMV** = **PreMV** $\cap$ **OM**,
(2) (Pmv) $\Longrightarrow$ (Δ_m), i.e. **PreMV** $\subset$ **MMV**,
(3) (Pqmv) $\Longleftrightarrow$ (Δ_m) + (Pom), i.e. **QMV** = **MMV** $\cap$ **OM**.

The connections between these algebras, and the transitive ones, were established in [21] (see Figure 2).

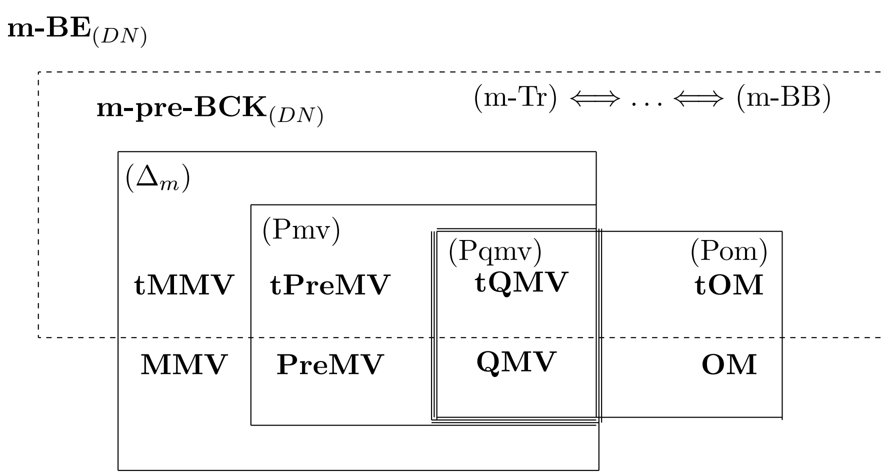

Figure 2: Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and (m-Tr)

Proposition 2.23. *([21], Proposition 3.22)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (G). Then:
(1) $\leq_m^P$ is reflexive also, hence it is an **order relation**.
(2) We have the equivalence:
$(x \odot y = x \Longleftrightarrow)\ x \leq_m^P y \Longleftrightarrow x \leq_m^M y\ (\Longleftrightarrow x \wedge_m^M y = x)$.

Theorem 2.24. *[21] We have:*
aPreMV = **aMMV** = **aQMV** = **atQMV** = **taQMV** = **MV** *and* **MV** $\subset$ **taOM**.

Recall, finally, some properties of OM algebras.

Proposition 2.25. *([18], Proposition 3.1)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. We have:

$$x \odot (y \vee_m^M x^-) = x \odot y, \tag{24}$$

$$x \ \leq_m^M \ y \Longrightarrow y^- \ \leq_m^M \ x^- \quad (order - reversibility\ of\ ^-), \tag{25}$$

$$x \ \leq_m^M \ y \Longrightarrow x \oplus z \ \leq_m^M \ y \oplus z \quad (monotonicity\ of\ \oplus), \tag{26}$$

$$x \ \leq_m^M \ y \Longrightarrow x \odot z \ \leq_m^M \ y \odot z \quad (monotonicity\ of\ \odot). \tag{27}$$

Corollary 2.26. *([18], Corollary 3.7)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. The binary relation $\leq_m^M$ is an order relation.

3 Orthomodular lattices

Recall the following definition [25].

Definition 3.1. *(Definition 1) (The dual one is omitted)*
A left-orthomodular lattice or an orthomodular left-lattice, or a left-OML for short, is a left-OL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ verifying: for all $x, y \in A^L$,
(Wom) $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OML** the class of all left-OMLs .

Following the equivalent Definition 2 of a left-OL (see Definition 2.17), we obtain immediately the equivalent definition:

Definition 3.2. *(Definition 2) (The dual one is omitted)*
A left-orthomodular lattice (left-OML) is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pom), i.e.

$$\mathbf{OML} = \mathbf{m-BE_{(DN)}} \ + \ (m - Pimpl) \ + \ (Pom) \ = \ \mathbf{OL} \ \cap \ \mathbf{OM}. \tag{28}$$

Further, we shall work with Definition 2 of left-OMLs. Hence, we have the connections from Figure 3.

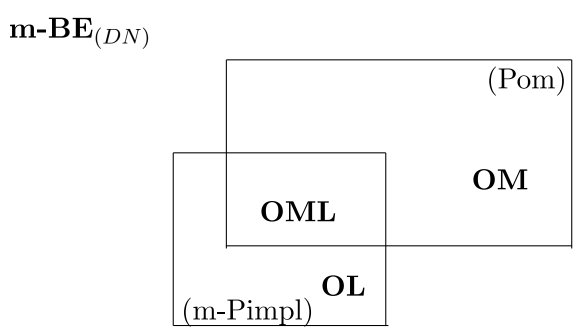

Figure 3: Resuming connections between **OL**, **OML** and **OM**

Recall ([6], Corollary 2.3.13) that:

$$\mathbf{OML} \ \subset \ \mathbf{QMV}, \tag{29}$$

the inclusion being strict, since there are examples of QMV algebras not verifying (m-Pimpl).

Proposition 3.3. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML. We have the equivalence:*

$$(x \odot y = x \overset{def.}{\Longleftrightarrow}) \ x \leq_m^P y \Longleftrightarrow x \leq_m^M y \ (\overset{def.}{\Longleftrightarrow} x \wedge_m^M y = x).$$

Proof. Suppose $x \leq_m^P y$, i.e. $x \odot y = x$. Then,
$x \wedge_m^M y \overset{(9)}{=} (x^- \odot y)^- \odot y = ((x \odot y)^- \odot y)^- \odot y$
$\overset{(DN)}{=} (((x^-)^- \odot y)^- \odot y)^- \odot y \overset{(9)}{=} (x^- \wedge_m^M y)^- \odot y$
$\overset{(16)}{=} ((x^-)^- \vee_m^M y^-)^- \odot y \overset{(m-Wcomm),(DN)}{=} y \odot (x \vee_m^M y^-)$
$\overset{(24)}{=} y \odot x = x \odot y = x$, since **OML** $\subset$ **OM**.

Conversely, suppose $x \leq_m^M y$, i.e. $x \wedge_m^M y = x$, i.e. $(x^- \odot y)^- \odot y = x$. Then,
$x \odot y = ((x^- \odot y)^- \odot y) \odot y \overset{(m-Wass)}{=} (x^- \odot y)^- \odot (y \odot y)$
$\overset{(G)}{=} (x^- \odot y)^- \odot y \overset{(9)}{=} x \wedge_m^M y = x$, since (m-Pimpl) implies (G), by Proposition 2.16. □

3.1 Connections between OML and PreMV, QMV, MMV, OM, OL

• **OML** + (Pmv) (Connections between **OML** and **PreMV**)

We establish the connections between the OMLs and the pre-MV algebras verifying (m-Pimpl).

Proposition 3.4. *(See Proposition 4.3)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,

$$(Pom)\ +\ (m-Pimpl)\ \Longrightarrow\ (Pmv).$$

Proof. Since (m-Pimpl) ($[(x \odot y^-)^- \odot x^-]^- = x$) is equivalent to $(x \odot y^-) \oplus x = x$, hence (by taking $X := x^-$) to $(x^- \odot y^-) \oplus x^- = x^-$, we obtain:
$x \odot (x^- \vee_m^M y) = x \odot ((x^- \odot y^-)^- \odot y^-)^- = (x^- \oplus ((x^- \odot y^-)^- \odot y^-)^=)^-$
$\overset{(DN)}{=} (x^- \oplus ((x^- \odot y^-)^- \odot y^-))^- \overset{(m-Pimpl),(Scomm)}{=} ((x^- \oplus (x^- \odot y^-)) \oplus ((x^- \odot y^-)^- \odot y^-))^-$
$\overset{(Sass),(Pcomm)}{=} (x^- \oplus ((y^- \odot x^-) \oplus ((y^- \odot x^-)^- \odot y^-)))^- \overset{(Pom)}{=} (x^- \oplus y^-)^- = x \odot y.$ □

Note that Proposition 3.4 says: **OML** $\subset$ **PreMV**, which follows by (29).

The following converse of Proposition 3.4 also holds:

Proposition 3.5. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(Pmv)\ +\ (m-Pimpl)\ \Longrightarrow\ (Pom).$$

Proof. (Following a proof by *Prover* 9 of length 25, lasting 0.11 seconds)

We know that (m-Pimpl) implies (G), and (G) implies:
(a) $x \odot (y \odot x) = y \odot x$;
indeed, $x \odot (y \odot x) \overset{(Pcomm),(Pass)}{=} y \odot (x \odot x) \overset{(G)}{=} y \odot x$.
Then, (m-Pimpl) ($((x \odot y^-)^- \odot x^-)^- = x$) implies, taking $Y := y^-$ and using (DN) and (Pcomm):
(b) $(x^- \odot (y \odot x)^-)^- = x$ and
(b') $x^- \odot (x \odot y)^- = x^-$.

On the other hand, (Pmv) ($x \odot (y^- \odot (x^- \odot y^-)^-)^- = x \odot y$) implies, by (Pcomm):
(c) $x \odot (y^- \odot (y^- \odot x^-)^-)^- = x \odot y$.

Now, by (a), (b) and (c), we obtain:
(d) $x \odot (x \odot (y \odot x)^-)^- = y \odot x$;
indeed, in (c), take $Y := y \odot x$ and $X := x$, to obtain:
(x) $x \odot ((y \odot x)^- \odot ((y \odot x)^- \odot x^-)^-)^- = x \odot (y \odot x) \overset{(a)}{=} y \odot x$;
since in (x), $((y \odot x)^- \odot x^-)^- \overset{(Pcomm)}{=} (x^- \odot (y \odot x)^-)^- \overset{(b)}{=} x$,

it follows that (x) becomes:
(x') $x \odot ((y \odot x)^- \odot x)^- = y \odot x$, i.e. (d) holds, by (Pcomm).
Now, by (b'), (d), we obtain:
(e) $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$;
indeed, in (d), take $X := (x \odot y)^-$ and $Y := x^-$ to obtain:
(y) $(x \odot y)^- \odot ((x \odot y)^- \odot (x^- \odot (x \odot y)^-)^-)^- = x^- \odot (x \odot y)^-$;
but, in (y), $x^- \odot (x \odot y)^- \overset{(b')}{=} x^-$, hence (y) becomes:
(y') $(x \odot y)^- \odot ((x \odot y)^- \odot x^{=})^- = x^-$, which becomes, by (DN):
(y'') $(x \odot y)^- \odot ((x \odot y)^- \odot x)^- = x^-$, which becomes, by (DN) and (Pcomm):
$((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)^- = x$, that is (Pom). □

Note that Proposition 3.5 says: **PreMV** $\cap$ **OL** $\subset$ **OM**.
By Propositions 3.4 and 3.5, we obtain:

Theorem 3.6. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(m - Pimpl) \implies ((Pom) \iff (Pmv))$$

or

$$(m - Pimpl) + (Pom) \iff (Pmv) + (m - Pimpl),$$

i.e. OMLs coincide with pre-MV algebras verifying (m-Pimpl).

Hence, Theorem 3.6 says:

$$\mathbf{OML} = \mathbf{PreMV} + (m - Pimpl) = \mathbf{PreMV} \cap \mathbf{OL}. \tag{30}$$

- **OML** + (Pqmv) (Connections between **OML** and **QMV**)

We establish now the connection between the OMLs and the QMV algebras verifying (m-Pimpl).

Proposition 3.7. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl).*
(i.e. in an involutive m-BE algebra, (Pom) + (m-Pimpl) $\implies$ (Pqmv).)

Proof. Since $\mathcal{A}^L$ is a left-OML, it is an involutive m-BE algebra verifying (m-Pimpl) and (Pom) (Definition 2). By Theorem 3.4, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl). □

Note that Proposition 3.7 says: **OML** $\subset$ **QMV**, which is (29). Note also that Proposition 3.4 follows from Proposition 3.7, since (Pqmv) $\implies$ (Pmv).
The following converse of Proposition 3.7 also holds.

Proposition 3.8. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (m-Pimpl). Then, $\mathcal{A}^L$ is a left-OML.*
(i.e. in an involutive m-BE algebra, (Pqmv) + (m-Pimpl) $\implies$ (Pom).)

Proof. Since $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl), it is an involutive m-BE algebra verifying (Pqmv) (hence (Pom), (Pmv)) and (m-Pimpl). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (m-Pimpl) and (Pom), i.e. it is a left-OML. □

Note that Proposition 3.8 says: **QMV** $\cap$ **OL** $\subset$ **OM**. Note also that Proposition 3.8 follows from Proposition 3.5.
By Propositions 3.7 and 3.8, we obtain:

Theorem 3.9. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(m - Pimpl) \implies ((Pom) \Leftrightarrow (Pqmv))$$

or

$$(m - Pimpl) + (Pom) \iff (Pqmv) + (m - Pimpl)$$

i.e. orthomodular lattices coincide with QMV algebras verifying (m-Pimpl).

Hence, Theorem 3.9 says:

$$\mathbf{OML} = \mathbf{QMV} + (m - Pimpl) = \mathbf{QMV} \cap \mathbf{OL}. \tag{31}$$

By the previous results (28), (29), (30) and (31), we obtain the connections from Figure 4.

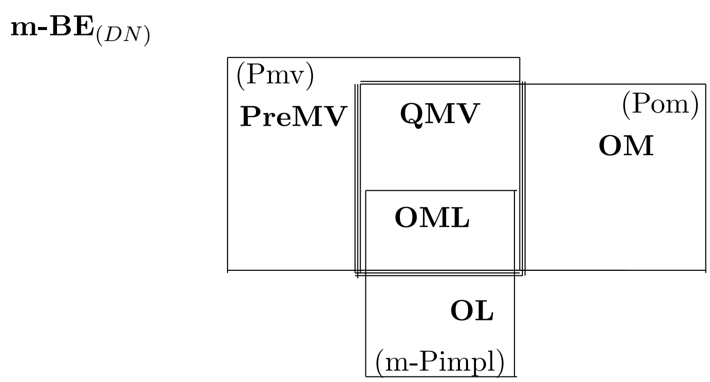

Figure 4: Resuming connections between **QMV**, **PreMV**, **OM**, **OL** and **OML**

- **OML** + (Δ_m) (Connections between **OML** and **MMV**)

Proposition 3.10. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(Pom) + (m - Pimpl) \implies (\Delta_m).$$

Proof. By Proposition 3.4, (Pom) + (m-Pimpl) imply (Pmv) and (Pmv) implies (Δ_m). □

Note that Proposition 3.10 says: **OML** $\subset$ **MMV**. which follows also by (29). Note also that Proposition 3.7 follows also from Proposition 3.10, since (Pom) + (Δ_m) imply (Pqmv) and that Proposition 3.10 follows from Proposition 3.7, since (Pqmv) implies (Δ_m).

Remark 3.11. *The following converse of Proposition 3.10 ((Δ_m) + (m-Pimpl) $\implies$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying (Δ_m) and (m-Pimpl) and not verifying (Pom).*

By the previous Remark, from the connections from Figure 4, we obtain the connections from Figure 5.

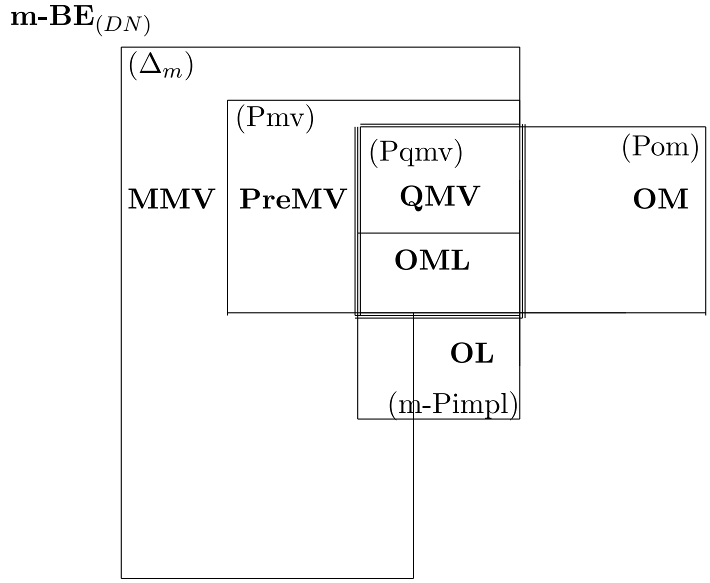

Figure 5: Resuming connections between **QMV**, **PreMV**, **MMV**, **OM**, **OL** and **OML**

Remark 3.12. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OL (Definition 2). Note that:*
- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is only reflexive ((m-Re) holds, by definition of m-BE algebra);
- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is only reflexive and antisymmetric;
- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is a **lattice order**, *with respect to $\wedge = \odot$, $\vee = \oplus$, denoted $\leq_m^O$, by Proposition 2.2.*

Remark 3.13. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML (Definition 2). Note that:*
- The initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is only reflexive;
- The binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an **order**, *by Corollary 2.26, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative;*
- The binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is a **lattice order**, *with respect to $\wedge = \odot$, $\vee = \oplus$, denoted $\leq_m^O$, by Proposition 2.2;*
- We have the equivalence $\leq_m^O \Longleftrightarrow \leq_m^M$, by Proposition 2.23; consequently, the tables of $\wedge$ and $\wedge_m^M$ are different, but they coincide for the comparable elements of A^L (with respect to $\leq_m^O$ and $\leq_m^M$, respectively).

3.2 The transitive and/or antisymmetric case

3.2.1 Antisymmetric orthomodular lattices: aOML = Boole

Denote by **aOML** the class of all antisymmetric left-OMLs. We prove that **aOML** does not exist as a proper class:

Theorem 3.14. *We have:*

$$\mathbf{aOML} = \mathbf{Boole}.$$

Proof. **aOML** = **m-BE**$_{(DN)}$ + (m-Pimpl) + (Pom) + (m-An) = **OL** + (Pom) + (m-An) = **Boole** + (Pom) = **Boole**, by Theorem 2.18. □

Remark: We have:

$$\mathbf{OML} \subset \mathbf{QMV} \quad \textit{and} \quad \mathbf{aOML} = \mathbf{Boole} \subset \mathbf{aQMV} = \mathbf{MV}.$$

3.2.2 Transitive orthomodular lattices: tOML = Boole

Denote by **tOML** the class of all transitive left-OMLs. We shall prove that **tOML** does not exist as a proper class (**tOML** = **Boole**, by Theorem 3.16).

Theorem 3.15. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom)\ +\ (m-Pimpl)\ +\ (m-BB) \implies (m-An).$$

Proof. (By *Prover*9**, in** 0.03 **seconds, the length of the proof being** 32**)**

Suppose: (i) $c_1 \odot c_2^- = 0$ and (j) $c_2 \odot c_1^- = 0$; we have to prove that $c_1 = c_2$.

First, (Pom): $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ means
$[(x \odot y)^- \odot ((x \odot y)^- \odot x)^-]^- = x$, hence by (Pcomm), (DN):

$$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-. \tag{32}$$

Second, (m-BB): $[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$, means, by (Pass):

$$(x \odot y)^- \odot [z \odot (y \odot (z \odot x^-)^-)] = 0. \tag{33}$$

Take $x := c_2^-$, $y := c_1$, $z := x$ in (33) to obtain:
$(c_2^- \odot c_1)^- \odot [x \odot (c_1 \odot (x \odot c_2)^-)] = 0$, hence by (i), (Neg0-1), (PU):
$x \odot (c_1 \odot (x \odot c_2)^-) = 0$, hence, by (Pass), (Pcomm):

$$c_1 \odot (x \odot (c_2 \odot x)^-) = 0. \tag{34}$$

Since (p-Pimpl) implies (G), then (G) $(x \odot x = x)$ implies $x \odot y = (x \odot x) \odot y \overset{(Pass)}{=} x \odot (x \odot y)$, hence we have:

$$x \odot (x \odot y) = x \odot y. \tag{35}$$

Take now $x := c_1$ in (34) to obtain: $c_1 \odot (c_1 \odot (c_2 \odot c_1)^-) = 0$, hence by (35) and (Pcomm):

$$c_1 \odot (c_1 \odot c_2)^- = 0. \tag{36}$$

Take now $x := c_1$, $y := c_2$ in (32) to obtain:
$(c_1 \odot c_2)^- \odot (c_1 \odot (c_1 \odot c_2)^-)^- = c_1^-$; then, by (36), (Neg0-1), (PU), we obtain:

$$(c_1 \odot c_2)^- = c_1^-, \quad \textit{hence} \tag{37}$$

$$c_1 \odot c_2 = c_1. \tag{38}$$

Now, from (m-Pimpl): $[(x \odot y^-)^- \odot x^-]^- = x$, we obtain by (Pcomm) and for $y := y^-$:

$$[x^- \odot (y \odot x)^-]^- = x. \tag{39}$$

Take now $x := c_2$, $y := c_1$ in (39) to obtain: $[c_2^- \odot (c_1 \odot c_2)^-]^- = c_2$, hence, by (37), $[c_2^- \odot c_1^-]^- = c_2$, hence, by (Pcomm):

$$(c_1^- \odot c_2^-)^- = c_2, \quad \textit{hence} \tag{40}$$

$$c_1^- \odot c_2^- = c_2^-. \tag{41}$$

Finally, take $x := c_1^-$, $y := c_2^-$ in (32) to obtain: $(c_1^- \odot c_2^-)^- \odot (c_1^- \odot (c_1^- \odot c_2^-)^-)^- = c_1$; hence, by (40), we obtain:
$c_2 \odot (c_1^- \odot c_2)^- = c_1$; hence, by (j), (Pcomm), (Neg0-1) and (PU), we obtain: $c_2 = c_1$. □

Note that Theorem 3.15 says: **tOML** $\subset$ **m-aBE**$_{(DN)}$. Hence, **tOML** $\subset$ **taOML**. But **taOML** = **aOML** + (m-Tr) = **Boole** + (m-Tr) = **Boole**, by Theorem 3.14. It follows that **tOML** = **Boole**. Thus, we have proved Theorem 3.16:

Theorem 3.16. *We have:*

$$\mathbf{tOML} \;=\; \mathbf{Boole}. \tag{42}$$

3.2.3 The transitive and antisymmetric case

If we make the following table:

No.	(m-Tr)	(m-Pimpl)	(Pqmv)	Type of m-BE$_{(DN)}$ algebra
(1)	0	0	0	proper m-BE$_{(DN)}$
(2)	0	0	1	proper QMV
(3)	0	1	0	proper OL
(4)	0	1	1	proper OM
(5)	1	0	0	proper m-pre-BCK$_{(DN)}$
(6)	1	0	1	proper tQMV
(7)	1	1	0	proper tOL
(8)	1	1	1	tOML = aOML = Boole

then, we obtain the resuming connections from Figures 6 and 7.

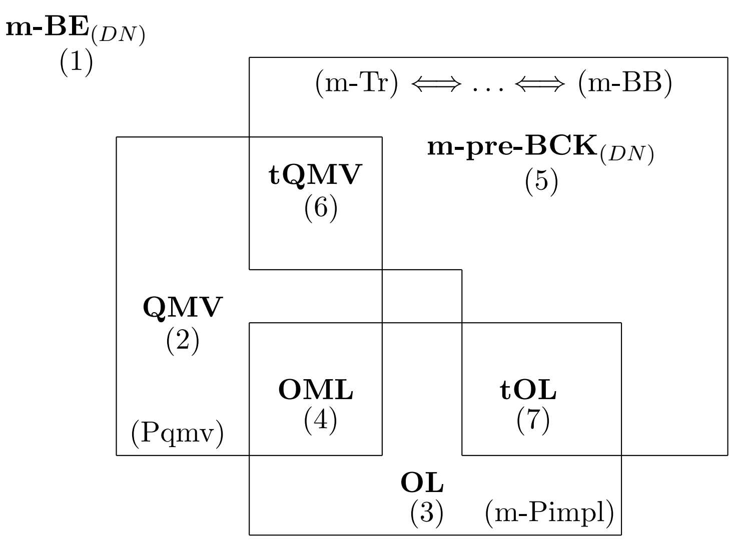

Figure 6: Resuming connections in **m-BE**$_{(DN)}$

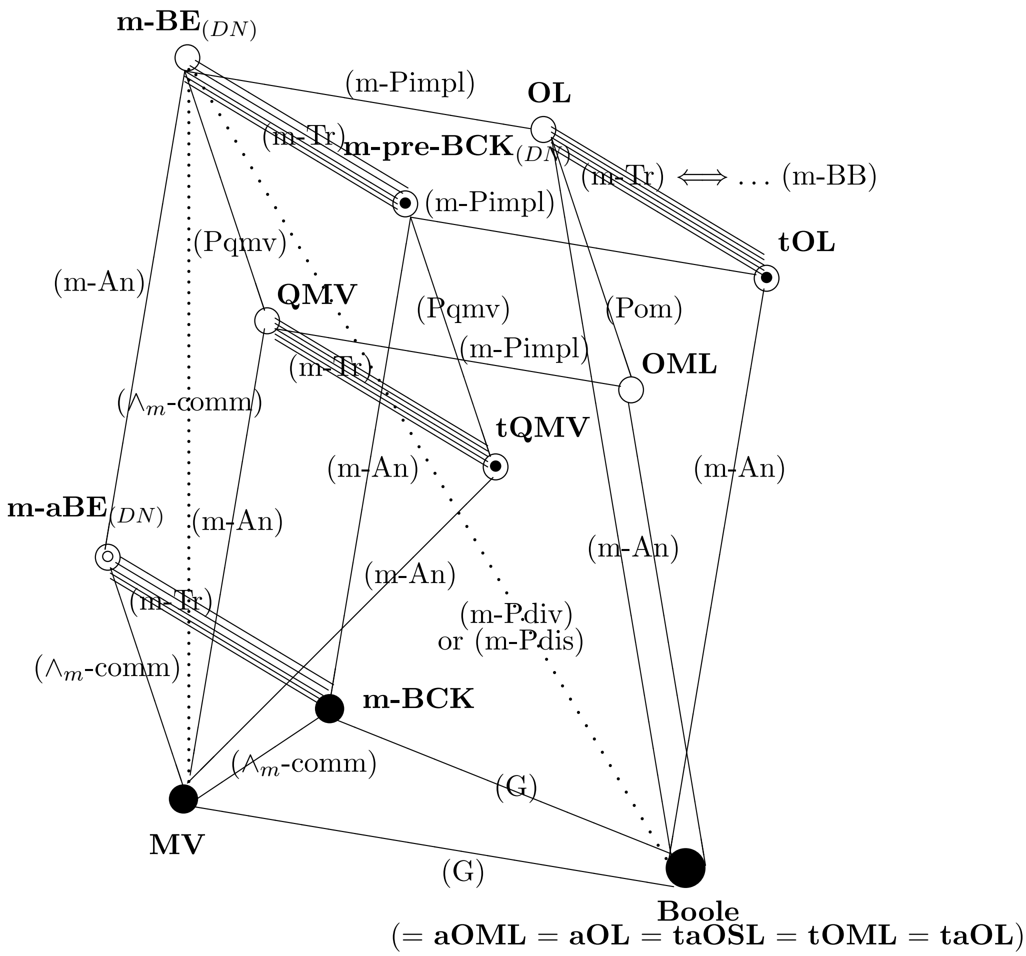

Figure 7: Resuming connections in this section

3.2.4 The transitive case: tOL $\subset$ tMMV

Theorem 3.17. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \ + \ (m - BB) \implies (\Delta_m).$$

Proof. Since (m-Pimpl) implies (m-Pabs-i) and since, by ([22], Theorem 5.13), (m-Pabs-i) + (m-BB) imply (Δ_m), it follows that (m-Pimpl) + (m-BB) imply (Δ_m). $\square$

Note that Theorem 3.17 says: **tOL** $\subset$ **MMV**, hence **tOL** $\subset$ **tMMV**, since (m-BB) $\Leftrightarrow$ (m-Tr). Now, by Theorems 3.16 and 3.17, from the connections from Figure 5, we obtain the connections from Figure 8.

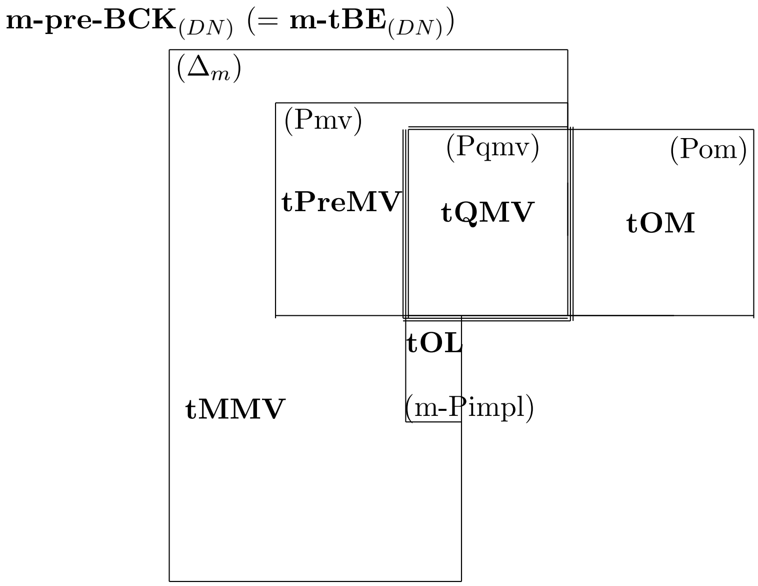

Figure 8: Resuming connections between **tQMV**, **tPreMV**, **tMMV**, **tOM** and **tOL**

3.3 Modular algebras: MOD $\subset$ OML

Recall the following definitions [25]:

(i) A latice $(L, \wedge, \vee)$ is *modular*, if for all $x, y, z \in L$,
(Wmod) $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ and, dually,

(i') the dual latice $(L, \vee, \wedge)$ is *modular*, if for all $x, y, z \in L$,
(Vmod) $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$.

Definition 3.18. *(Definition 1) (The dual case is omitted) [25]*

A modular left-ortholattice is a left-OL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ whose lattice $(A^L, \wedge, \vee)$ is modular.

We shall denote by **MODOL** the class of all modular left-ortholattices.
Recall also [25] that any modular ortholattice is an orthomodular lattice, i.e.

$$\mathbf{MODOL} \subset \mathbf{OML}. \tag{43}$$

Following the equivalent definition of OLs, we obtain the following equivalent definition.

Definition 3.19. *(Definition 2) (The dual one is omitted)*

A modular left-ortholattice is an involutive left-m-BE algebra $(A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pmod), where: for all $x, y, z \in A^L$,
(Pmod) $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, i.e.
MODOL $=$ **m-BE**$_{(DN)}$ + *(m-Pimpl)* + *(Pmod)* $=$ **OL** + *(Pmod)*.

Then, we introduce the following notion:

Definition 3.20.

(i) A left-modular algebra or a modular left-algebra, or a left-MOD algebra for short, is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ verifying: for all $x, y, z \in A^L$,
(Pmod) $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$.

(i') Dually, a right-modular algebra or a modular right-algebra, or a right-MOD algebra for short, is an involutive right-m-BE algebra $\mathcal{A}^R = (A^L, \oplus, ^- = ^{-^R}, 0)$ verifying: for all $x, y, z \in A^R$,
(Smod) $x \oplus (y \odot (x \oplus z)) = (x \oplus y) \odot (x \oplus z)$.

We shall denote by **MOD** the class of all left-MOD algebras and by $\mathbf{MOD}^R$ the class of all right-MOD algebras. Hence, $\mathbf{MOD} = \mathbf{m\text{-}BE}_{(DN)}$ + (Pmod).

Then,

$$\mathbf{MODOL} = \mathbf{OL} + (Pmod) = \mathbf{OL} \cap \mathbf{MOD}. \tag{44}$$

Proposition 3.21. *(The dual one is omitted)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pmod) \implies (Pom).$$

Proof. (**Following a proof by Prover9 of length** 14**, lasting** 0.05 **seconds**)
(Pmod), i.e. $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, is equivalent with
(a) $x \odot (y^- \odot (x \odot z)^-)^- = ((x \odot y)^- \odot (x \odot z)^-)^-$, i.e. with:
(a') $((x \odot y)^- \odot (x \odot z)^-)^- = x \odot (y^- \odot (x \odot z)^-)^-$.
Then,
$((x \odot y)^- \odot (x \odot z)^-)^- \overset{(Pcomm)}{=} ((x \odot z)^- \odot (x \odot y)^-)^- \overset{(a')}{=} x \odot (z^- \odot (x \odot y)^-)^-$,
hence we obtain:
(b) $x \odot (y^- \odot (x \odot z)^-)^- = x \odot (z^- \odot (x \odot y)^-)^-$.
Take now $Z := (x \odot y)^-$ in (b) to obtain:
$x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- = x \odot ((x \odot y)^= \odot (x \odot y)^-)^-$
$\overset{(DN)}{=} x \odot ((x \odot y) \odot (x \odot y)^-)^- \overset{(m-Re)}{=} x \odot 0^- \overset{(Neg0-1)}{=} x \odot 1 \overset{(PU)}{=} x$;
hence, we have:
(c) $x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- = x$.
Now, (Pom), i.e. $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$, is equivalent with:
(d) $((x \odot y)^- \odot ((x \odot y)^- \odot x)^-)^- = x$, which by (Pcomm) means:
(d') $((x \odot y)^- \odot (x \odot (x \odot y)^-)-)^- = x$;
hence, we must prove that (d') holds.

Indeed, $((x \odot y)^- \odot (x \odot (x \odot y)^-)-)^- \overset{(a')}{=} x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- \overset{(c)}{=} x$, hence (d') holds, i.e. (Pom) holds. □

Note that Proposition 3.21 says: $\mathbf{MOD} \subset \mathbf{OM}$.

Proposition 3.22. *(The dual one is omitted)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pmod) \implies (m-Pimpl).$$

Proof. (**Following a proof by Prover9 of length** 16**, lasting** 0.00 **seconds**)
(Pmod), i.e. $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, is equivalent with
(a) $x \odot (y^- \odot (x \odot z)^-)^- = ((x \odot y)^- \odot (x \odot z)^-)^-$, i.e. with:
(a') $((x \odot y)^- \odot (x \odot z)^-)^- = x \odot (y^- \odot (x \odot z)^-)^-$.
Now, take in (a') $Y := 1$ and $Z := y$ to obtain, by (PU), (Neg1-0), (Pcomm), (m-L):
$(x^- \odot (x \odot y)^-)^- = ((x \odot 1)^- \odot (x \odot z)^-)^- \overset{(a')}{=} x \odot (1^- \odot (x \odot z)^-)^- = x \odot (0 \odot (x \odot z)^-)^- = x \odot 0^- = x \odot 1 = x$,
hence:
(b) $(x^- \odot (x \odot y)^-)^- = x$.
Note that (m-Pimpl), i.e. $((x \odot y^-)^- \odot x^-)^- = x$, follows from (b), by (Pcomm). □

Note that Proposition 3.22 says: $\mathbf{MOD} \subset \mathbf{OL}$, hence, $\mathbf{MOD} = \mathbf{OL} \cap \mathbf{MOD} \overset{(44)}{=} \mathbf{MODOL}$. Thus, we have:

$$\mathbf{MOD} = \mathbf{MODOL}. \tag{45}$$

By Propositions 3.21 and 3.22, we obtain obviously:

Theorem 3.23. *(The dual one is omitted)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,

$$(Pmod) \implies (Pom) + (m - Pimpl).$$

By above Theorem 3.23, which says: **MOD** $\subset$ **OM** $\cap$ **OL** $=$ **OML**, by (28), we reobtain immediately the recalled known result from (43): **MODOL** (= **MOD**) $\subset$ **OML** ($\subset$ **OL**).

Recall [25] that the inclusion is strict.

Since **OML** $\subset$ **QMV**, by (28), we obtain:

$$\mathbf{MOD}\ (= \mathbf{MODOL})\ \subset\ \mathbf{OML}\ \subset\ \mathbf{QMV}. \tag{46}$$

Hence, we have:

$$\mathbf{aMOD}\ =\ \mathbf{aOML}\ =\ \mathbf{Boole}\ \subset\ \mathbf{aQMV}\ =\ \mathbf{MV}\quad \text{and} \tag{47}$$

$$\mathbf{tMOD}\ =\ \mathbf{tOML}\ =\ \mathbf{Boole}\ \subset\ \mathbf{tQMV}. \tag{48}$$

Remark 3.24. *Recall that any OL that is distributive is a Boolean algebra, by definitions. Consequently, any OML that is distributive is a Boolean algebra and any modular algebra that is distributive is a Boolean algebra.*

4 Orthomodular softlattices, widelattices

Starting from the two generalizations of ortholattices (**OL**): the orthosoftlattices (**OSL**) and the orthowidelattices (**OWL**) (Definition 2.17 and Figure 1), we introduce, in separate subsections, two corresponding generalizations of orthomodular lattices (OMLs): the orthomodular softlattices and the orthomodular widelattices.

4.1 Orthomodular softlattices: OMSL

We introduce the following notion.

Definition 4.1. *(Definition 1) (The dual one is omitted)*
A left-orthomodular softlattice or an orthomodular left-softlattice, or a left-OMSL for short, is a left-OSL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ verifying: for all $x, y \in A^L$,
(Wom) $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OMSL** the class of all left-OMSLs. Following the equivalent Definition 2 of a left-OSL (see Definition 2.17), we obtain immediately an equivalent definition:

Definition 4.2. *(Definition 2) (The dual one is omitted)*
A left-OMSL is a left-OSL verifying (Pom), i.e. is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (G) and (Pom), i.e.

$$\mathbf{OMSL}\ =\ \mathbf{m-BE_{(DN)}}\ +\ (G)\ +\ (Pom)\ =\ \mathbf{OSL}\ \cap\ \mathbf{OM}. \tag{49}$$

Further, we shall work with Definition 2 of left-OMSLs. Hence, we have the connections from Figure 9.

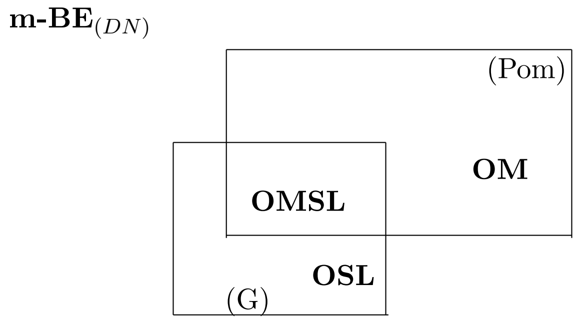

Figure 9: Resuming connections between **OSL**, **OMSL** and **OM**

Denote by **tOMSL** the class of all transitive left-OMSLs. We shall prove that **OMSL** and **tOMSL** do not exist (as proper classes) (**OMSL** = **OML**, by (53), hence **tOMSL** = **Boole**, by Theorem 3.16).

4.1.1 Connections between OMSL and PreMV, QMV, MMV, OM, OSL

• **OMSL** + (Pmv) (Connections between **OMSL** and **PreMV**)

We establish the connections between the OMSLs and the pre-MV algebras verifying (G).

Proposition 4.3. *(See Proposition 3.4)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra, Then,

$$(Pom)\ +\ (G)\ \Longrightarrow\ (Pmv).$$

Proof. (following a proof by *Prover*9**, of length** 24**, lasting** 0.36 **seconds)**
First, from (Pom) $(((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)^- = x)$, by (DN), we obtain:
(a) $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$.
Then, (a) implies:
(b) $x^- \odot (x \odot y)^- = x^-$;
indeed, $x^- \odot (x \odot y)^- \overset{(Pcomm)}{=} (x \odot y)^- \odot x^-$
$\overset{(a)}{=} (x \odot y)^- \odot ((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)$
$\overset{(Pass)}{=} ((x \odot y)^- \odot (x \odot y)^-) \odot (x \odot (x \odot y)^-)^-$
$\overset{(G)}{=} (x \odot y)^- \odot (x \odot (x \odot y)^-)^- \overset{(a)}{=} x^-$.
Then, (b) implies (c), by (DN):
(c) $x \odot (x^- \odot y)^- = x$.
On the other hand, (a) implies (d), by interchanging x with y:
(d) $(y \odot x)^- \odot (y \odot (y \odot x)^-)^- = y^-$,
and (d) implies (e), by taking $X := x^-$ and by (Pcomm):
(e) $(x^- \odot y)^- \odot (y \odot (x^- \odot y)^-)^- = y^-$.
Finally, (c) and (e) imply:
(f) $x \odot y^- = x \odot (y \odot (x^- \odot y)^-)^-$;
indeed, $x \odot y^- \overset{(e)}{=} x \odot ((x^- \odot y)^- \odot (y \odot (x^- \odot y)^-)^-)$
$\overset{(Pass)}{=} (x \odot (x^- \odot y)^-) \odot (y \odot (x^- \odot y)^-)^-$
$\overset{(c)}{=} x \odot (y \odot (x^- \odot y)^-)^-$; thus, (f) holds.

By taking $Y := y^-$ in (f), we obtain, by (DN):
$x \odot y = x \odot (y^- \odot (x^- \odot y^-)^-)^-$, that is (Pmv). □

Note that Proposition 3.4 follows from Proposition 4.3, since (m-Pimpl) implies (G). Note also that Proposition 4.3 says: **OMSL** $\subset$ **PreMV**.

The following converse of Proposition 4.3 also holds:

Proposition 4.4. *(See Proposition 3.5)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,

$$(Pmv) \ + \ (G) \implies (Pom).$$

Proof. (following a proof by *Prover*9**, of length** 27**, lasting** 0.12 **seconds)**

From (Pmv) ($x \odot (y^- \odot (x^- \odot y^-)^-)^- = x \odot y$), by taking $Y := y^-$ and by (DN), we obtain:
(a) $x \odot (y \odot (x^- \odot y)^-)^- = x \odot y^-$.

On the other hand, from (G) ($x \odot x = x$), we obtain:
(b) $x \odot (x \odot y) = x \odot y$;
indeed, $x \odot (x \odot y) \overset{(Pass)}{=} (x \odot x) \odot y \overset{(G)}{=} x \odot y$; hence, by (Pcomm), we obtain:
(b') $x \odot (y \odot x) = y \odot x$.

Now, from (b) and (a) we obtain:
(c) $x \odot (x^- \odot y)^- = x$;
indeed, in (a), take $X := x$ and $Y := x^- \odot y$ to obtain:
(x) $x \odot ((x^- \odot y) \odot (x^- \odot (x^- \odot y))^-)^- = x \odot (x^- \odot y)^-$;
but, the part from (x): $x^- \odot (x^- \odot y) \overset{(b)}{=} x^- \odot y$, hence (x) becomes:
(x') $x \odot ((x^- \odot y) \odot (x^- \odot y)^-)^- = x \odot (x^- \odot y)^-$,
which by (m-Re) and (Neg0-1) becomes:
(x'') $x \odot 1 = x \odot (x^- \odot y)^-$,
which, by (PU), becomes (c).

Now, from (c), by (Pcomm), we obtain:
(c') $x \odot (y \odot x^-)^- = x$ and
from (c), by taking $X := x^-$ we obtain:
(c'') $x^- \odot (x \odot y)^- = x^-$.

Now, from (c') and (a), we obtain:
(d) $x \odot (x \odot (y \odot x)^-)^- = y \odot x$;
indeed, in (a), take $X := x$ and $Y := (y \odot x^=)^-$ to obtain:
(y) $x \odot ((y \odot x^=)^- \odot (x^- \odot (y \odot x^=)^-)^-)^- = x \odot (y \odot x^=)^=$;
but, the part from (y) $x^- \odot (y \odot x^=)^- \overset{(c')}{=} x^-$, hence (y) becomes, by (DN):
(y') $x \odot ((y \odot x)^- \odot x^=)^- = x \odot (y \odot x)$;
but (y'), by (DN) and (b') becomes:
(y'') $x \odot ((y \odot x)^- \odot x)^- = y \odot x$;
and (y''), by (Pcomm), becomes (d).

Now, from (c'') and (d), we obtain:
(e) $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$;
indeed, in (d), take $X := (x \odot y)^-$ and $Y := x^-$ to obtain:
(u) $(x \odot y)^- \odot ((x \odot y)^- \odot (x^- \odot (x \odot y)^-)^-)^- = x^- \odot (x \odot y)^-$;
but, the parts from (u) $x^- \odot (x \odot y)^- \overset{(c'')}{=} x^-$, hence (u) becomes:
(u') $(x \odot y)^- \odot ((x \odot y)^- \odot x^=)^- = x^-$,
which by (DN) and (Pcomm) becomes:
$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$, that is (e).

Finally, from (e), by (DN), we obtain:
$((x \odot y)^{-} \odot (x \odot (x \odot y)^{-})^{-})^{-} = x$, that is (Pom). □

Note that Proposition 3.5 follows from Proposition 4.4, since (m-Pimpl) $\Longrightarrow$ (G). Note also that Proposition 4.4 says: **PreMV** $\cap$ **OSL** $\subset$ **OM**.

By Propositions 4.3 and 4.4, we obtain:

Theorem 4.5. *(See Theorem 3.6)*
Let $\mathcal{A}^L = (A^L, \odot, ^{-}, 1)$ be an involutive m-BE algebra. Then,

$$(G) \implies ((Pom) \Leftrightarrow (Pmv))$$

or

$$(G) + (Pom) \iff (Pmv) + (G),$$

i.e. OMSLs coincide with pre-MV algebras verifying (G).

Hence, Theorem 4.5 says:

$$\mathbf{OMSL} = \mathbf{PreMV} + (G) = \mathbf{PreMV} \cap \mathbf{OSL}. \tag{50}$$

Note that Theorem 3.6 follows from Theorem 4.5, since (m-Pimpl) implies (G).

- **OMSL** + (Pqmv) (Connections between **OMSL** and **QMV**)

We establish now the connection between the OMSLs and the QMV algebras verifying (G).

Proposition 4.6. *(See Proposition 3.7)*
Let $\mathcal{A}^L = (A^L, \odot, ^{-}, 1)$ be a left-OMSL. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (G).
(i.e. in an involutive m-BE algebra, (Pom) + (G) $\Longrightarrow$ (Pqmv).)

Proof. Since $\mathcal{A}^L$ is a left-OMSL, it is an involutive m-BE algebra verifying (G) and (Pom) (Definition 2). By Proposition 4.3, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (G). □

Note that Proposition 4.6 says:

$$\mathbf{OMSL} \subset \mathbf{QMV}, \tag{51}$$

the inclusion being strict, since there are examples of QMV algebras not verifying (G). Note also that Proposition 3.7 follows from Proposition 4.6 and also that Proposition 4.3 follows from Proposition 4.6, since (Pqmv) implies (Pmv).

The following converse of Proposition 4.6 holds.

Proposition 4.7. *(See Proposition 3.8)*
Let $\mathcal{A}^L = (A^L, \odot, ^{-}, 1)$ be a left-QMV algebra verifying (G). Then, $\mathcal{A}^L$ is a left-OMSL.
(i.e. in an involutive m-BE algebra, (Pqmv) + (G) $\Longrightarrow$ (Pom).)

Proof. Since $\mathcal{A}^L$ is a left-QMV algebra verifying (G), it is an involutive m-BE algebra verifying (Pmv), (Pom) and (G) (Definition 2). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (G) and (Pom), i.e. it is a left-OMSL. □

Note that Proposition 4.7 says: **QMV** $\cap$ **OSL** $\subset$ **OM**. Note also that Proposition 3.8 follows from Proposition 4.7, since (m-Pimpl) implies (G), and also that Proposition 4.7 follows from Proposition 4.4, since (Pqmv) implies (Pmv).

By Propositions 4.6 and 4.7, we obtain:

Theorem 4.8. *(See Theorem 3.9)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,

$$(G) \implies ((Pom) \Leftrightarrow (Pqmv))$$

or

$$(G) + (Pom) \Longleftrightarrow (Pqmv) + (G)$$

i.e. orthomodular softlattices coincide with QMV algebras verifying (G).

Hence, Theorem 4.8 says:

$$\mathbf{OMSL} = \mathbf{QMV} + (G) = \mathbf{QMV} \cap \mathbf{OSL}. \tag{52}$$

Note that Theorem 3.9 follows from Theorem 4.8, since (m-Pimpl) implies (G).
By the previous results (49), (50), (51) and (52), we obtain the connections from Figure 10.

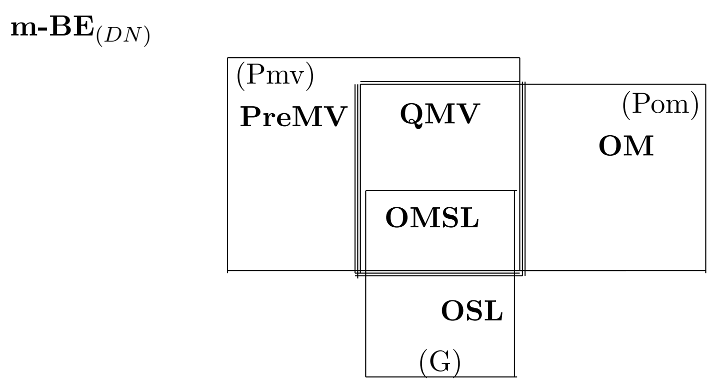

Figure 10: Resuming connections between **QMV**, **PreMV**, **OM**, **OSL** and **OMSL**

- **OMSL** + (Δ_m) (Connections between **OMSL** and **MMV**)

Proposition 4.9. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(Pom) + (G) \implies (\Delta_m).$$

Proof. By Proposition 4.3, (Pom) + (G) implies (Pmv) and (Pmv) implies (Δ_m). $\square$
Note that Proposition 4.9 says: **OMSL** $\subset$ **MMV**. Note also that Proposition 3.10 follows from Proposition 4.9, since (m-Pimpl) implies (G), that Proposition 4.6 follows from Proposition 4.9, since (Pom) + (Δ_m) imply (Pqmv), and that Proposition 4.9 follows also from Proposition 4.6, since (Pqmv) implies (Δ_m).

Remark 4.10. *The following converse of Proposition 4.9 ($(\Delta_m) + (G) \Longrightarrow (Pom)$) does not hold: there are examples of involutive m-BE algebras verifying (Δ_m) and (G) and not verifying (m-Pimpl) and (Pom).*

By the previous Remark, from the connections from Figure 10, we obtain the connections from Figure 11.

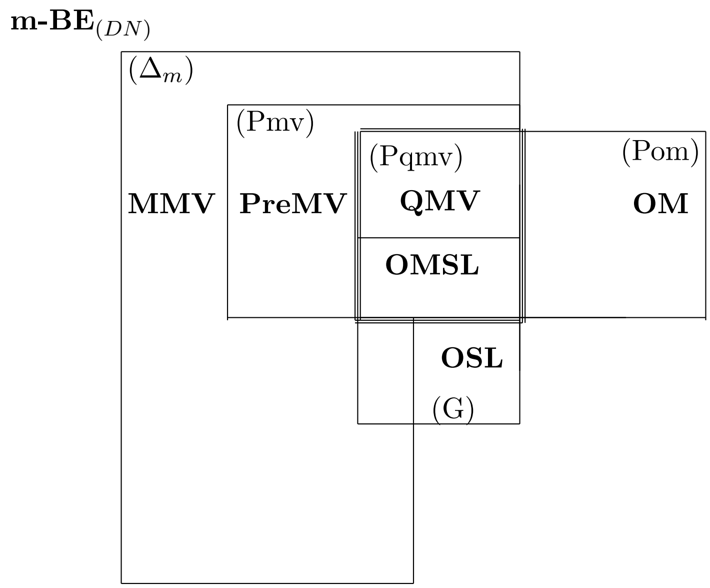

Figure 11: Resuming connections between **QMV**, **PreMV**, **MMV**, **OM**, **OSL** and **OMSL**

4.1.2 OMSL = OML

Proposition 4.11. *We have:*
(mPom1) (Pom) + (Pcomm) + (Neg0-1) + (PU) + (DN) $\Longrightarrow$ *(m-Re)* [21]
(mPom2) (Pom) + (G) + (Pass) + (DN) $\Longrightarrow$ *(m-Pimpl).*

Proof. (mPom2) : **(By** *Prover*9**, in** 0.01 **seconds, the length of the proof being** 15**)**
First, we have: $(a)\ x \odot y \stackrel{(G)}{=} (x \odot x) \odot y \stackrel{(Pass)}{=} x \odot (x \odot y)$.
Then, in (a), take $X := (x\odot y)^-$ and $Y := ((x\odot y)^- \odot x)^-$ to obtain: $(b)\ X \odot Y = (x\odot y)^- \odot ((x\odot y)^- \odot x)^- \stackrel{(Pom)}{=} x^-$. Then, $x^- = X \odot Y \stackrel{(a)}{=} X \odot (X \odot Y) \stackrel{(b)}{=} X \odot x^- = (x \odot y)^- \odot x^-$; hence, $((x \odot y)^- \odot x^-)^- = (x^-)^- \stackrel{(DN)}{=} x$, i.e. (m-Pimpl) holds. □

We know already, by Proposition 2.16, that:

Proposition 4.12. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \implies (G),$$

i.e. **OL** $\subset$ **OSL**.

Proposition 4.13. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pom) + (G) \implies (m - Pimpl),$$

i.e. **OMSL** $\subset$ **OL**.

Proof. By (mPom2). □

By Propositions 4.12 and 4.13, we obtain:

Theorem 4.14. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies ((m - Pimpl) \Leftrightarrow (G))$$

or

$$(Pom) + (m - Pimpl) \iff (Pom) + (G).$$

By Theorem 4.14 and the equivalent definitions (Definition 2) of left-OMLs and of left-OMSLs, we obtain: **OML**= **OM** + (G) = **OSL** + (Pom) = **OSL** ∩ **OM** = **OMSL**, by (49). Hence, we have:

$$\mathbf{OMSL} = \mathbf{OML}. \tag{53}$$

By (28), (49) and (53), we obtain the connections from Figure 12.

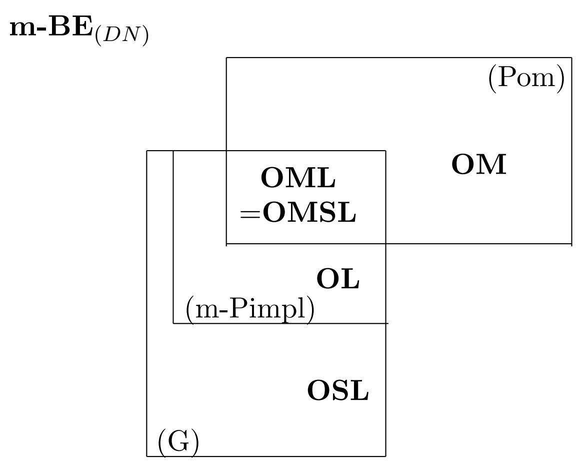

Figure 12: Resuming connections between **OML** = **OMSL**, **OL**, **OSL** and **OM**

Finally, since **OML** = **OMSL**, it follows, by Theorems 3.9 and 4.8:

Corollary 4.15. *We have:*

$$\mathbf{OML} = \mathbf{OMSL} = \mathbf{QMV} + (m - Pimpl) = \mathbf{QMV} \cap \mathbf{OL} = \mathbf{QMV} + (G) = \mathbf{QMV} \cap \mathbf{OSL}. \tag{54}$$

Corollary 4.16. *(See [6], Theorem 2.3.12)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Consider the set of all idempotent elements of A^L (i.e. elements verifying (G)):

$$Id(A^L) = \{x \in A^L \mid x \odot x = x\}.$$

Then, $(Id(A^L), \odot, ^-, 1)$ is a left-OML.

Proof. Note that $(Id(A^L), \odot, ^-, 1)$ is a subalgebra of $\mathcal{A}^L$ verifying (G). Then apply above Corollary 4.15. □

Moreover,
- There are examples of involutive m-BE algebras verifying (G) and not verifying (Δ_m), (m-Pimpl) and (Pom);
- There are examples of involutive m-BE algebras verifying (m-Pimpl) and not verifying (Δ_m) and (Pom).

By the connections from Figures 4, 10 and 12, we obtain the connections from Figure 13.

By the connections from Figures 5, 11 and 13, we obtain the connections from Figure 14.

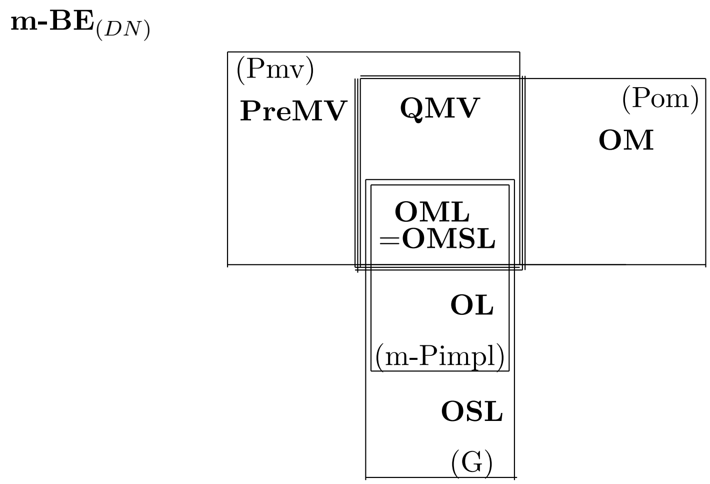

Figure 13: Resuming connections between **QMV**, **PreMV**, **OSL**, **OL**, **OM** and **OML** = **OMSL**

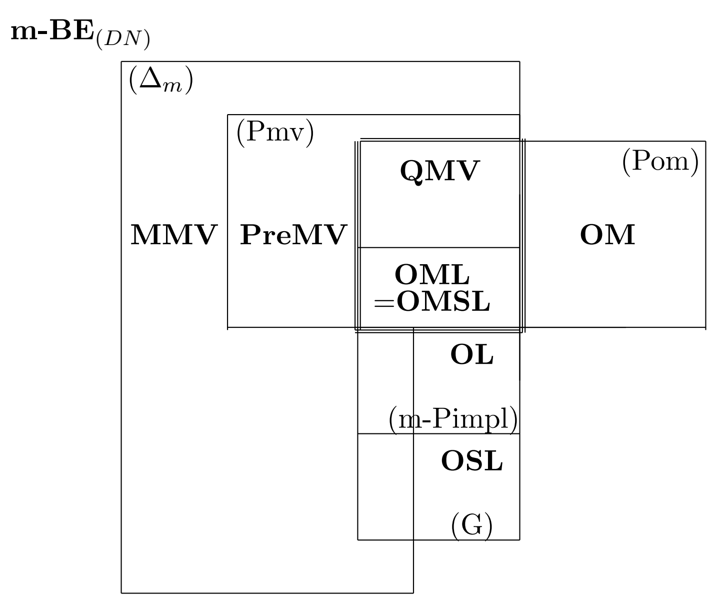

Figure 14: Resuming connections between **QMV**, **PreMV**, **MMV**, **OL**, **OSL** and **OML** = **OMSL**

4.1.3 The transitive case: tOSL $\subset$ tMMV

Theorem 4.17. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(G) \ + \ (m-BB) \implies (\Delta_m).$$

Proof. (following a proof by *Prover*9 **in** 10.75 **seconds, the length of the proof being** 28**)**
First, (G) ($x \odot x = x$) implies:

$$x \odot (x \odot y) = x \odot y. \tag{55}$$

Indeed, $x \odot (x \odot y) \overset{(Pass)}{=} (x \odot x) \odot y \overset{(G)}{=} x \odot y$.
Second, (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) implies:

$$x \odot (y \odot ((x \odot z^-)^- \odot (z \odot y)^-)) = 0. \tag{56}$$

Indeed, interchange x with z in (m-BB) to obtain:
(x) $[(z \odot y)^- \odot (x \odot y)] \odot (x \odot z^-)^- = 0$;
then, in (x), apply (Pass) and (Pcomm) to obtain:
(x') $[(x \odot y) \odot (x \odot z^-)^-] \odot (z \odot y)^- = 0$;
then apply (Pass) to obtain (56).
Also (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) implies:

$$(x \odot y)^- \odot (z \odot (x \odot (z \odot y^-)^-)) = 0. \tag{57}$$

Indeed, interchange x with y in (m-BB) to obtain, by (Pcomm):
$[(x \odot y)^- \odot (z \odot x)] \odot (z \odot y^-)^- = 0$;
then apply (Pass) to obtain (57).
Now, from (57), we obtain:

$$x \odot (y \odot (x \odot (y^- \odot z)^-)^-) = 0. \tag{58}$$

Indeed, in (57) take $X := x$ and $Y := x^- \odot y$ to obtain:
(y) $(x \odot (x^- \odot y))^- \odot (z \odot (x \odot (z \odot (x^- \odot y)^-)^-)) = 0$;
but, in (y), $x \odot (x^- \odot y) \overset{(Pass)}{=} (x \odot x^-) \odot y \overset{(m-Re)}{=} 0 \odot y \overset{(Pcomm)}{=} y \odot 0 \overset{(m-L)}{=} 0$, hence (y) becomes:
(y') $0^- \odot (z \odot (x \odot (z \odot (x^- \odot y)^-)^-)) = 0$,
which by (Neg0-1) and (PU) becomes:
(y'') $z \odot (x \odot (z \odot (x^- \odot y)^-)^-) = 0$;
now, in (y”) take $X := y$, $Y := z$ and $Z := x$ to obtain:
$x \odot (y \odot (x \odot (y^- \odot z)^-)^-) = 0$, that is (58).
Now, from (58) and (55), we obtain:

$$x \odot (x \odot (y \odot x^-)^-)^- = 0. \tag{59}$$

Indeed, in (55) take $X := x$ and $Y := (x \odot (x^- \odot y)^-)^-$ to obtain:
(u) $x \odot (x \odot (x \odot (x^- \odot y)^-)^-) = x \odot (x \odot (x^- \odot y)^-)^-$;
also in (58) take $X := x$, $Y := x$ and $Z := y$ to obtain:
(v) $x \odot (x \odot (x \odot (x^- \odot y)^-)^-) = 0$;
then, (u) becomes, by (v):
(u') $0 = x \odot (x \odot (x^- \odot y)^-)^-$,
which by (Pcomm) becomes (59).

Now, from (m-BB) and (59) we obtain:

$$x \odot (y \odot (x \odot ((z \odot x^-)^- \odot y))^-) = 0. \quad (60)$$

Indeed, in (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) take $X := x \odot (y \odot x^-)^-$, $Y := z$ and $Z := x$ to obtain:
(w) $[((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z)] \odot (x \odot (x \odot (y \odot x^-)^-)^-)^- = 0$;
but, in (w), the part $x \odot (x \odot (y \odot x^-)^-)^- = 0$, by (59); hence, (w) becomes:
(w') $[((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z)] \odot 0^- = 0$,
which by (Neg0-1) and (PU) becomes:
(w'') $((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z) = 0$,
which by (Pcomm), (Pass) becomes:
(w''') $(x \odot z) \odot (x \odot ((y \odot x^-)^- \odot z))^- = 0$,
which by interchanging y with z and by (Pass) becomes:
$x \odot (y \odot (x \odot ((z \odot x^-)^- \odot y))^-) = 0$, that is (60).

Now, from (56) and (60), we obtain:

$$(x \odot (x \odot y^-)^-) \odot (y \odot (y \odot x^-)^-)^- = 0. \quad (61)$$

Indeed, in (60), take $X := x$, $Y := (x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-$ and $Z := z$ to obtain:
(z) $x \odot (((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) \odot (x \odot ((z \odot x^-)^- \odot Y))^-) = 0$,
where the part of (z):
$A \overset{notation}{=} x \odot ((z \odot x^-)^- \odot Y) = x \odot ((z \odot x^-)^- \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-)) = 0$;
indeed, in (56) take $X := x$, $Y := (z \odot x^-)^-$ and $Z := y$ to obtain:
$x \odot ((z \odot x^-)^- \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-)) = 0$, i.e. $A = 0$;
hence, (z) becomes:
(z') $x \odot (((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) \odot 0^-) = 0$;
then, by (Neg0-1) and (PU), (z') becomes:
(z'') $x \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) = 0$,
and (z'') by (Pass) and by taking $z = y$ becomes:
$(x \odot (x \odot y^-)^-) \odot (y \odot (y \odot x^-)^-)^- = 0$, that is (61).

Finally, from (61), by interchanging x with y, we obtain:
$(y \odot (y \odot x^-)^-) \odot (x \odot (x \odot y^-)^-)^- = 0$, that is (Δ_m). □

Note that Theorem 4.17 says: **tOSL** $\subset$ **MMV**. Hence, **tOSL** $\subset$ **tMMV**.

Note also that Theorem 3.17 follows from Theorem 4.17, since (m-Pimpl) implies (G).

By Theorems 3.17 and 4.17 and by the connections from Figures 8 and 14, we obtain the connections from Figure 15.

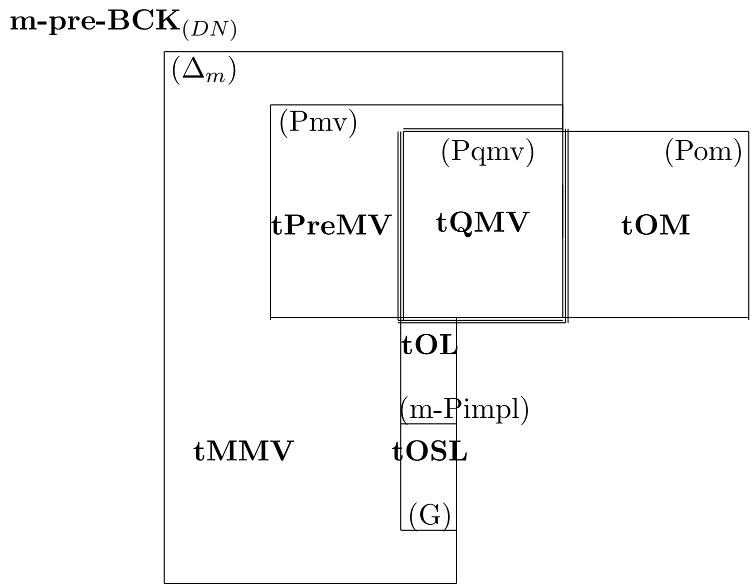

Figure 15: Resuming connections between **tQMV**, **tMMV**, **tOSL** and **tOL**

4.2 Orthomodular widelattices: OMWL

We introduce the following notion.

Definition 4.18. *(Definition 1) (The dual one is omitted)*
A left-orthomodular widelattice or an orthomodular left-widelattice, or a left-OMWL for short, is a left-OWL verifying: for all $x, y \in A^L$,
(Wom) $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OMWL** the class of all left-OMWLs. Following the equivalent Definition 2 of a left-OWL (see Definition 2.17), we obtain immediately an equivalent definition:

Definition 4.19. *(Definition 2) (The dual one is omitted)*
A left-OMWL is a left-OWL verifying (Pom), i.e. is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pabs-i) and (Pom), i.e.

$$\mathbf{OMWL} = \mathbf{m-BE_{(DN)}} + (m-Pabs-i) + (Pom) = \mathbf{OWL} \cap \mathbf{OM}. \quad (62)$$

Further, we shall work with Definition 2 of OMWLs. Hence, we have the connections from Figure 16.

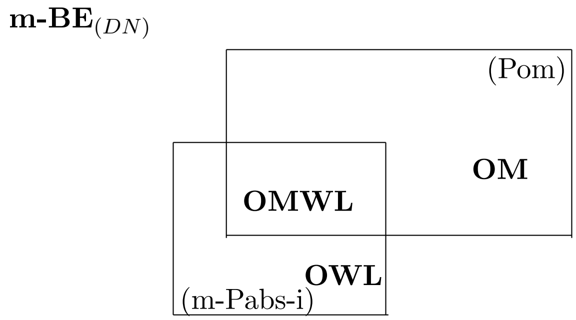

Figure 16: Resuming connections between **OWL**, **OMWL** and **OM**

4.2.1 Connections between OMWL and PreMV, QMV, MMV, OM, OWL

• **OMWL** + (Pmv) (Connections between **OMWL** and **PreMV**)

The next Proposition 4.21 (saying that (Pom) and (m-Pabs-i) imply (Pmv)) was proved by *Prover9* in 17.06 seconds and the proof produced by *Prover9* has the length 23. We divide the proof produced by *Prover9* into the proof of Lemma 4.20 and Proposition 4.21.

Lemma 4.20. *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra verifying (Pom) (i.e. an OM algebra). Then, we have:*

$$(x \odot y)^- \odot (y \odot (y \odot x)^-)^- = y^-, \tag{63}$$

$$(x \odot y)^- \odot [(y \odot (y \odot x)^-)^- \odot z] = y^- \odot z, \tag{64}$$

$$(x \odot (y \odot z))^- \odot (x \odot (y \odot (x \odot (y \odot z))^-))^- = (x \odot y)^-, \tag{65}$$

$$(x \odot y)^- \odot (z \odot (x \odot (x \odot y)^-)^-) = z \odot x^-, \tag{66}$$

$$(x \odot y^-)^- \odot [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^- = ((y \odot z)^- \odot x)^-. \tag{67}$$

Proof. (63): From (Pom), by interchanging x with y and by (Pcomm).
(64): From (63), by "multiplying" by z.
(65): From (Pom), taking $X := x \odot y$ and $Y := z$ and by (Pass).
(66): By "multiplying" (Pom) by z, and by (Pcomm), (Pass).
(67): In (65), take $X := (y \odot z)^-$, $Y := x$, $Z := (y \odot (y \odot z)^-)^-$ to obtain:

$$[(y\odot z)^-\odot(x\odot(y\odot(y\odot z)^-)^-)]^-\odot[(y\odot z)^-\odot(x\odot[(y\odot z)^-\odot(x\odot(y\odot(y\odot z)^-)^-)]^-)]^- = ((y\odot z)^-\odot x)^-. \tag{68}$$

On the other hand, in (66), take $X := y$, $Y := z$, $Z := x$ to obtain:

$$(y \odot z)^- \odot (x \odot (y \odot (y \odot z)^-)^-) = x \odot y^-. \tag{69}$$

Now, from (68), by (69), we obtain:
$(x \odot y^-)^- \odot ((y \odot z)^- \odot (x \odot (x \odot y^-)^-))^- = ((y \odot z)^- \odot x)^-$, i.e. (67) holds. □

Proposition 4.21. *(See Proposition 3.4)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,

$$(Pom) \ + \ (m-Pabs-i) \ \Longrightarrow \ (Pmv).$$

Proof. (**By** *Prover*9)

• First, from (m-Pabs-i) ($x \odot (x^- \odot (x^- \odot y^-))^- = x$), by taking $Y := y^-$, we obtain:

$$x \odot (x^- \odot (x^- \odot y))^- = x. \tag{70}$$

• Now, we prove:

$$x \odot (y \odot (y^- \odot ((x \odot y)^- \odot z))^-) = x \odot y. \tag{71}$$

Indeed, in (70), take $X := x \odot y$, Y:= $(y \odot (y \odot x)^-)^- \odot z$ to obtain:

$$(x \odot y) \odot ((x \odot y)^- \odot ((x \odot y)^- \odot [(y \odot (y \odot x)^-)^- \odot z]))^- = x \odot y. \tag{72}$$

Now, from (72), by (64), we obtain:

$$(x \odot y) \odot ((x \odot y)^- \odot (y^- \odot z))^- = x \odot y. \tag{73}$$

From (73), by (Pass), (Pcomm), we obtain:
$x \odot (y \odot (y^- \odot ((x \odot y)^- \odot z))^-) = x \odot y$, i.e. (71) holds.

• Now, we prove:

$$x \odot (y^- \odot (y \odot ((y \odot z)^- \odot x)^-)^-) = x \odot y^-. \tag{74}$$

Indeed, in (71), take $X := x$, $Y := y^-$, $Z := [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^-$ to obtain:

$$x \odot (y^- \odot (y \odot ((x \odot y^-)^- \odot [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^-))^-) = x \odot y^-. \tag{75}$$

From (75), by (67), we obtain:
$x \odot (y^- \odot (y \odot ((y \odot z)^- \odot x)^-)^-) = x \odot y^-$, i.e. (74) holds.

• Now, we prove:

$$x^- \odot (y \odot (y \odot x)^-)^- = x^- \odot y^-. \tag{76}$$

Indeed, in (74), take $X := (x \odot (x \odot y)^-)^-$, $Y := y$, $Z := x$ to obtain:

$$(x \odot (x \odot y)^-)^- \odot (y^- \odot [y \odot ((y \odot x)^- \odot (x \odot (x \odot y)^-)^-)^-]^-) = (x \odot (x \odot y)^-)^- \odot y^-. \tag{77}$$

In (63), take $X := y$, $Y := x$, to obtain:

$$(y \odot x)^- \odot (x \odot (x \odot y)^-)^- = x^-. \tag{78}$$

Then, from (77), by (78), we obtain:

$$(x \odot (x \odot y)^-)^- \odot (y^- \odot (y \odot x^=)^-) = (x \odot (x \odot y)^-)^- \odot y^-. \tag{79}$$

From (79), by (DN), we obtain:
$(x \odot (x \odot y)^-)^- \odot (y^- \odot (y \odot x)^-) = (x \odot (x \odot y)^-)^- \odot y^-$, hence, by (Pcomm), (Pass), we obtain:
$y^- \odot ((x \odot y)^- \odot ((x \odot y)^- \odot x)^-) = (x \odot (x \odot y)^-)^- \odot y^-$, hence by (Pom), we obtain:
$y^- \odot x^- = (x \odot (x \odot y)^-)^- \odot y^-$, hence, by interchanging x, y, we obtain:
$x^- \odot y^- = (y \odot (y \odot x)^-)^- \odot x^-$, hence, by (Pcomm), $x^- \odot (y \odot (y \odot x)^-)^- = x^- \odot y^-$, i.e. (76) holds.

• Now, finally, from (76), by $X := x^-$, $Y := y^-$ and (DN), (Pcomm), we obtain:
$x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$, i.e. (Pmv) holds. □

Note that Proposition 3.4 follows from Proposition 4.21, since (m-Pimpl) implies (m-Pabs-i).

Note also that Proposition 4.21 says: **OMWL** $\subset$ **PreMV**.

Remark 4.22. *The following converse of Proposition 4.21 ((Pmv) + (m-Pabs-i) $\Longrightarrow$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying (Pmv) and (m-Pabs-i) and not verifying (Pom).*

• **OMWL** + (Pqmv) (Connections between **OMWL** and **QMV**)

We establish now the connection between the OMWLs and the QMV algebras verifying (m-Pabs-i).

Proposition 4.23. *(See Proposition 3.7)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OMWL. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pabs-i). (i.e. in an involutive m-BE algebras, (Pom) + (m-Pabs-i) $\Longrightarrow$ (Pqmv).)

Proof. Since $\mathcal{A}^L$ is a left-OMWL, it is an involutive m-BE algebra verifying (m-Pabs-i) and (Pom) (Definition 2). By Proposition 4.21, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pabs-i). □

Note that Proposition 4.23 says:

$$\textbf{OMWL} \subset \textbf{QMV}, \tag{80}$$

the inclusion being strict since there are examples of QMV algebras not verifying (m-Pabs-i). Note also that Propositions 3.7 and 4.21 follow from Proposition 4.23.

The following converse of Proposition 4.23 holds.

Proposition 4.24. *(See Proposition 3.8)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (m-Pabs-i). Then, $\mathcal{A}^L$ is a left-OMWL. (i.e. in involutive m-BE algebras, (Pqmv) + (m-Pabs-i) $\Longrightarrow$ (Pom).)

Proof. Since $\mathcal{A}^L$ a left-QMV algebra verifying (m-Pabs-i), it is an involutive left-m-BE algebra verifying (Pqmv) (hence (Pmv), (Pom)) and (m-Pabs-i) (Definition 2). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (m-Pabs-i) and (Pom), i.e. it is a left-orthomodular widelattice. □

Note that Proposition 4.24 says: **QMV** ∩ **OWL** ⊂ **OM**. Note also that Proposition 3.8 follows from Proposition 4.24, since (m-Pimpl) $\Longrightarrow$ (m-Pabs-i).

By Propositions 4.23 and 4.24, we obtain:

Theorem 4.25. *(See Theorem 3.9)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,

$$(m-Pabs-i) \implies ((Pom) \Leftrightarrow (Pqmv))$$

or

$$(m-Pabs-i) + (Pom) \iff (Pqmv) + (m-Pabs-i),$$

i.e. orthomodular widelattices coincide with QMV algebras verifying (m-Pabs-i).

Note that Theorem 4.25 says:

$$\textbf{OMWL} = \textbf{QMV} + (m-Pabs-i) = \textbf{QMV} \cap \textbf{OWL}. \tag{81}$$

Note also that Theorem 3.9 follows from Theorem 4.25.

By (62), (81) and Remark 4.22, we obtain the connections from Figure 17.

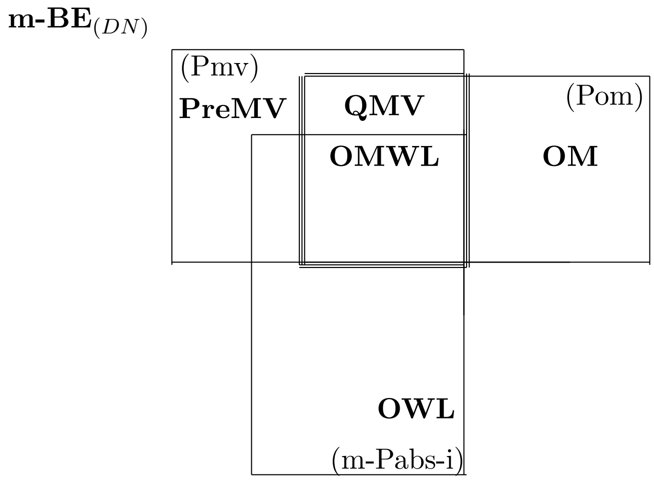

Figure 17: Resuming connections between **QMV**, **PreMV**, **OWL**, **OM** and **OMWL**

- **OMWL** + (Δ_m) (Connections between **OMWL** and **MMV**)

Proposition 4.26. *(See Proposition 3.10)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,

$$(Pom) \ + \ (m - Pabs - i) \ \Longrightarrow \ (\Delta_m).$$

Proof. By Proposition 4.21, (Pom) + (m-Pabs-i) imply (Pmv) and (Pmv) implies (Δ_m), thus (Pom) + (m-Pabs-i) imply (Δ_m). □

Note that Proposition 4.26 says: **OMWL** $\subset$ **MMV**.

Note also that Proposition 3.10 follows from Proposition 4.26, since (m-Pimpl) implies (m-Pabs-i), that Proposition 4.23 follows also from Proposition 4.26, since (Pom) + (Δ_m) imply (Pqmv), and that Proposition 4.26 follows also from Proposition 4.23, since (Pqmv) implies (Δ_m).

Remark 4.27. *The following converse of Proposition 4.26 ((Δ_m) + (m-Pabs-i) $\Longrightarrow$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying (Δ_m) and (m-Pabs-i) and not verifying (m-Pimpl) and (Pom).*

By the previous Remark and by the connections from Figure 17, we obtain the connections from Figure 18.

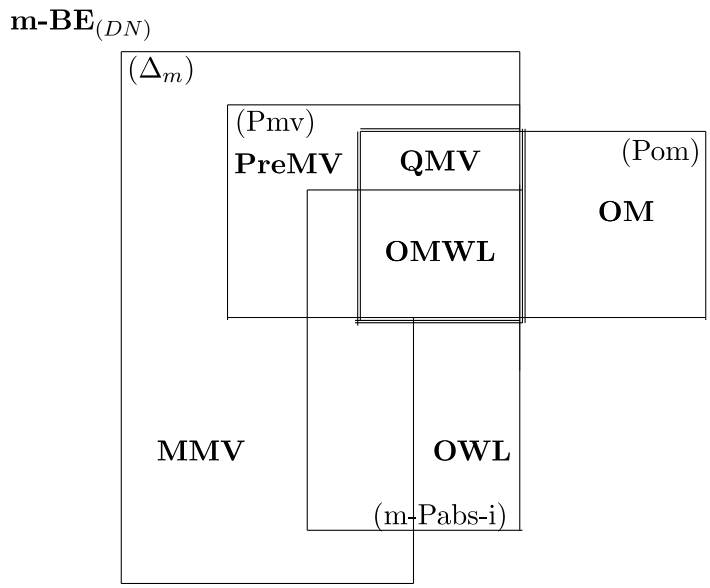

Figure 18: Resuming connections between **QMV**, **PreMV**, **MMV**, **OWL** and **OMWL**

4.2.2 OML $\subset$ OMWL

We know (by Proposition 2.16) that:

Proposition 4.28. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \implies (m - Pabs - i),$$

i.e. **OL** $\subset$ **OWL**.

Proposition 4.29. *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pom) + (G) \implies (m - Pabs - i).$$

Proof. By Proposition 4.13, (Pom) + (G) imply (m-Pimpl), and by Proposition 4.28, (m-Pimpl) implies (m-Pabs-i). □

Note that Proposition 4.29 follows from Proposition 4.13.

Note also that Proposition 4.29 says: **OML** (= **OMSL**) $\subset$ **OWL**, hence,

$$\mathbf{OML}\ (= \mathbf{OMSL})\ \subset\ \mathbf{OMWL}, \tag{82}$$

the inclusion being strict, since there are examples of OMWLs not verifying (G).

Note also that **OML** (= **OMSL**) $\subset$ **OMWL** means (see 23):

$$\mathbf{OML} = \mathbf{OMSL}\ \cap\ \mathbf{OMWL}.$$

By (28), (62) and (82), we obtain the connections from the Figure 19.

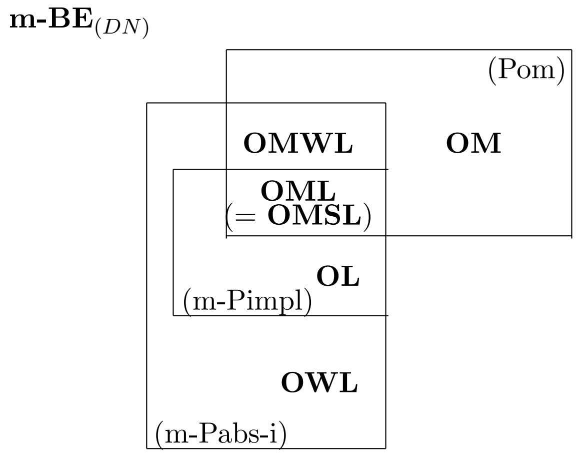

Figure 19: Resuming connections between **OMWL**, **OML**, **OL**, **OWL** and **OM**

Since **OML** = **OMSL** $\subset$ **OMWL**, by Theorems 4.14 and 4.29, and **OMWL** $\subset$ **QMV**, by (80), we obtain:

$$\mathbf{MOD} \subset \mathbf{OML} = \mathbf{OMSL} \subset \mathbf{OMWL} \subset \mathbf{QMV}.$$

By the connections from Figures 4, 17 and 19, we obtain the connections from Figure 20.

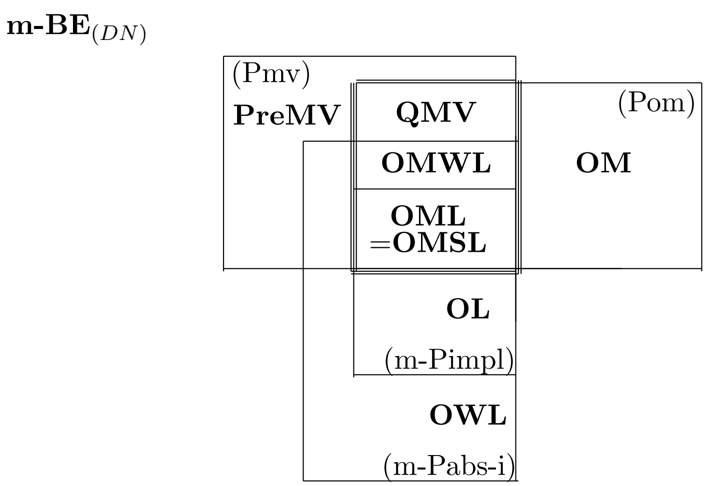

Figure 20: Resuming connections between **QMV**, **PreMV**, **OML**, **OWL**, **OL**, **OM** and **OMWL**

By the connections from Figures 5, 18 and 20, we obtain the connections from Figure 21.

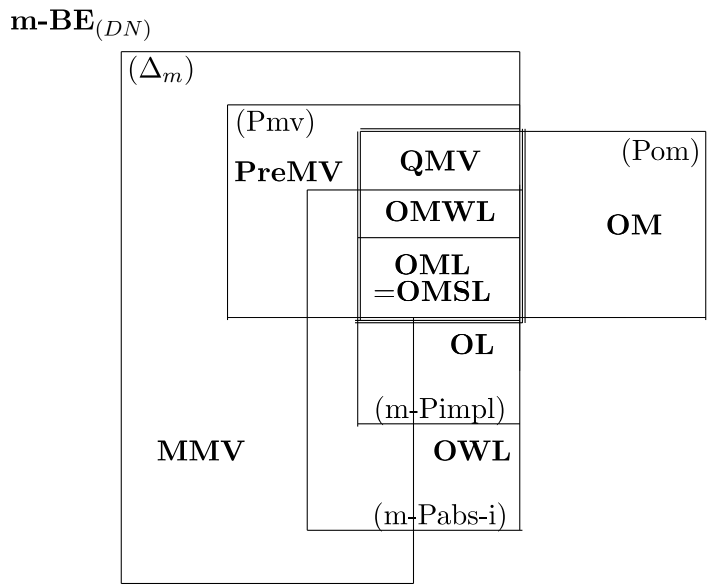

Figure 21: Resuming connections between **QMV**, **PreMV**, **MMV**, **OML**, **OL**, **OWL** and **OMWL**

4.2.3 The transitive and/or antisymmetric case

• **The transitive case: tOWL $\subset$ tMMV**

Denote by **tOMWL** the class of all transitive left-OMWLs.

Theorem 4.30. *(See Theorem 4.17)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,

$$(m - Pabs - i) \ + \ (m - BB) \implies (\Delta_m).$$

Note that this theorem is Theorem 5.13 from [22], proved by *Prover*9. It says that: **tOWL** $\subset$ **MMV**. Hence, **tOWL** $\subset$ **tMMV**.

If, additionally, (Pom) holds, then, as expected: **tOMWL** $\subset$ **tQMV**.

Note that Theorem 3.17 follows also from Theorem 4.30, since (m-Pimpl) implies (m-Pabs-i).

By (42), by Theorems 3.17 and 4.30 and the connections from Figure 21, we obtain the connections from Figure 22.

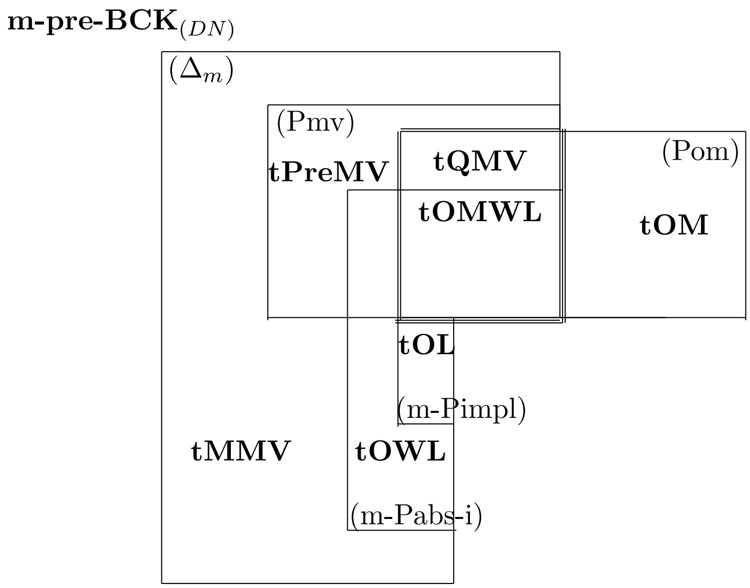

Figure 22: Resuming connections between **tQMV**, **tMMV**, **tOWL** and **tOL**

• **The transitive and the antisymmetric case**

Denote by **aOMWL** the class of all antisymmetric left-OMWLs.

Theorem 4.31. *We have:*

$$\mathbf{aOMWL} = \mathbf{taOMWL}.$$

Proof. Since **OMWL** $\subset$ **QMV**, by adding (m-An), we obtain: **aOMWL** $\subset$ **aQMV** $=$ **MV**, by Theorem 2.24, and since any MV algebra verifies (m-Tr), it follows that **aOMWL** $=$ **taOMWL**. □

While **tOMWL** $\subset$ **tOWL**, we obtain the following results.

Theorem 4.32. *We have:*
(i) **taOWL** $\subset$ **MV**; *(ii)* **taOMWL** $\subset$ **MV**; *(iii)* **taOWL** $=$ **taOMWL**.

Proof. (i) Since **tOWL** $\subset$ **tMMV**, by applying (m-An), we obtain:
taOWL $\subset$ **taMMV** $=$ **MV**, by Theorem 2.24.

(ii) Since **tOMWL** $\subset$ **tQMV**, by applying (m-An), we obtain:
taOMWL $\subset$ **taQMV** $=$ **MV**, by Theorem 2.24.

(iii) Since any MV algebra verifies (Pom), it follows by (i) that **taOWL** $=$ **taOMWL**. □

Theorem 4.33. *We have:*

$$\mathbf{taOWL} = \mathbf{taOMWL} = \mathbf{aOMWL} \subset \mathbf{MV}.$$

Proof. By Theorems 4.31, 4.32. □

• **Final remarks** We have:

tOMWL	$\subset$	**tQMV**
(m-An) $\downarrow$		$\downarrow$ (m-An)
taOMWL $=$ **taOWL**	$\subset$	**MV**.

The tOMWLs (inside the tQMV algebras) will be deeply analysed in next paper [19], in connection with the taOWLs (inside the MV algebras).

Conflict of Interest: The author declares no conflict of interest.

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Pure Ideals in Residuated Lattices

Mihaela Istrata

Abstract. Ideals in MV algebras are, by definition, kernels of homomorphism. An ideal is the dual of a filter in some special logical algebras but not in non-regular residuated lattices. Ideals in residuated lattices are defined as natural generalizations of ideals in MV algebras. Spec(L), the spectrum of a residuated lattice L, is the set of all prime ideals of L, and it can be endowed with the spectral topology. The main scope of this paper is to characterize Spec(L), called the stable topology. In this paper, we introduce and investigate the notion of *pure* ideal in residuated lattices, and using these ideals we study the related spectral topologies.

Also, using the model of MV algebras, for a De Morgan residuated lattice L, we construct the *Belluce lattice* associated with L. This will provide information about the pure ideals and the prime ideals space of L. So, in this paper we generalize some results relative to MV algebras to the case of residuated lattices.

AMS Subject Classification 2020: MSC 22A30, MSC 03B50, MSC 03G25, MSC 06D35, MSC 06B30.
Keywords and Phrases: De Morgan residuated lattice, pure ideal, prime ideal, spectral topology, stable topology.

1 Introduction

In fuzzy logic theory, residuated lattices play an important role because they provide an algebraic framework to fuzzy logic and fuzzy reasoning. From a logical point of view, various filters and ideals correspond to various sets of provable formulae. The notion of the ideal has been introduced in many algebraic structures such as lattices, rings of MV algebras. By definition, the ideals of MV algebras are kernels of homomorphisms. An ideal is the dual of a filter in some special logical algebras but not in non-regular residuated lattices. For terminology and theory of residuated Lattices we refer the reader to the papers (see [16], [18]).

For a residuated lattice, L, $\mathcal{P}(L)$, the set of all prime ideals of L, can be endowed with the spectral topology τ_L in the same manner as in the case of commutative rings of bounded distributive lattice.

For an ideal I of L, $V(I) = \{P \in Spec(L) : I \nsubseteq P\}$ is open in $(\mathcal{P}(L), \tau_L)$ and $\overline{V}(I) = \mathcal{P}(L) \backslash V(I) = \{P \in \mathcal{P}(L) : I \subseteq P\}$ is closed; Thus $V(I)$ is *stable under descent* and $\overline{V}(I)$ is stable under ascent. So, clopen sets are stable, that is, these are simultaneous stable under ascent and descent.

The characterization of open stable sets relies on the concept of *pure ideal* (see also, [7]) for commutative rings with the unit, (see [8]) for bounded distributive lattices, and (see [3], [6]) for MV algebras).

The scope of this paper is to introduce and investigate pure ideals in residuated lattices, using the model of MV algebras.

In Section 2 and Section 3 we recall basic results about residuated lattices and ideals in residuated lattices and we give new characterizations for prime and maximal ideals.

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In Section 4, we introduce the notion of *pure ideal*. Their properties and characterizations are obtained. We will use pure ideals in Section 6 to characterize the stable open sets relative to the spectral topology.

Using the model of MV algebras, (see [2]), in Section 5, for a De Morgan residuated lattice L, we construct the *Belluce lattice* [L] associated with L. The Belluce lattice will provide some insight about pure ideals and prime ideals space of L (see Theorem 5.8, Corollary 6.2, Corollary 6.5). The Belluce lattice [L] is a Boolean algebra iff L is a hyperarchimedean De Morgan residuated lattice, (see Theorem 5.4).

Section 6 contains topological results relative to the spectral topology τ_L and the stable topology S_L, coarser than the spectral one. For a De Morgan residuated lattice L, $\mathcal{P}(L)$, and Spec([L]) are homeomorphic, and (see Corollary 6.2) the stable topology S_L coincides with the spectral topology τ_L iff L is a hyperarchimedean, (see Theorem 6.4, Corollary 6.5, Corollary 6.6, Corollary 6.7) study the connections between pure ideals of L and open stable subsets of $\mathcal{P}(L)$.

2 Preliminaries

A residuated lattice is an algebra $(L, \wedge, \vee, \odot, \to, 1)$ of type $(2, 2, 2, 2, 0)$ satisfying the following axioms:

(RL_1) $(L, \wedge, \vee)$ is a bounded lattice (the partial order is denoted by $\leq$);

(RL_2) $(L, \odot, 1)$ is a commutative monoid;

(RL_3) For every $x, y, z \in L$, $x \odot z \leq y$ iff $z \leq x \to y$ for any $x, y, z \in L$ (residuation).

A residuated lattice L is called an MTL algebra if $(x \to y) \vee (y \to x) = 1$ for every $x, y \in L$, (see [12], [13], [16]) and is called a *De Morgan residuated lattice* if $(x \wedge y)^* = x^* \vee y^*$, for every $x, y \in L$, (see [16], [18]). Examples of De Morgan residuated lattices are Boolean algebras, MV algebras, BL algebras, MTL algebras, Girard algebras.

MV algebras are particular cases of residuated lattices, (see [16]). A residuated lattice L is an MV algebras if it satisfies the additional condition:$(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$.

Example 2.1. (See [12]) Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, and a, b incomparable. L is a commutative residuated lattice with the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Example 2.2. (See [12]) Let $L = \{0, b, c, d, 1\}$ with $0 < b, c < d < 1$ but b, c are incomparable. L is a commutative residuated lattice with the following operations:

$\to$	0	b	c	d	1
0	1	1	1	1	1
b	d	1	d	1	1
c	d	d	1	1	1
d	d	d	d	1	1
1	0	b	c	d	1

$\odot$	0	b	c	d	1
0	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	0	c
d	0	0	0	0	d
1	0	b	c	d	1

Let L be a residuated lattice. For $x \in L$ and $x \geq 0$ we denote $x^0 = 1, x^n = x^{n-1} \odot x$ for $n \geq 1, x^* = x \to 0$ and $x^{**} = (x^*)^*$.

Recall (see [1]) that an element $x \in L$ is called *complemented* if there is an element $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$; y is the complement of x.

If we denote by $B(L)$ the set of all complemented elements in the lattice $(L, \wedge, \vee, 0, 1)$, then $B(L)$ is a Boolean subalgebra of L, called *the Boolean center of* L and $e \in B(L)$ iff $e \vee e^* = 1$, (see [16]).

For $x, y, z \in L$ we have the following rules of calculus, (see [14], [16], [18]):

(c_1) $x \to 1 = 1$ and $1 \to x = x, x \to x = 1$;

(c_2) $x \leq y$ iff $x \to y = 1$ and $x \leq y \to x, x \odot (x \to y) \leq y$;

(c_3) If $x \leq y$ then $z \odot x \leq z \odot y, z \to x \leq z \to y, y \to z \leq x \to z, y^* \leq x^*$;

(c_4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(c_5) $0^* = 1, 1^* = 0, x \odot x^* = 0, x \odot 0 = 0, x \leq (x^*)^*$;

(c_6) $(x \vee y)^* = x^* \wedge y^*$ and $(x \wedge y)^* \geq x^* \vee y^*$;

(c_7) $x \to y^* = y \to x^* = (x^*)^* \to y^* = (x \odot y)^*$;

(c_8) $(x \to y)^{**} \leq x^{**} \to y^{**}, (x \odot y)^{**} = x^{**} \odot y^{**}$;

(c_9) $x \vee y = 1$ implies $x \odot y = x \wedge y$ and $x^n \vee y^n = 1$, for every $n \geq 1$;

(c_{10}) for $x \geq 1, x^n \in B(L)$ iff $x \vee (x^n)^* = 1$.

In a residuated lattice L, for $x, y \in L$ we define $x \oplus y = x^* \to y$ and $x \boxplus y = (x^* \odot y^*)^* = x^* \to y^{**}$. We remark that $x \boxplus y = x \oplus y^{**}$ and for $x \in L$, we will use the notation $(n+1)x := nx \boxplus x$, for a natural number $n \geq 1$.

Let L be a commutative residuated lattice, for $x, y, z \in L$ and $m, n \geq 1$ we have the rules of calculus, (see [5] and [14]):

(c_{11}) $x, y \leq x \oplus y, (x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(c_{12}) $x \boxplus y = y \boxplus x, (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$;

(c_{13}) $x \wedge (y \boxplus z) \leq (x^{**} \wedge y^{**}) \boxplus (x^{**} \wedge z^{**})$ and $(mx) \wedge (ny) \leq (mn)(x^{**} \wedge y^{**})$.

Lemma 2.3. *If L is a De Morgan residuated lattices and $x, y, z \in L$, then*

(c_{14}) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$*;*

Proof. To prove (c_{14}) we have to show that $(x \wedge y)^* \to z = (x^* \to z) \wedge (y^* \to z)$. To do this we prove that

(i) $(x \wedge y)^* \to z \leq x^* \to z, y^* \to z$;

(ii) If $t \leq x^* \to z, y^* \to z \Rightarrow t \leq (x \wedge y)^* \to z$.

We have $x \wedge y \leq x \Rightarrow x^* \leq (x \wedge y)^* \Rightarrow (x \wedge y)^* \to z \leq x^* \to z$ and similarly $(x \wedge y)^* \to z \leq x^* \to z$. Because L is a De Morgan residuated lattice, we have $x^* \leq t \to z, y^* \leq t \to z \Rightarrow (x \wedge y)^* = x^* \vee y^* \leq t \to z \Rightarrow (x \wedge y)^* \leq t \to z \Rightarrow t \leq (x \wedge y)^* \to z$. $\square$

Lemma 2.4. *Let $x, y, z \in L$ and $n \geq 2$. Then:*

(c_{15}) $x \oplus (y \oplus z) = y \oplus (x \oplus z)$ *and* $1 \oplus x = x \oplus 1 = 1$ *and* $x \boxplus x^* = 1$*;*

(c_{16}) *$x^* \odot y^* = (x \boxplus y)^*$ and $[(x^*)^n]^* = nx$;*

(c_{17}) *If L is a De Morgan residuated lattice $x \wedge (y \oplus z) \leq (x \wedge z) \oplus (x \wedge z)$, $x \wedge y = x \wedge z = 0$ then $x \wedge (y \oplus z) = 0$.*

Proof. (c_{15}) $x \oplus (y \oplus z) = x^* \to (y^* \to z) = (y^* \to (x^* \to z)) = y \oplus (x \oplus z)$.

Also, $1 \oplus x = 1^* \to x = 0 \to x = 1$, $x \oplus 1 = x^* \to 1 = 1$ and $x \boxplus x^* = (x^* \odot x^{**})^* = 1$.

(c_{16}) $x^* \odot y^* = (x^* \odot y^*)^{**} = (x^* \to y^{**})^* = (x \boxplus y)^*$. The proof that $[(x^*)^n]^* = nx$ for arbitrary n is a mathematical induction argument. $2x = x \boxplus x = x^* \to x^{**} = (x^* \odot x^*)^* = [(x^*)^2]^*$. If we suppose that $nx = [(x^*)^n]^*$, then $(n+1)x = x \boxplus (nx) = x^* \to (nx)^{**} = x^* \to [(x^*)^n]^* = [(x^*)^{n+1}]^*$.

(c_{17}) From (c_{14}) we have $(x \wedge y) \oplus (x \wedge z) = [x \oplus (x \wedge z)] \wedge [y \oplus (x \wedge z)] = (x \oplus x) \wedge (x \oplus z) \wedge (y \oplus z) \wedge (y \oplus z) \geq x \wedge (y \oplus z)$ since by (c_{11}), $x \oplus x, x \oplus z, y \oplus x \geq x$. If $x \wedge y = x \wedge z = 0$, then $x \wedge (y \oplus z) \leq 0 \oplus 0 = 0^* \to 0 = 1 \to 0 = 0$, so $x \wedge (y \oplus z) = 0$. □

3 Ideals in residuated lattices

Let L be a residuated lattice. A nonempty subset I of a residuated lattice L will be called an ideal of L, (see [13], [14]) if it satisfies:

(I_1) If $x \leq y$ and $y \in I$, then $x \in I$;

(I_2) If $x, y \in I$, then $x \oplus y \in I$.

An ideal I called *proper* if $I \neq L$ (that is, $1 \notin I$). We denoted by $Id(L)$ the set of all ideals of L. If $I \in Id(L)$, then $0 \in I$ and $x \in I$ iff $x^{**} \in I$, (see [14]). Also, since $x, y \leq x \vee y \leq x \oplus y$, if $x, y \in I$ then $x \vee y \in I$, so I is a Lattice ideal.

Remark 3.1. *$I \in Id(L)$ iff it satisfies the conditions (I_1) and $(I_2^{'})$: $x, y \in I$ implies $x \boxplus y \in I$. Indeed , if $I \in Id(L)$ then $x, y \in I$ implies $y^{**} \in I$, so, $x \oplus y^{**} = x \boxplus y \in I$. Conversely, if $I \subseteq L$ satisfies the conditions (I_1) and $(I_2^{'})$, then $x \oplus y \leq x \boxplus y$, for every $x, y \in I$, so, $x \oplus y \in I$ and $I \in Id(L)$.*

Let L be a residuated lattice and $I \in Id(L)$. In (see [14]), on L is defined as a congruence relation $x \sim_I y$ iff $(x \to y)^*$, $(y \to x)^* \in I$. Moreover, $I = \{x \in L : x \sim_I 0\}$.

As an immediate consequence we have:

Let L be a residuated lattice. For $x \in L$ we denote by x/I the congruence class of x concerning to $\sim_I$ by x/I and the quotient set $L/\sim_I$ by L/I. Since $\sim_I$ is a congruence on L, L/I becomes a residuated lattice with the natural operations induced from those of L.

Clearly, in L/I, $\mathbf{0} = 0/I = \{x \in L : x \in I\}$, $\mathbf{1} = 1/I = \{x \in L : x^* \in I\}$ and for $x, y \in L$, $x/I \leq y/I$ iff $(x \to y)^* \in I$.

For a nonempty subset S of L, we denoted by $(S]$ the ideal of L generated by S and $x \in L$ we denoted by $(x] = (\{x\}]$.

Also, for $I \in Id(L)$ and $x \in L$ we denote by $I(x) = (I \cup \{x\}]$.

Proposition 3.2. *(See [5], [4]) Let L be a residuate lattice, $S \subseteq L$ a nonempty subset, $x, y \in L$ and $I \in Id(L)$. Then:*

(i) $(S] = \{z \in L : z \leq s_1 \boxplus ... \boxplus s_n$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$ and $(x] = \{z \in L : z \leq nx$, for some $n \geq 1\}$;

(ii) $I(x) = \{z \in L : z \leq i \boxplus nx$, for some $i \in L$ and $n \geq 0\}$and $I(x \wedge y) \subseteq I(x) \cap I(y) \subseteq I(x^{**} \wedge y^{**})$;

(iii) $(Id(L), \subseteq)$ is a complete Brouwerian lattice, where for $I_1, I_2 \in I_d(L), I_1 \wedge I_2 = I_1 \cap I_2$ and $I_1 \vee I_2 = (I_1 \cup I_2]$.

Remark 3.3. *If $e \in B(L)$, then $(e] = \{z \in L : z \leq e\}$, since $e \boxplus e = e^* \to e^{**} = e^* \to e = e$, so $ne = e$, for every $n \geq 1$.*

In a residuated lattice L, the order of an element $x \in L$, denoted by $ord(x)$, is the smallest natural number n such that $x^n = 0$ and we write $ord(x) = n$. If no such n exists (that is, $x^n \neq 0$ for every $n \geq 1$) we say that the order of x is *infinite* and we write $ord(x) = \infty$.

A residuate lattice L is called *locally finite* if every non-unit element of L has finite order.

Lemma 3.4. *Let L be a residuated lattice and $x \in L$. Then there is $I \in Id(L)$ proper such that $x \in I$ iff $ord(x^*) = \infty$.*

Proof. Let $I \in Id(L)$ proper ideal and $x \in I$ such that $ord(x^*) \neq \infty$. Then there is $n \geq 1$ such that $(x^*)^n = 0$ so, $[(x^*)^n]^* = 1$. From (c_{16}), $[(x^*)^n]^* = nx \in I$, thus, $1 \in I$, a contradiction so $ord(x^*) = \infty$.

Conversely, suppose that $ord(x^*) = \infty$. If $(x]$ is not proper then $1 \in (x]$, thus, $1 = nx$, so $0 = (nx)^*$, for some $n \geq 1$. Using (c_{16}), $(x^*)^n = 0$, so $ord(x^*) \neq \infty$, a contradiction. Thus, $(x]$ is proper. □

Using Lemma 3.4, we deduce that:

Proposition 3.5. *If L is a residuated lattice and $x \in L$, then $(x]$ is proper iff $ord(x^*) = \infty$*

In a residuated lattice L, an ideal $P \in Id(L)$ is called *prime*, (see [15]) if $P \neq L$ and P is a prime element in $(Id(L), \subseteq)$, that is, if $I, J \in Id(L)$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

We denote by $\mathcal{P}(L)$ the set of prime of L. Since $(Id(L), \subseteq)$ is a distributive lattice, meet-irreductible and meet-prime elements coincide, so, $P \in \mathcal{P}(L)$ iff $[I, J \in Id(L)$ with $I \cap J = P$, implies $I = P$ or $J = P]$.

Theorem 3.6. *Let L be a residuated lattice and $P \in Id(L)$. Then $P \in \mathcal{P}(L)$ iff $[x^{**} \wedge y^{**} \in P$ implies $x \in P$ or $y \in P]$.*

Proof. Let $P \in \mathcal{P}(L)$ and $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. By Proposition 3.2, $P(x) \cap P(y) = P(x^{**} \wedge y^{**}) = P$. Since $P \in \mathcal{P}(L)$ we deduce that $P(x) = P$ or $P(y) = P$, that is, $x \in P$ or $y \in P$.

Conversely, let $I, J \in Id(L)$ such that $I \cap J \subseteq P$. If we suppose that $I \nsubseteq P$ and $J \nsubseteq P$, then there are $x \in I$ and $y \in J$ such that $x, y \notin P$. Then $x^{**} \in I, y^{**} \in J$ so $x^{**} \wedge y^{**} \in I \cap J \subseteq P$. By hypothesis, $x \in P$ or $y \in P$, a contradiction. □

Theorem 3.7. *Let L be a residuated lattice and $P \in Id(L)$. We consider the following assertions:*

(i) $P \in \mathcal{P}(L)$;

(ii) *If $x \wedge y \in P$, then $x \in P$ or $y \in P$;*

(iii) *For every $x, y \in L, (x \to y)^* \in P$ or $(y \to x)^* \in P$;*

(iv) *L/P is a chain.*

Then $(ii), (iii), (iv) \Rightarrow (i)$ but $(i) \not\Rightarrow (ii), (iii), (iv)$.

Proof. $(ii) \Rightarrow (iii)$. Let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. Since $x \wedge y \leq x^{**} \wedge y^{**}$ we deduce that $x \wedge y \in P$. From hypothesis, $x \in P$ or $y \in P$. Using Theorem 3.6, we conclude that $P \in \mathcal{P}(L)$.

$(iii) \Rightarrow (i)$. Let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$ and we suppose that $(x \to y)^* \in P$.It follows that $(x \to y)^* \oplus (x^{**} \wedge y^{**}) = (x \to y)^{**} \to (x^{**} \wedge y^{**}) \in P$. From (c_8), $(x \to y)^{**} \leq x^{**} \to y^{**}$, so $(x^{**} \to y^{**}) \to (x^{**} \wedge y^{**}) \leq (x \to y)^{**} \to (x^{**} \wedge y^{**})$. Since P is an ideal and $x^{**} \leq (x^{**} \to y^{**}) \to (x^{**} \wedge y^{**})$, we deduce that $x^{**} \in P$, thus $x \in P$. Similarly, if $(y \to x)^* \in P$ we obtain $y \in P$, so $P \in \mathcal{P}(L)$.

$(iv) \Rightarrow (i)$. Suppose that L/P is a chain and let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. Then $x^{**}/P \wedge y^{**}/P = 0/P$, so $x^{**}/P = 0/P$ or $y^{**}/P = 0/P$. We deduce that, $x^{**} \in P$ or $y^{**} \in P$, so, $x \in P$ or $y \in P$. Hence $P \in \mathcal{P}(L)$.

$(i) \nRightarrow (ii), (iii), (iv)$. If we consider the residuated lattice $L = \{0, b, c, d, 1\}$ from Example 2.2, it is easy to see that $0^{**} = 0, b^{**} = c^{**} = d^{**} = d$ and $1^{**} = 1$. Obviously, $P = \{0\} \in \mathcal{P}(L)$ because if $x^{**} \wedge y^{**} = 0$ implies $x = 0$ or $y = 0$. But $b \wedge c = 0 \in P$ and $b, c \notin P$, thus $(i) \nRightarrow (ii)$.

Also, $(i) \nRightarrow (iii)$ since $(b \to c)^* = (c \to b)^* = d^* = d \notin P$

Also, for $b/P = \{x \in L : (b \to x)^* = (x \to b)^* = 0\} = \{x \in L : b \to x = x \to b = 1\} = \{b\}$ and $c/P = \{x \in L : (c \to x)^* = (x \to c)^* = 0\} = \{x \in L : c \to x = x \to c = 1\} = \{c\}$. But $\{b\} \nsubseteq \{c\}$ and $\{c\} \nsubseteq \{b\}$, so, L/P is not a chain, thus, $(i) \nRightarrow (iv)$. □

If L is a De Morgan residuated lattice then $P \in \mathcal{P}(L)$ iff $[x \wedge y \in P$ implies $x \in P$ or $y \in P]$, (see [11]).

Corollary 3.8. *Let L be an MTL algebra and $P \in Id(L)$. Then the following conditions are equivalent:*

(i) $P \in \mathcal{P}(L)$;

(ii) If $x \wedge y \in P$, *then* $x \in P$ *or* $y \in P$;

(iii) For every $x, y \in L, (x \to y)^* \in P$ *or* $(y \to x)^* \in P$;

(iv) L/P *is a chain;*

(v) For $x, y \in L$, *if* $x \wedge y = 0$, *then* $x \in P$ *or* $y \in P$;

(vi) For every $x, y \in L, x \odot y^* \in P$ *or* $x^* \odot y \in P$.

Proof. $(i) \Rightarrow (ii)$. (See [11]).

$(ii) \Rightarrow (iii)$. From $(x \to y) \vee (y \to x) = 1$, for every $x, y \in L$, we deduce that $(x \to y)^* \wedge (y \to x)^* = 0 \in P$. Thus, $(x \to y)^* \in P$ or $(y \to x)^* \in P$.

$(iii) \Rightarrow (i)$ From Theorem 3.7.

$(iv) \Rightarrow (ii)$. If L/P is a chain and $x \wedge y \in P$ then $x/P \wedge y/P = 0/P$, so $x/P = 0/P$ or $y/P = 0/P$, that is, $x \in P$ or $y \in P$.

$(ii) \Rightarrow (v)$. Obviously, $x \wedge y = 0 \in P$, so $x \in P$ or $y \in P$.

$(v) \Rightarrow (iv)$. Let $x/P, y/P \in L/P$; since $(x \to y)^* \wedge (y \to x)^* = 0 \in P$, we deduce that $(x \to y)^* \in P$ or $(y \to x)^* \in P$, so $x/P \leq y/P$ or $y/P \leq x/P$.

$(i) \Rightarrow (iv)$. Since $(x \odot y^*)^{**} \wedge (x^* \odot y)^{**} = (y^* \to x^*)^* \wedge (x^* \to y^*) = [(y^* \to x^*) \vee (x^* \to y^*)]^* = 1^* = 0 \in P$, we deduced that $x \odot y^* \in P$ or $x^* \odot y \in P$.

$(vi) \Rightarrow (i)$. Suppose that $x^{**} \wedge y^{**} \in P$ and $x \odot y^* \in P$. It follows that $(x \odot y^*) \oplus (x^{**} \wedge y^{**}) \in P$. From (c_{14}),$(x \odot y^*) \oplus (x^{**} \wedge y^{**}) = [(x \odot y^*) \oplus x^{**}] \wedge [(x \odot y^*) \oplus y^{**}] \geq x \wedge x = x$, since $(x \odot y^*) \oplus x^{**} = (x \odot y^*)^* \to x^{**} \geq x^{**} \geq x$ and $(x \odot y^*) \oplus y^{**} = (x \odot y^*)^* \to y^{**} = (x \to y^{**}) \to y^{**} \geq x$. We conclude that $x \in P$, so, $P \in \mathcal{P}(L)$.

Similarly, if $x^* \odot y \in P$, we obtain that $y \in P$, so, P is a prime ideal of L. □

In general, in a residuated lattice L , if $P \in \mathcal{P}(L)$ and I is a proper ideal such that $P \subseteq I$, then I is not prime. Also, the set of proper ideals including a prime ideal is not a chain, (see [5]).

Theorem 3.9. *If L is an MTL algebra then:*

(i) Every proper ideal of L that contains a prime ideal is prime;

(ii) For every prime ideal P of L, the set $\mathfrak{I}_P = \{I \in Id(L) : P \subseteq I$ *and* $I \neq L\}$ *is totally ordered by inclusion.*

Proof. (i). Let $P \in \mathcal{P}(L)$ and I a proper ideal of L such that $P \subseteq I$ and $x, y \in L$. From Corollary 3.8, (vi), $x \odot y^* \in P$ or $y \odot x^* \in P$. Since $P \subseteq I$, we obtain $x \odot y^* \in I$ or $y \odot x^* \in I$, so $I \in \mathcal{P}(L)$.

(ii). Let $I_1, I_2 \in \mathfrak{I}$ and suppose that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$. Then, there are $x_1, x_2 \in L$ such that $x_1 \in I_1 \backslash I_2$ and $x_2 \in I_2 \backslash I_1$. Since P is prime, $x_1 \odot x_2^* \in P \subseteq I_2$ or $x_2 \odot x_1^* \in P \subseteq I_1$. We deduce that $x_2 \oplus (x_1 \odot x_2^*) = x_2^* \rightarrow (x_1 \oplus x_2^*) \in I_2$ or $x_1 \oplus (x_1^* \odot x_2) = x_1^* \rightarrow (x_1^* \odot x_2^*) \in I_2$ or $x_1 \oplus (x_1^* \odot x_2) = x_1^* \rightarrow (x_1^* \odot x_2) \in I_1$. But $x_1 \leq x_2 \oplus (x_1 \odot x_2^*)$ and $x_2 \leq x_1 \oplus (x_1^* \odot x_2)$, so $x_1 \in I_2$ or $x_2 \in I_1$, a contradiction.
□

Remark 3.10. *(i) In a residuated lattice L, if $(P_i)_{i \in I} \subseteq \mathcal{P}(L)$ is a totally ordered family of prime ideals of L then $P = \cap_{i \in I} P_i \in Spec(L)$ and $Q = \vee_{i \in I} P_i \in Spec(L)$. Indeed, let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$, if by contrary $x \notin P$ and $y \notin P$ then there are $i_1, i_2 \in I$ such that $x \notin P_{i_1}$ and $y \notin P_{i_2}$. Since P_{i_1}, P_{i_2} are prime ideals and $x^{**} \wedge y^{**} \in P_{i_1}, P_{i_2}$ then $x \in P_{i_2}$ and $y \in P_{i_1}$. Since the family $(P_i)_{i \in I}$ is totally ordered, then $P_{i_1} \subseteq P_{i_2}$ or $P_{i_2} \subseteq P_{i_1}$. If $P_{i_1} \subseteq P_{i_2}$ then $y \in P_{i_2}$, a contradiction. Similarly, if $P_{i_2} \subseteq P_{i_1}$. It follows that $x \in P$ or $y \in P$, that is, $P \in \mathcal{P}(L)$. Also, we remark that $Q = \cup_{i \in I} P_i$ and the proof for $Q \in \mathcal{P}(L)$ is obvious.*

(ii) In general, an intersection of prime ideals in a residuated lattice is not necessary a prime ideal. For example, if we consider the residuated lattice L from Example 2.1, then $Id(L) = \{\{0\}, \{0, a\}, \{0, b\}, L\}$ and $\mathcal{P}(L) = \{\{0, a\}, \{0, b\}\}$. $\{0\} = \{0, a\} \cap \{0, b\} \notin \mathcal{P}(L), a^{**} \wedge b^{**} = 0$ but $a, b \neq 0$.

Theorem 3.11. (Prime ideal theorem , see [5]) *Let L be a residuated lattice. If $I \in Id(L)$ and F is a filter of the lattice $(L, \wedge, \vee, 0, 1)$ such that $I \cap F = \emptyset$, then there is $P \in \mathcal{P}(L)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Obviously, in a residuated lattice, any proper ideal of L can be extended to a prime ideal.

Corollary 3.12. *Let L be a residuated lattice and $x \in L$. Then $ord(x^*) < \infty$ iff $x \notin P$ for every $P \in \mathcal{P}(L)$.*

Proof. Suppose that $ord(x^*) < \infty$ and there exists $P \in \mathcal{P}(L)$ such that $x \in P$. Thus, there is $n \geq 1$ such that $(x^*)^n = 0$. Hence $1 = [(x^*)^n]^* = nx \in P$, so $P = L$, a contradiction. Conversely, we suppose that $x \notin P$ for every $P \in \mathcal{P}(L)$ and $ord(x^*) = \infty$. By Proposition 3.5 and Theorem 3.11, $(x]$ is proper so, there is $P \in \mathcal{P}(L)$ such that $(x] \subseteq P$, hence $x \in P$, is a contradiction. □

As immediate consequences of Theorem 3.11 we have:

Corollary 3.13. *If L is a residuated lattice then $\cap\{P \in \mathcal{P}(L)\} = \{0\}$ and for every $I \in Id(L), I = \cap\{P \in \mathcal{P}(L) : I \subseteq P\}$.*

Proof. If $x \neq 0$ there is a prime ideal $P \in \mathcal{P}(L)$ such that $x \notin P$, so $x \notin \cap\{P \in \mathcal{P}(L)\}$. □

Proposition 3.14. *Let L be a residuated Lattice, $L_1 \subseteq L$ a subalgebra of L and $P_1 \in \mathcal{P}(L_1)$. Then there exists $P \in \mathcal{P}(L)$ such that $P_1 = P \cap L_1$.*

Proof. Let I be the ideal generated by P_1 in L. Then $I = \{x \in L : x \leq x_1 \boxplus ... \boxplus x_n$, for some $x_1, ..., x_n \in P\}$. Then $I \cap (L_1 \backslash P_1) = \emptyset$. Indeed, if there is $i \in I \cap (L_1 \backslash P_1)$, then $i \in I, i \in L_1$ and $i \notin P_1$. From $i \in I$, there exists $p \in P_1$ such that $i \leq p$, hence $i \in P_1$, is a contradiction.

Clearly, $0 \notin L_1 \backslash P_1$ and $1 \in L_1 \backslash P_1$. Let $x, y \in L_1 \backslash P_1$. Then $x, y \notin P_1$ so $x \wedge y \notin P_1$ (since P_1 is prime in L_1). Thus, $x \wedge y \in L_1 \backslash P_1$, hence $L_1 \backslash P_1$ is a $\wedge -$ *closed* subset of L. By Theorem 3.11, there exists $P \in \mathcal{P}(L)$ such that $I \subseteq P$ and $P \cap (L_1 \backslash P_1) = \emptyset$, hence $P \cap L_1 \subseteq P_1$. Then $P_1 \subseteq I \cap L_1 \subseteq P \cap L_1 \subseteq P_1$, so $P_1 = P \cap L_1$.
□

We recall that an ideal M of a residuated lattice L is called *maximal*, (see [5], [14]), if it is proper and is not contained in any other proper ideal of L, i.e., for every ideal $I \neq L$, if $M \subseteq I$, then $M = I$.

We denote by $\mathcal{M}(L)$ the set of maximal ideals of L. Obviously, $\mathcal{M}(L) \subseteq \mathcal{P}(L)$.

Also, if M is a proper ideal of a residuated lattice L, then $M \in \mathcal{M}(L)$ iff for every $x \in L, x \notin M$ iff $(nx)^* \in M$, for some $n \geq 1$, (see [5], [15]).

Theorem 3.15. *Let L be a residuated lattice and $M \in Id(L)$ be a proper ideal. Then $M \in \mathcal{M}(L)$ iff L/M is locally finite.*

Proof. Suppose that $M \in \mathcal{M}(L)$ and let $x/M \neq 1/M$. Then $x^* \notin M$, so there is a natural number $n \geq 1$ such that $(nx^*)^* = [(x^{**})^n]^{**} \in M$. Since $M \in Id(L)$, $(x^{**})^n \in M$, so $x^n \in M$. We deduce that $x^n/M = (x/M)^n = 0/M$, so, L/M is locally finite.

Conversely, let $I \in Id(L)$, $I \neq M$ be an ideal of L such that $M \subset I$. Then there is $x \in I \backslash M$, so, $x^*/M \neq 1/M$ (since if we suppose that $x^*/M = 1/M$, thus $x^{**} \in M$, so $x \in M$). But L/M is locally finite, thus $(x^*/M)^n = 0/M$, for some $n \geq 1$. We conclude that $(x^*)^n \in M \subset I$. Since I is an ideal and $x \in I$, then $nx = [(x^*)^n]^* \in I$, so $(x^*)^n \oplus [(x^*)^n]^* = [(x^*)^n]^* \rightarrow [(x^*)^n]^* = 1 \in I$. Thus $I = L$ and $M \in \mathcal{M}(L)$.
□

As an immediate consequence of Zorn's lemma, every proper ideal of L can be extended to a maximal ideal.

Theorem 3.16. *Every prime ideal of an MTL algebra L is contained in a unique maximal ideal of L.*

Proof. For $P \in \mathcal{P}(L)$, the set $\mathfrak{I}_P = \{I \in Id(L) : P \subseteq I \text{ and } I \neq L\}$ is totally ordered by inclusion, from Theorem 3.9. Therefore, $\overline{P} = \cup_{I \in \mathfrak{I}_P}$ is proper, since $1 \notin \overline{P}$, so $\overline{P}$ is the only maximal ideal containing P. □

We recall that a residuated lattice L is called *local* if it has a unique maximal ideal (see [16]).

Proposition 3.17. *Let L be a residuated lattice and $\mathfrak{I} = \{x \in L : ord(x^*) = \infty\}$. The following assertions are equivalent:*

(i) $\mathfrak{I} \in Id(L)$;

(ii) $(\mathfrak{I}]$ *is a proper ideal of L;*

(iii) *L is local;*

(iv) $\mathcal{M}(L) = \{\mathfrak{I}\}$.

Proof. $(i) \Rightarrow (ii)$. Suppose $\mathfrak{I} \in Id(L)$ implies $(\mathfrak{I}] = \mathfrak{I} \neq L$ since $1 \notin \mathfrak{I}$.

$(ii) \Rightarrow (i)$. Obviously, $0 \in \mathfrak{I}$. Let $x, y \in L$ such that $x \leq y$ and $y \in \mathfrak{I}$. Then $ord(y^*) = \infty$. Since $y^* \leq x^*$ we deduce that $ord(x^*) = \infty$, thus, $x \in \mathfrak{I}$. Let now, $x, y \in \mathfrak{I}$. Since $\mathfrak{I} \subseteq (\mathfrak{I}]$ we have $x, y \in (\mathfrak{I}]$. If we suppose by contrary that $x \boxplus y \notin \mathfrak{I}$, then there is $n \geq 1$ such that $[(x \boxplus y)^*]^n = 0$. But $[(x \boxplus y)^*]^n = (x^* \odot y^*)^n = (x^*)^n \odot (y^*)^n = 0$. Thus, $1 = [(x^*)^n \odot (y^*)^n]^* = [(x^*)^n]^{**} \rightarrow [(y^*)^n]^* = (nx)^* \rightarrow (ny) = (nx) \oplus (ny)$, a contradiction since $(\mathfrak{I}]$ is proper.

We conclude that $\mathfrak{I} \in Id(L)$.

$(iv) \Rightarrow (iii)$. Clearly.

$(i) \Rightarrow (iv)$. To prove that $\mathfrak{I}$ is maximal, let $x \in L$ such that $x \notin \mathfrak{I}$. Then $(x^*)^n = 0$ for some $n \geq 1$. Thus, $(nx)^* = [(x^*)^n]^{**} = 0^{**} = 0 \in \mathfrak{I}$, so $\mathfrak{I} \in Max(L)$. To prove that $\mathfrak{I}$ is the unique maximal ideal of L, we consider $I_1 \in Id(L)$ such that $I_1 \neq L$. If by contrary, $I_1 \nsubseteq \mathfrak{I}$, then there is $x \in I_1$ such that $x \notin \mathfrak{I}$. Then $(x^*)^n = 0$ for some $n \geq 1$, hence $1 = [(x^*)^n]^* = nx \in I_1$ and $I_1 = L$, a contradiction. Therefore $\mathfrak{I}$ contains all the proper ideals of L, thus, $\mathfrak{I}$ is the unique maximal ideal of L.

$(iii) \Rightarrow (iv)$ and (i). Let M be the unique maximal ideal of L. Since Proposition 3.5 every element $x \in \mathfrak{I}$ generates a proper ideal $(x]$ which can be extended to a maximal ideal M_x, we obtain $M = M_x$, so for every $x \in \mathfrak{I}, x \in M$ hence $\mathfrak{I} \subseteq M$. Since M is proper, from Lemma 3.4, $M \subseteq \mathfrak{I}$, hence $M = \mathfrak{I}$. □

Theorem 3.18. *In a local residuated lattice L, for every $x \in L, ord(x) < \infty$ or $ord(x^*) < \infty$.*

Proof. Suppose that there exists $x \in L$ such that $x^n > 0$ and $(x^*)^n > 0$ for every $n \geq 1$. Thus, $(x^{**})^n > 0$ for every $n \geq 1$. Then $x, x^* \in (\mathfrak{I}]$ so $x \boxplus x^* = 1 \in (\mathfrak{I}]$, so, $(\mathfrak{I}] = L$ in contradiction with Proposition 3.17.
□

4 Pure ideals in residuated lattices

Let L be a residuated lattice. For $x \in L$ we denote $x^{\perp} = \{y \in L : x \wedge y = 0\}$.

Lemma 4.1. *Let L be a De Morgan residuated lattice and $x, y \in L$, $e \in B(L)$. Then:*

(i) $x^{\perp} \in Id(L)$ *and* $x \leq y$ *implies* $y^{\perp} \subseteq x^{\perp}$*;*

(ii) $x^{\perp} = L$ *iff* $x = 0$*;*

(iii) $x^{\perp} \cap y^{\perp} = (x \oplus y)^{\perp} = (x \vee y)^{\perp}$ *and* $e^{\perp} = (e^*]$.

(iv) $x^{\perp} \cap y^{\perp} = (x \boxplus y)^{\perp}$.

Proof.(i) Let $t, z \in L$ such that $t \leq z$ and $z \in x^{\perp}$. Then $x \wedge z = 0$. Since $x \wedge t \leq x \wedge z = 0$, we deduce that $t \in x^{\perp}$. Also, if $t, z \in x^{\perp}$, then $x \wedge z = x \wedge y = 0$. Using (c_{17}), $x \wedge (t \oplus z) = 0$, so $t \oplus z \in x^{\perp}$ and $x^{\perp} \in Id(L)$. Now, suppose that $x \leq y$ and let $z \in y^{\perp}$. Then $z \wedge x \leq z \wedge y = 0$, so $z \wedge x = 0$, thus , $z \in x^{\perp}$.

(ii) $x^{\perp} = L$ iff $1 \in x^{\perp}$ iff $1 \wedge x = 0$ iff $x = 0$.

(iii). From $x, y \leq x \oplus y$, we deduce that $x, y \leq x \vee y \leq x \oplus y$. Using (i), $(x \oplus y)^{\perp} \subseteq (x \vee y)^{\perp} \subseteq x^{\perp} \cap y^{\perp}$. Now $z \in (x \oplus y)^{\perp}$. Then $x \wedge z = y \wedge z = 0$. Using (c_{17}), $z \wedge (x \oplus y) = 0$, so, $z \in (x \oplus y)^{\perp}$ and $x^{\perp} \cap y^{\perp} \subseteq (x \oplus y)^{\perp}$ and we have obtained the equalities.

Finally, for $e \in B(L)$, since $e \wedge e^* = 0$ we deduce that $e^* \in e^{\perp}$ so, $(e^*] \subseteq e^{\perp}$. Let $x \in e^{\perp}$. Then $x \wedge e = 0$. Since $e^* \in B(L), x \wedge e^* = x \odot (x \to e^*) = x \odot (x \odot e)^* = x \odot 0^* = x \odot 1 = x$, so $x \leq e^*$, that is, $x \in (e^*]$, thus, $e^{\perp} = (e^*]$.

(iv) From $x, y \leq x \boxplus y$ we deduce $(x \boxplus y)^{\perp} \subseteq x^{\perp} \cap y^{\perp}$. Now we consider $z \in x^{\perp} \cap y^{\perp}$. Then $x \wedge z = y \wedge z = 0$ From (c_{13}), $z \wedge (x \boxplus y) \leq (z^{**} \wedge x^{**}) \boxplus (z^{**} \wedge y^{**}) = (z \wedge x)^{**} \boxplus (z \wedge y)^{**} = 0 \boxplus 0 = 0^* \to 0^{**} = 1 \to 0 = 0$. We deduce that $z \in (x \boxplus y)^{\perp}$, thus, $x^{\perp} \cap y^{\perp} = (x \boxplus y)^{\perp}$. $\square$

For a residuated lattice L and $I \in Id(L)$ we denote by $\sigma(I) = \{x \in L$: there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1\}$. For MV-algebras, (see [6]).

Also, for a distributive lattice $(\mathcal{L}, \wedge, \vee, 0, 1)$ we denote by $Id(\mathcal{L})$ the set of ideals of L, $Spec(\mathcal{L})$ the set of prime ideals and by $Max(\mathcal{L})$ the set of maximal ideals of $\mathcal{L}$. About notations involving lattices and their spectral topologies, (see [8]).

We recall, (see [8], [9]), that if L is a distributive lattice $\mathcal{L}$, if $I \in Id(L)$, then $\sigma(I) = \{x \in L :$ there are $i \in I$ and $y \in x^{\perp}$ such that $i \vee y = 1\} \in Id(\mathcal{L})$ and $\sigma(I) \subseteq I$. Moreover, an ideal $I \in Id(\mathcal{L})$ is called *pure* if $\sigma(I) = I$, (see [8], [9]).

We denote by $Pure(\mathcal{L})$ the set of pure ideal of $\mathcal{L}$.

Remark 4.2. *In a residuated lattice L, if $I \in Id(L)$, then $\sigma(I) = I'$ where $I' = \{x \in L$: there are $i \in I$ and $y \in x^{\perp}$ such that $i \boxplus y = 1\}$. Obviously, $\sigma(I) \subseteq I'$ since $i \oplus y \leq i \boxplus y$. Conversely, let $x \in I'$. Then there are $i \in I$ and $y \in x^{\perp}$ such that $1 = i \boxplus y = i \oplus y^{**}$. Since $x^{\perp} \in Id(L)$ and $y \in x^{\perp}$ we deduce that $y^{**} \in x^{\perp}$, so $x \in \sigma(I)$ and $I' \subseteq \sigma(I)$.*

Theorem 4.3. *Let L be a De Morgan residuated lattice and $I, J \in Id(L)$. Then*

(i) $\sigma(I) \in Id(L)$ *and* $\sigma(I) \subseteq I$*;*

(ii) $I \subseteq J$ *implies* $\sigma(I) \subseteq \sigma(J)$*;*

(iii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$ *and* $\sigma(I) \vee \sigma(J)) \subseteq \sigma(I \vee J)$.

(iv) $\sigma(I) \neq \{0\}$ *then there is* $i \in I$ *such that* $ord(i^{**}) = \infty$.

Proof. (i). Let $x_1, x_2 \in L, x_1 \leq x_2$ and $x_2 \in \sigma(I)$, then there are $i \in I$ and $y \in x_2^{\perp}$ such that $i \oplus y = 1$.

Since $x_2^{\perp} \subseteq x_1^{\perp}$ so $y \in x_1^{\perp}$. We deduce that $x_1 \in \sigma(I)$.

For $x_1, x_2 \in \sigma(I)$, there are $i_1, i_2 \in I$ and $y_1 \in x_1^{\perp}, y_2 \in x_2^{\perp}$ such that $i_1 \oplus y_1 = i_2 \oplus y_2 = 1$. Denoting $i = i_1 \oplus i_2 \in I$ and $y = y_1 \wedge y_2$, we have $y \wedge (x_1 \oplus x_2) \leq (y \wedge x_1) \oplus (y \wedge x_2) = 0 \oplus 0 = 0$, so $y \in (x_1 \oplus x_2)^{\perp}$.

Also, $i \oplus y = (i_1 \oplus i_2) \oplus (y_1 \wedge y_2) = i_1 \oplus [(i_2 \oplus y_1) \wedge (i_2 \oplus y_2)] = i_1 \oplus [(i_2 \oplus y_1) \wedge 1] = i_1 \oplus (i_2 \oplus y_1) = i_2 \oplus (i_1 \oplus y_1) = i_2 \oplus 1 = 1$, so $x_1 \oplus x_2 \in \sigma(I)$, that is $\sigma(I) \in Id(L)$.

To prove that $\sigma(I) \subseteq I$, let $x \in \sigma(I)$. Then there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1$. We have $i^{**} = i \oplus 0 = i \oplus (x \wedge y) = (i \oplus x) \wedge (i \oplus y) = (i \oplus x) \wedge 1 = i \oplus x$. Hence $x \leq i^{**}$, so $x \in I$and $\sigma(I) \subseteq I$.

(ii) Obviously.

(iii). By (ii) $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$. Let $x \in \sigma(I) \cap \sigma(J)$. Then there are $i \in I, j \in J, y_1, y_2 \in x^{\perp}$ such that $i \oplus y_1 = j \oplus y_2 = 1$. Since $x^{\perp}, I, J$ are ideals we deduce that $y = y_1 \oplus y_2 \in x^{\perp}$ and $k = i \wedge j \in I \cap J$. Then $k \oplus y = (i \wedge j) \oplus y = (i \oplus y) \wedge (j \oplus y) = [i \oplus (y_1 \oplus y_2)] \wedge [j \oplus (y_1 \oplus y_2 \wedge)] = [(i \oplus y_1) \oplus y_2] \wedge [y_1 \oplus (j \oplus y_2)] = (1 \oplus y_2) \wedge (y_1 \oplus 1) = 1 \wedge 1 = 1$. We deduce that $x \in \sigma(I \cap J)$, so $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$. Hence $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$. From (ii), we obtain $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$.

(iv). For $x \in \sigma(I), x \neq 0$, there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1$. Then $i^* \to y = 1$, so $i^* \leq y$ and $(y^*)^n \leq (i^{**})$, for every $n \geq 1$. Obviously, if we prove that $ord(y^*) = \infty$, then $ord(i^{**}) = \infty$. From $x \wedge y = 0$ we deduce that $x^* \vee y^* = 1$, so, from (c_9), $(x^*)^n \vee (y^*)^n = 1$, for every $n \geq 1$. If suppose by contrary that $(y^*)^n = 0$ for some $n \geq 1$, then $(x^*)^n = 1$, so, $x^* = 1$ and $x^{**} = 0$. Thus, $x = 0$, a contradiction. □

Corollary 4.4. *If L is a local De Morgan residuated lattices and $I \in Id(L)$ is proper,then $\sigma(I) = \{0\}$.*

Proof. Suppose $\sigma(I) \neq \{0\}$. From Theorem 4.3 (iv), there is $i \in I$ such that $ord(i^{**}) = \infty$. Since L is local, by Theorem 3.18, $ord(i^*) < \infty$, so, $(i^*)^n = 0$ for some $n \geq 1$. Thus, $1 = [(i^*)^n]^* = ni \in I$, so $I = L$, a contradiction. □

Definition 4.5. *An ideal I of a residuated lattice L is called pure in L if $\sigma(I) = I$.*

For a residuated lattice L, we denote by $Pure(L)$ the set of pure ideals of L.

Remark 4.6. *For a residuated lattice L,*

(*i***)** *$\{0\}, L \in Pure(L)$. Indeed, since $\{0\} \subseteq \sigma(\{0\}) \subseteq \{0\}$ we deduce that $\sigma(\{0\}) = \{0\}$. Also, since for every $x \in L$ there are $1 \in L$ and $0 \in x^{\perp}$ such that $1 \oplus 0 = 1$ we deduce that $x \in \sigma(L)$, so $\sigma(L) = L$.*

(ii) If $I, J \in Pure(L)$, then $I \cap J$ and $I \vee J \in Pure(L)$. Indeed, $\sigma(I) = I$ and $\sigma(J) = J$, so by Theorem 4.3, $\sigma(I \cap J) = \sigma(I) \cap \sigma(J) = I \cap J$, hence $I \cap J$ is a pure ideal in L. Also, we deduce that $I \vee J = \sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$, so, $\sigma(I \vee J) = I \vee J$, hence $I \vee J$ is pure in L.

By Corollary 4.4 we deduce that:

Corollary 4.7. *If L is a local MTL algebra, then the unique pure ideals in L are $\{0\}$ and L.*

Example 4.8. If we consider the residuated lattice $L = \{0, a, b, c, 1\}$ from Example 2.1 then $0^{\perp} = L, a^{\perp} = \{0, b\}, b^{\perp} = \{0, a\}$ and $1^{\perp} = c^{\perp} = \{0\}$. It is easy to prove that every ideal of L is a pure ideal, so $Pure = Id(L)$.

5 The Belluce lattice associated with a De Morgan residuated lattice

In this section, we consider L a De Morgan residuated lattice L.

On L we define the relation $\equiv (mod\mathcal{P}(L))$ on L by $x \equiv y(mod\mathcal{P}(L))$ iff for every $P \in \mathcal{P}(L), x \in P$ iff $y \in P$. Thus, $x \equiv y(mod\mathcal{P}(L))$ iff no prime $P \in \mathcal{P}(L)$ can separate x and y.

Lemma 5.1. *$\equiv (mod\mathcal{P}(L))$ is an equivalence relation compatible with $\wedge$ and $\vee$.*

Proof. Obviously, $\equiv (mod\mathcal{P}(L))$ is an equivalence relation on L. Let $x, y, z, t \in L$ such that $x \equiv y(mod\mathcal{S})$ and $z \equiv t(mod\mathcal{S})$. Also, let $P \in \mathcal{S}$ such that $x \vee z \in P$. Since $x, z \leq x \vee z$ then $x, z \in P$, $y, t \in P$ and $y \vee t \in P$. Then $y \oplus t \in P$. But P is an ideal and $y \vee t \leq y \oplus t$, so $y \vee t \in P$. Suppose now $x \wedge z \in P$, since P is prime then $x \in P$ or $z \in P$. Thus $y \in P$ or $t \in P$. In either case $y \wedge t \in P$. So, $\equiv (mod\mathcal{P}(L))$ is compatible with $\vee$ and $\wedge$. □

For every $x \in L$ we denote by $[x]$ the equivalence class of x and by $[L]_{\mathcal{S}}$ the set of these equivalence classes.

In this case, we denote $[L]_{\mathcal{S}}$ by $[L]$. On $[L]$ we define $[x] \wedge [y] = [x \wedge y], [x \vee y] = [x] \vee [y], \mathbf{0} = [0] = \cap\{P : P \in \mathcal{P}(L)\} = \{0\}$ and $\mathbf{1} = [1] = \{x \in L : x \notin P$, for every $P \in \mathcal{P}(L)\}$. Also, we define $[x] \leq [y]$ iff $[x] \wedge [y] = [x]$ iff $[x] \vee [y] = [y]$. Obviously, the relation $\leq$ is well defined and $([L], \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

Using the model of MV algebra, (see [2], [3]), $[L]$ will called *Belluce lattice associated with L.*

Lemma 5.2. *Let $x, y \in L$ then:*

(i) $x \leq y$ implies $[x] \leq [y]$;

(ii) $[x] = \mathbf{0}$ iff $x = 0$ and $[x] = \mathbf{1}$ iff $ord(x^*) < \infty$;

(iii) $[x \vee y] = [x \oplus y] = [x \boxplus y]$, so $[nx] = [x]$, for every $n \geq 1$.

Proof. (i). $x \leq y$ implies $x \wedge y = x$, so, $[x] = [x \wedge y] = [x] \wedge [y]$. Thus, $[x] \leq [y]$. (ii) $x = 0$, implies $[x] = \mathbf{0}$. Conversely, let $x \in L$ such that $[x] = \mathbf{0}$, then $x \in \cap\{P : P \in \mathcal{P}(L)\} = \{0\}$, since $0 \in P$ for every $P \in \mathcal{P}(L)$. Thus, $[x] = \mathbf{0}$ iff $x = 0$.

Now, let $x \in L$ such that $ord(x^*) < \infty$. Then there exists $n \geq 1$ such that $(x^*)^n = 0$, so, $(x^*)^n \in P$ for every $P \in \mathcal{P}(L)$. Hence $x \notin P$ for every $P \in \mathcal{P}(L)$, since if we suppose that there is $P \in \mathcal{P}(L)$ such that $x \in P$, then $(nx) \boxplus (x^*)^n \in P$. But $(nx) \boxplus (x^*)^n = [(x^*)^n]^* \boxplus (x^*)^n = [(x^*)^n]^{**} \to [(x^*)^n]^{**} = 1 \in P$, a contradiction. Hence $[x] = \mathbf{1}$. Conversely, suppose that $[x] = \mathbf{1}$ but $ord(x^*) = \infty$. Then, using Proposition 3.5 $(x]$ is proper there is $P \in \mathcal{P}(L)$ such that $(x] \subseteq P$. Thus $x \in P$, a contradiction. We conclude that $[x] = \mathbf{1}$ iff $ord(x^*) < \infty$.

(iii). Let $P \in \mathcal{P}(L)$ if $x \vee y \in P$, then $x, y \in P$, so $x \boxplus y \in P$. Conversely, since $x \vee y \leq x \oplus y \leq x \boxplus y \in P$ if $x \boxplus y \in P$ then $x \vee y \in P$. Using (i), $[x \vee y] = [x \oplus y] = [x \boxplus y]$. Obviously, $[nx] = [x]$, for every $n \geq 1$ since $[x \boxplus y] = [x \vee y]$. □

Theorem 5.3. *$([L], \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a distributive lattice.*

Proof. For $x, y, z \in L$, we have $[x] \vee ([y] \vee [z]) = ([x] \vee [y]) \wedge ([x] \vee [z])$ iff $[x \vee (y \wedge z)] = [(x \vee y) \wedge (x \vee z)]$. To prove this equality, let $P \in \mathcal{P}(L)$ such that $x \vee (y \wedge z) \in P$. Since $P \in \mathcal{P}(L)$, we have $x, y \in P$ or $x, z \in P$. If $x, y \in P$ then $(x \vee y) \wedge (x \vee z) \in P$ and similarly, if $x, z \in P$. Conversely, if $(x \vee y) \wedge (x \vee z) \in P$, then $x \vee y \in P$ or $x \vee z \in P$. We deduce that $x \in P, y \in P$ or $x \in P, z \in P$. Hence $x, y \wedge z \in P$, so $x \vee (y \wedge z) \in P$. We deduce that $[L]$ is distributive bounded lattice. □

As in case of MV algebra, (see [3]), for residuated lattice L, an element $x \in L$ is called *archimedean* if there is $n \geq 1$ such that $nx \in B(L)$. The residuated lattice L is called *hyperarchimedean* if all its elements are archimedean.

Remembering that a De Morgan residuated lattice L is hyperarchimedean iff $\mathcal{P}(L) = \mathcal{M}(L)$, (see [9]), we have:

Theorem 5.4. *Let L be a De Morgan residuated lattice. Then $[L]$ is a Boolean algebra iff L is hyperarchimedean.*

Proof. If $[L]$ is a Boolean algebra, then for every $x \in L$, there is $y \in L$ such that $[x] \vee [y] = \mathbf{1}$ and $[x] \wedge [y] = \mathbf{0}$. From $[x] \vee [y] = \mathbf{1}$ we deduce that $[x \vee y] = \mathbf{1}$, so by Theorem 5.3, $ord(x \vee y)^* = ord(x^* \wedge y^*) < \infty$, hence there is $n \geq 1$ such that $(x^* \wedge y^*)^n = 0$. Since $[x] \wedge [y] = [0]$ then $x \wedge y = 0$, hence $x^* \vee y^* = 1$. From (c_9), $(x^*)^n \vee (y^*)^n = 1$. Also, $(x^*)^n \wedge (y^*)^n = (x^*)^n \odot (y^*)^n = (x^* \odot y^*)^n \leq (x^* \wedge y^*)^n = 0$, hence $(x^*)^n \in B(L)$, so $[(x^*)^n]^* = nx \in B(L)$ and L is hyperarchimedean.

Conversely, suppose that L is hyperarchimedean. From Theorem 5.3, $[L]$ is a bounded distributive lattice and for every $x \in L$ there is $n \geq 1$ such that $nx \in B(L)$ i.e., $(nx) \vee (nx)^* = 1$ and $(nx) \wedge (nx)^* = 0$. Then $[x] \vee [(nx)^*] = [1] = \mathbf{1}$ and $[x] \wedge [(nx)^*] = [0] = \mathbf{0}$, so, $[L]$ is a Boolean algebra.
□

For $I \in Id(L)$ and $J \in Id([L])$, we denote $I^* = \{[x] : x \in I\}$ and $J_* = \cup\{[x] : [x] \in J\}$.

Proposition 5.5. *(i) If $I \in Id(L)$, then $I^* \in Id(L)$; Moreover, if $P \in \mathcal{P}(L)$, then $P^* \in Spec([L])$;*

(ii) If $J \in Id([L])$, then $J_* \in Id(L)$; Moreover, if $Q \in Spec([L])$, then $Q_* \in \mathcal{P}(L)$;

(iii) If $I_1, I_2 \in Id(L)$ and $I_2 \in \mathcal{P}(L)$, then $I_1 \subseteq I_2$ iff $I_1^* \subseteq I_2^*$;

(iv) If $J_1, J_2 \in Id([L])$, then $J_1 \subseteq J_2$ iff $(J_1)_* \subseteq (J_2)_*$.

Proof. (i) Let $x, y \in L$ such that $[x] \leq [y]$ and $[y] \in I^*$. Thus, there is $y_1 \in I$ such that $[y] = [y_1]$. Then $[x] = [x] \wedge [y] = [x] \wedge [y_1] \in I^*$, since $y_1 \in I$ and $x \wedge y_1 \leq y_1$. If $[x], [y] \in I^*$, then there are $x_1, y_1 \in I$ such that $[x] = [x_1]$ and $[y] = [y_1]$. Hence $x_1 \vee y_1 \in I$ and $[x] \vee [y] = [x_1] \vee [y_1] = [x_1 \vee y_1] \in I^*$, so $I^* \in Id([L])$.

Also, if $P \in \mathcal{P}(L)$, then $P \neq L$, we deduce $P^* \neq [L]$. If by contrary, $P^* = [L]$ then $\mathbf{1} \in P^*$, so $1 \in P$ and $P = L$, a contradiction. Let $x, y \in L$ such that $[x] \wedge [y] \in P^*$. Then $[x \wedge y] \in P^*$, so $x \wedge y \in P$. Since $P \in \mathcal{P}(L), x \in P$ or $y \in P$. We deduce that $[x] \in P^*$ or $[y] \in P^*$, that is $P^* \in Spec([L])$.

(ii). Let $x, y \in L$ such that $x \leq y$ and $y \in J_*$(hence $[y] \in J$). Then by Lemma 5.2, $(i), [x] \leq [y]$ and since $[y] \in J$ then $[x] \in J$, so $x \in J_*$. If $x, y \in J_*$ then $[x], [y] \in J$ so $[x] \vee [y] = [x \vee y] \in J$. Since $[x \vee y] = [x \oplus y]$, we obtain that $[x \oplus y] \in J$, so $x \oplus y \in J_*$ and $J_* \in Id(L)$. Also, for $Q \in Spec([L])$, if $Q_* = L$, then $1 \in Q_*$, so, $1 \in [x]$. Thus, $[1] = [x] \in Q$, so $Q = [L]$, a contradiction. Let $x, y \in L$ such that $x \wedge y \in Q_*$. Then $[x \wedge y] = [x] \wedge [y] \in Q$. Since $Q \in Spec([L]), [x] \in Q$ or $[y] \in Q$, so $x \in Q_*$ or $y \in Q_*$. Thus, $Q_* \in \mathcal{P}(L)$.

(iii) Suppose that $I_1 \subseteq I_2$ and we consider $x \in I_1$ such that $[x] \in I_1^*$; then $x \in I_2$, so $[x] \in I_2^*$ that is , $I_1^* \subseteq I_2^*$. Suppose now that $I_1^* \subseteq I_2^*$ and let $x \in I_1$. Then $[x] \in I_1^* \subseteq I_2^*$ so $[x] \in I_2^*$. Then there is $y \in I_2$ such that $[x] = [y]$. Since $I_2 \in \mathcal{P}(L)$ and $y \in I_2$ we deduce that $x \in I_2$, so $I_1 \subseteq I_2$.

(iv) Suppose $J_1 \subseteq J_2$ and let $x \in (J_1)_*$. Thus, $[x] \in J_1$. Then $[x] \in J_2$ so $x \in (J_2)_*$. We deduce $(J_1)_* \subseteq (J_2)_*$. Conversely, suppose $(J_1)_* \subseteq (J_2)_*$ and let $[x] \in J_1$. Then $x \in (J_1)_* \subseteq (J_2)_*$, thus $x \in (J_2)_*$. Hence $[x] \in J_2$, so $J_1 \subseteq J_2$. □

The following results hold:

Proposition 5.6. *Let $I \in Id(L), J \in Id([L])$ and $x \in L$. Then*

(i) $x \in \sigma(I)$ implies $[x] \in \sigma(I^*)$;

(ii) If $[x] \in \sigma(I^*)$, then there exists $z \in [x]$ such that $z \in \sigma(I)$;

(iii) $[x] \in \sigma(J)$ iff $x \in \sigma(J_*)$;

(iv) $(\sigma(I))^* = \sigma(I^*)$ and $(\sigma(J))_* = \sigma(J_*)$.

Proof. (i). $x \in \sigma(I) \subseteq I$ implies $x \in I$, so $[x] \in I^*$. From $x \in \sigma(I)$ there are $i \in I$ and $y \in x^\perp$ such that $i \boxplus y = 1$. Hence $[1] = [i \boxplus y] = [i \vee y] = [i] \vee [y]$ and $[x] \wedge [y] = [x \wedge y] = [0]$. Since $[i] \in I^*$ and $[y] \in [x]^\perp$ we deduce that $[x] \in \sigma(I^*)$.

(ii). For $[x] \in \sigma(I^*) \subseteq I^*$ there is $z \in [x] \cap I$ such that $[x] = [z]$.

Since $[L]$ is a distributive lattice and $[x] \in \sigma(I^*)$ there are $[i] \in I^*, [y] \in [x]^{\perp}$ such that $[i] \vee [y] = [1]$ and $[x] \wedge [y] = [0]$. Thus, $\mathbf{0} = [z] \wedge [y] = [z \wedge y]$ so, $z \wedge y = 0$. We conclude that $y \in z^{\perp}$.

Since $[1] = [i] \vee [y] = [i \vee y] = [i \boxplus y]$, we deduce that $i \boxplus y \notin P$ for every $P \in \mathcal{P}(L)$. Using Corollary 3.12, $ord((i \boxplus y)^*) < \infty$ so there is $n \geq 1$ such that $[(i \boxplus y)^*]^n = 0$. Since $n, [i] = [ni] \in I^*$ we deduce that there is $t \in [ni] \cap I$ such that $[t] = [ni]$.

To prove that $ord([(ny) \boxplus t]^*) < \infty$, we show that $(ny) \boxplus t \notin P$ for every $P \in \mathcal{P}(L)$. If $(ny) \boxplus t \in Q$ for some $Q \in \mathcal{P}(L)$ then $ny, t \in Q$. Since $t \in Q$ we deduce that $ni \in Q$, so $(ni) \boxplus (ny) = n(i \boxplus y) = n \cdot 1 = 1 \in Q$, a contradiction.

Then there is a natural number m such that $ord([(ny) \boxplus t]^*) = m$, so, $1 = \{[(ny \boxplus t)^*]^m\}^* = m[(ny) \boxplus t] = (mny) \boxplus (mt)$, with $mt \in I$. Since $y \in z^{\perp}$ and $z^{\perp} \in Id(L)$, we deduce that $mny \in z^{\perp}$. Hence $z \in \sigma(I)$.

(iii) First, suppose $[x] \in \sigma(J) \subseteq J$. Then $[x] \in J$ and $x \in J_*$. From $[x] \in \sigma(J)$ there are $[j] \in J$ and $[y] \in [x]^{\perp}$ such that $[j] \vee [y] = [1]$. Thus $[1] = [j \vee y] = [j \boxplus y]$, so $j \boxplus y \notin P$ for every $P \in \mathcal{P}(L)$, that is, $ord((j \boxplus y)^*) < \infty$. Then $[(j \boxplus y)^*]^n = 0$ for some $n \geq 1$, so $1 = \{[(j \boxplus y)^*]^n\}^* = n(j \boxplus y) = (nj) \boxplus (ny)$. Also, from $[y] \in [x]^{\perp}$ we deduce that $[0] = [x] \wedge [y] = [x \wedge y]$, so $x \wedge y = 0$. Since $j \in J_*, y \in x^{\perp}$ and $J_*, x^{\perp} \in Id(L)$. We obtain that $nj \in J_*, ny \in x^{\perp}$, so $x \in \sigma(J_*)$. Conversely, let $x \in L$ such that $x \in \sigma(J_*) \subseteq J_*$. Then $x \in J_*$ and $[x] \in J$. Moreover there are $j \in J_*, y \in x^{\perp}$ such that $j \boxplus y = 1$. We have that $[j] \vee [y] = [j \vee y] = [j \boxplus y] = [1]$ and $[y] \in [x]^{\perp}$, since $x \wedge y = 0$ implies $[x] \wedge [y] = [0]$. Hence, $[x] \in \sigma(J)$.

(iv) Let $[x] \in (\sigma(I))^*$. Then $[x] = [x_1]$ with $x_1 \in \sigma(I)$. From Proposition 5.6, (i), $[x_1] \in \sigma(I^*)$, so $(\sigma(I))^* \subseteq \sigma(I^*)$. Conversely, let $x \in L$ such that $[x] \in \sigma(I^*)$. By Proposition 5.6, (ii), there exists $z \in [x]$ such that $z \in \sigma(I)$. We deduce that $[z] \in (\sigma(I))^*$. But $z \in [x]$ so $[z] = [x]$. Then $[x] \in (\sigma(I))^*$, so $\sigma(I^*) \subseteq (\sigma(I))^*$. Thus, $(\sigma(I))^* = \sigma(I^*)$.

Finally, $x \in (\sigma(J))_*$, then $[x] \in \sigma(J)$, so $x \in \sigma(J_*)$ and $(\sigma(J))_* \subseteq \sigma(J_*)$. Conversely, if $x \in \sigma(J_*)$ then $[x] \in \sigma(J)$. Implies $x \in (\sigma(J))_*$ so $\sigma(J_*) \subseteq (\sigma(J))_*$. We conclude that $(\sigma(J))_* = \sigma(J_*)$. $\square$

Theorem 5.7. (i) *If $I \in Id(L)$, then $(I^*)_* = I$;*

(ii) *If $J \in Id([L])$, then $(J_*)^* = J$;*

((iii) *If $M \in Max(L)$, then $M^* \in Max([L])$.*

Proof. (i). Clearly, $I \subseteq (I^*)_*$. Let $x \in (I^*)_*$. Then $x \in \cup\{[y] : [y] \in I^*\}$, so there exists $y_0 \in I$ such that $x \in [y_0]$. Since $I = \cap\{P \in Spec(L) : I \subseteq P\}$ so for every $P \in \mathcal{P}(L)$ such that $I \subseteq P$ we deduce $y_0 \in P$ so $x \in P$. Thus, $(I^*)_* \subseteq \cap\{P \in \mathcal{P}(L) : I \subseteq P\} = I$, so $(I^*)_* \subseteq I$. Hence $(I^*)_* = I$.

(ii). For $x \in L, [x] \in (J_*)^*$ iff $[x] \in J$, so, $(J_*)^* = J$.

(iii). Obviously, M^* is a proper ideal in $[L]$. Let, $J \in Id([L])$ such that $M^* \subseteq J$. Then $(M^*)_* \subseteq J_*$ so, $M \subseteq J_*$. Thus, $J_* = L$ or $J_* = M$. If $J_* = L$, then $J = [L]$. If $J_* = M$, then $J = (J_*)^* = M^*$. Thus $M^* \in Max([L])$. $\square$

Theorem 5.8. *The assignment $P \rightsquigarrow P^*$ is an one-one map from $\mathcal{P}(L)$ to $Spec([L])$. This mapping carries $\mathcal{M}(L)$ onto in $Max([L])$.*

Proof. Let $P, Q \in \mathcal{P}(L)$ such that $P^* = Q^*$. Using Proposition 5.5 and Theorem 5.7 , $P^*, Q^* \in Spec([L])$ and $P = (P^*)_* = (Q^*)_* = Q$. If $R \in Spec([L])$, then $R_* \in Spec(L)$ and $(R_*)^* = R$. Let $M \in \mathcal{M}(L)$. From Theorem 5.7, $M^* \in Max([L])$. Let $I \in Max([L])$ and J a proper ideal of L such that $I_* \subseteq J$. Then $I = (I_*)^* \subseteq J^* \neq [L]$. Hence $I = J^*$. If $x \in J$, then $[x] \in I$ so $x \in I_*$. Thus $J = I_*$, so $I_* \in \mathcal{M}(L)$ and this map carries $\mathcal{M}(L)$ onto in $Max([L])$. $\square$

Theorem 5.9. *Let $I \in Id(L)$ and $J \in Id([L])$. Then*

(i) *$\sigma(I) \in Pure(L)$;*

(ii) If $I \in Pure(L)$ then $I^* \in Pure([L])$;

(iii) If $\sigma(I) \in \mathcal{P}(L)$ then $I \in Pure(L)$ iff $I^* \in Pure([L])$;

(iv) $J \in Pure([L])$ iff $J_* \in Pure(L)$.

Proof. (i) Dualizing Lemma 3.3 from ([9]) we obtain that $\sigma(I^*)$ is pure, that is, $\sigma(I^*) = \sigma(\sigma(I^*))$. Now, from Proposition 5.6, Theorem 5.7 we obtain $\sigma(I) = \sigma(\sigma(I))$, that is, $\sigma(I)$ is a pure ideal.

(ii). $I \in Pure(L)$ implies $\sigma(I) = I$. By Proposition 5.6, $I^* = (\sigma(I))^* = \sigma(I^*)$.

(iii). From Proposition 5.6, $(\sigma(I))^* = \sigma(I^*) = I^*$ and using Proposition 5.5 we obtain $I \in Pure(L)$.

(iv) $J \in Pure([L])$ implies $\sigma(J) = J$, so, by Proposition 5.6, $J_* = (\sigma(J))_* = \sigma(J_*)$. Thus, $J_* \in Pure(L)$. Conversely, $J_* \in Pure(L)$ implies, using Proposition 5.6, $J_* = \sigma(J_*) = (\sigma(J))_*$. From Proposition 5.5, $J \in Pure([L])$. □

6 The spectral topology on a residuated lattice

In ([15]), for a residuated lattice L, $\mathcal{P}(L)$ was endowed with the spectral topology as in case of bounded distributive lattices. For $I \in Id(L)$ we denote $V(I) = \{P \in Spec(L) : I \nsubseteq P\}$. Then $\tau_L = \{V(I) : I \in Id(L)\}$ is a topology on $\mathcal{P}(L)$, called the spectral topology. Moreover, the mapping $V : Id(L) \to \tau_L$ defined above is a bijection. Also, for every $x \in L$, we denote $V(x) = \{P \in Spec(L) : x \notin P\}$. We recall that the family $\{V(x) : x \in L\}$ is a basis for the topology τ_L on $\mathcal{P}(L)$ and the compact open subsets of $\mathcal{P}(L)$ are exactly the sets of the form $V(x)$.

Now, let L be a De Morgan residuated lattice. We compare the spectral topologies on $\mathcal{P}(L)$ and $Spec([L])$. Since $\{V(x)\}_{x\in L}$ generate the spectral topology τ_L on $\mathcal{P}(L)$, we consider the family of sets $V([x]) = \{Q \in Spec([L]) : [x] \notin Q\}$ which determines a topology on $[L]$.

For a subsets $\mathcal{S} \subseteq \mathcal{P}(L)$ we denote $\mathcal{S}^* = \{P^* \in \mathcal{S}\}$.

Theorem 6.1. *Let L be a De Morgan residuated lattice and $x, y \in L$. Then*

(i) $(V(x))^* = V([x])$ and $(V(x))^* = (V(y))^*$ implies $V(x) = V(y)$;

(ii) $(V(x) \cap V(y))^* = (V(x))^* \cap (V(y))^*$;

(iii) $(\cup_{x\in I} V(x))^* = \cup_{x\in I}(V(x))^*$, for $I \subseteq L$.

Proof.(i) Let $R^* \in (V(x))^* = \{P^* : P \in V(x)\}$. Then $x \notin R$, so $[x] \notin R^*$. Thus, $R^* \in V([x])$. Conversely, let $I \in V([x])$. Then by Proposition 5.5 and Theorem 5.7, $I = P^*$ for some $P \in \mathcal{P}(L)$. So $[x] \notin P^*$, hence $x \notin (P^*)_* = P$. So $P \in V(x)$ and $P^* = I \in (V(x))^*$. Finally, $(V(x))^* = (V(y))^*$ implies $V([x]) = V([y])$. So for every $P \in Spec(L)$ we have $[x] \notin P^*$ iff $[y] \notin P^*$. This implies $x \notin P$ iff $y \notin P$ since $P = P^*_*$. Therefore $V(x) = V(y)$.

(ii) From ([15]) $V(x) \cap V(y) = V(x^{**} \wedge y^{**})$. Thus, by, (i), $(V(x) \cap V(y))^* = (V(x^{**} \wedge y^{**}))^* = V([x^{**} \wedge y^{**}]) = V([(x \wedge y)^{**}]) = V([x \wedge y]) = V([x]) \cap V([y]) = (V(x))^* \cap (V(y))^*$.

(iii)Let $P^* \in (\cup_{x\in I} V(x))^*$. Then $P \in \cup_{x\in I} V(x)$, so, for some $x \in I, P \in V(x)$ Thus $P^* \in (V(x))^*$. So $(\cup_{x\in I} V(x))^* \subseteq \cup_{x\in I}(V(x))^*$. Conversely, if $P^* \in \cup_{x\in I}(V(x))^*$ then $P^* \in (V(x))^*$ for some $x \in I$ so $P \in V(x) \subseteq \cup_{x\in I} V(x)$. Hence $P^* \in (\cup_{x\in I} V(x))^*$ and $(\cup_{x\in I} V(x))^* \supseteq \cup_{x\in I}(V(x))^*$.
□

To summarize, we have:

Corollary 6.2. *If L is a De Morgan residuated lattice, then*

(i) the map $V(x) \rightsquigarrow (V(x))^*$ is one-one, onto and preserves arbitrary unions and finite intersections;

(ii) the prime ideal spaces $\mathcal{P}(L)$ *and* $Spec([L])$ *are homeomorphic.*

Since in a residuated lattice L, for $I \in Id(L), V(I) = \{P \in \mathcal{P}(L) : I \nsubseteq P\}$ is open in $\mathcal{P}(L)$ and $\overline{V}(I) = \mathcal{P}(L) \backslash V(I) = \{P \in \mathcal{P}(L) : I \subseteq P\}$ is closed, then obviously, $V(I)$ is stable under descent (that is, if $P \in V(I), Q \in \mathcal{P}(L)$ and $Q \subseteq \mathcal{P}(L)$ and $P \subseteq Q$, then $Q \in \overline{V}(I)$ and $\overline{V}(I)$ is stable under ascent (that is, if $P \in \overline{V}(I), Q \in \mathcal{P}(L)$ and $P \subseteq Q$ then $Q \in \overline{V}(I)$).

So, the sets simultaneous open and closed (*clopen sets* in $\mathcal{P}(L)$, are *stable*, that is, are stable under ascent and descent.

As in the case of MV algebras, by *stable topology* for L, we mean a collection S_L of stable open subsets $V(I)$ of $\mathcal{P}(L)$, that is $S_L = \{V(I) : I \in Id(L)\}$ and $V(I)$ is stable under ascent.

Proposition 6.3. *Let L be a residuated lattice and* $I \in Id(L)$*. Then* $V(I)$ *is stable in* $\mathcal{P}(L)$ *iff* $V(I^*)$ *is stable in* $Spec([L])$*.*

Proof. Suppose that $V(I)$ is stable in $\mathcal{P}(L)$ and let $P, Q \in Spec([L])$ such that $P \subseteq Q$ and $P \in V(I^*)$. Then $I^* \nsubseteq P$ and by Theorem 5.7 we deduce that $I = (I^*)_* \nsubseteq P_*$, so $P_* \in V(I)$. Since $P_* \nsubseteq Q_*$ and $V(I)$ is stable, then $Q_* \in V(I)$. But $Q_* \in V(I)$ iff $I \nsubseteq Q_*$. Then $I^* \nsubseteq Q_* = Q$ so $Q \in V(I^*)$. Thus, $V(I^*)$ is stable in $\mathcal{P}(L)$. Conversely, suppose that $V(I^*)$ is stable in $Spec([L])$ then for $P, Q \in \mathcal{P}(L)$ such that $P \subseteq Q$ and $P \in V(I)$. We have $I \nsubseteq P$. Thus $I^* \nsubseteq P^*$, so $P^* \in V(I^*)$. Since $P^* \subseteq Q^*$ and $V(I^*)$ is stable in $Spec([L])$ then $Q^* \in V(I^*)$. But $Q^* \in V(I^*)$ iff $I^* \nsubseteq Q^*$ iff $I \nsubseteq Q$. Thus, $Q \in V(I)$, that is ,$V(I)$ is stable in $\mathcal{P}(L)$. □

Theorem 6.4. *Let L be a De Morgan residuated lattice and* $I \in Id(L)$*. Then* $I \in Pure(L)$ *iff* $V(I)$ *is stable in* $\mathcal{P}(L)$*.*

Proof. Suppose that $I \in \mathcal{P}(L)$ and let $P, Q \in \mathcal{P}(L)$ such that $P \subseteq Q$ and $P \in V(I)$. Then $I \nsubseteq P$, so there exists $i_0 \in I$ such that $i_0 \notin P$. Since $I = \sigma(I)$, then $i_0 \in \sigma(I)$, so $i_0^{**} \in \sigma(I)$. Then there are $i \in I$ and $y \in (i_0^{**})^{\perp}$ such that $i \oplus y = 1$. Since $y^{**} \in (i_0^{**})^{\perp}$ we deduce that $i_0^{**} \wedge y^{**} = 0 \in P$. But $i_0 \notin P$ so, $y \in P \subseteq Q$, thus $y \in Q$. If by the contrary, $Q \notin V(I)$ then $I \subseteq Q$ so $i \in Q$. From $y, i \in Q$ we deduce that $i \oplus y = 1 \in Q$. Hence $Q = L$, a contradiction.

Conversely, we suppose that $V(I)$ is stable in $\mathcal{P}(L)$. If by contrary I is not pure in L, then there is $x_0 \in I$ such that $x_0 \notin \sigma(I)$, so $x_0 \neq 0$. From (see [14], Corollary 23), there is a minimal prime ideal P such that $\sigma(I) \subseteq P$ and $x_0 \notin P$. Thus $I \nsubseteq P$, hence $P \in V(I)$. Since $x_0 \notin \sigma(I)$, then for every $i \in I$ and $x_0^{\perp}$ we have $i \boxplus y \neq 1$. This implies that $i \notin x_0^{\perp} \vee I$, that is $x_0^{\perp} \vee I$ is proper in L. From Theorem 3.11, there is $Q \in Spec(L)$ such that $x_0^{\perp} \vee I \subseteq Q$. But $\sigma(I) \subseteq I \subseteq Q$ and by minimally of P, $P \subseteq Q$. Since $V(I)$ is stable, we deduce $Q \in V(I)$. But $I \subseteq Q$, hence $Q \notin V(I)$, a contradiction. Thus, $\sigma(I) = I$ and I is pure L. □

From Proposition 6.3 and Theorem 6.4 we obtain:

Corollary 6.5. *Let L be a De Morgan residuated lattice and* $I \in Id(L)$*. Then the following are equivalent:*

(i) $I \in Pure(L)$*;*

(ii) $V(I)$ *is stable in* $\mathcal{P}(L)$*;*

(iii) $V(I^*)$ *is stable in* $Spec([L])$*.*

Corollary 6.6. *For a residuated lattice L, the assignment* $I \rightsquigarrow V(I)$ *is a bijection between from* $Pure(L)$ *and the set of stable open subsets of* $\mathcal{P}(L)$*.*

Corollary 6.7. *Let L be a De Morgan residuated lattice. Then the spectral topology coincides with a stable topology on* $\mathcal{P}(L)$ *iff L is hyperarchimedean.*

Proof. By Theorem 5.4, L is hyperarchimedean iff $[L]$ is a Boolean algebra. Using Corollary 6.2 and Theorem 4, (see [8]) we deduce the conclusion. □

Conflict of Interest: The author declares no conflict of interest.

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$\top$-Nets and $\top$-Filters

Gunther Jäger

Abstract. In this paper, we develop a theory of $\top$-nets and study their relation to $\top$-filters. We show that convergence in strong L-topological spaces can be described by both $\top$-nets and $\top$-filters and both concepts are equivalent in the sense that definitions and proofs that are given using $\top$-filters can also be given using $\top$-nets and vice versa.

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1 Introduction

There are usually two ways in which convergence in topology is studied. One way makes use of so-called nets or Moore-Smith sequences. These were introduced by Moore and Smith [24] and made popular with the textbook of Kelley [17]. The other way uses filters and these were introduced by Cartan [4] and made popular e.g. by Kowalsky [18] and Bourbaki [3]. Bartle pointed out that both notions are equivalent in the sense that a definition, proposition, or proof based on nets can also be given using filters and vice versa [1].

In the lattice-valued case — for different lattice backgrounds — both approaches have been generalized and used from the very beginning of fuzzy topology. Lowen [22] developed a convergence theory based on prefilters and at around the same time, Pu and Liu [25] developed a convergence theory using fuzzy nets. The relationship between these two approaches was clarified in [23]. Höhle developed a theory of $\top$-filters [10] and L-filters [11, 12]. Convergence theories based on this concept were developed e.g. in [14, 15, 5, 20]. A further notable contribution is due to Yao [28] who defined and studied LM-nets and discussed the relationship to LM-filters.

Recently, new interest in Höhle's $\top$-filters evolved [7, 29, 31] as they can be used for a convergence theory for strong L-topological spaces [5, 32] or conical neighborhood spaces [21, 19]. They are also applied to study probabilistic uniform spaces [10, 7, 30] and $\top$-uniform convergence spaces [16].

In this paper, we provide a suitable theory of $\top$-nets and show with examples that this concept can also be fruitfully applied in cases where $\top$-filters have been used so far. In this sense, we again obtain equivalence between $\top$-nets and $\top$-filters.

The paper is organized as follows. In a preliminary section, we describe the lattice context used in this paper and collect the basic underlying theory and results that we use later on. The next section gives the new concepts of a $\top$-net and — most important for the equivalence mentioned above — the definition of

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a $\top$-subnet. The relationship between $\top$-nets and $\top$-filters is developed. This is followed by a section on applications of both $\top$-nets and $\top$-filters in the theory of strong L-topological spaces and a section on a diagonal principle based on $\top$-nets. Then we briefly glimpse the use of $\top$-sequences and finally we draw some conclusions.

2 Preliminaries

Let $(L,\le)$ be a complete lattice with distinct top and bottom elements $\top \neq \bot$. We can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \le \bigvee D$ there is $\delta \in D$ such that $\alpha \le \delta$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta \,:\, \beta \lhd \alpha\}$ for any $\alpha \in L$, [26]. For more details and results on lattices, we refer to [9].

The triple $\mathsf{L} = (L,\le,*)$, where $(L,\le)$ is a complete lattice with order relation $\le$, is called a *commutative and integral quantale* if $(L,*)$ is a commutative semigroup with the top element of L as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i\in J}\alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$, see e.g. [13].

In a quantale, we can define an *implication* by $\alpha \to \beta = \bigvee\{\delta \in L \,:\, \alpha * \delta \le \beta\}$. Then $\delta \le \alpha \to \beta \iff \delta * \alpha \le \beta$. A commutative and integral quantale is an *MV-algebra* [11] if $(\alpha \to \beta) \to \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$.

We will in this paper always assume that $\mathsf{L} = (L,\le,*)$ is a commutative and integral quantale and that the lattice $(L,\le)$ is completely distributive with the additional property that $\alpha,\beta \lhd \top$ implies $\alpha \vee \beta \lhd \top$, see [8]. While for a good part of the theory the weaker assumption $\top = \bigvee\{\alpha \,:\, \alpha \lhd \top\}$ is sufficient we will need the complete distributivity in particular for the concept of a $\top$-subnet and here, for the important Theorem 3.7.

An L-set in X is a mapping $a : X \longrightarrow L$ and we denote the set of L-sets in X by L^X. The lattice operations are extended pointwisely from L to L^X.

For $a, b \in L^X$ we denote $[a,b] = \bigwedge_{x\in X}(a(x) \to b(x))$. $[\cdot,\cdot]$ is sometimes called the *fuzzy inclusion order* [2]. We collect some of the properties that we will need later.

Lemma 2.1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \le b$ *if and only if* $[a,b] = \top$*;*

(ii) $a \le a'$ *implies* $[a',b] \le [a,b]$ *and* $b \le b'$ *implies* $[a,b] \le [a,b']$*;*

(iii) $[a,c] \wedge [b,c] = [a \vee b, c]$*;*

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$.

Definition 2.2. [29, 10]
A subset $\mathbb{F} \subseteq L^X$ is called a $\top$-*filter* if

($\top$-F1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{F}$;

($\top$-F2) $a, b \in \mathbb{F}$ implies $a \wedge b \in \mathbb{F}$;

($\top$-F3) $\bigvee_{b\in\mathbb{F}}[b,c] = \top$ implies $c \in \mathbb{F}$.

We denote the set of all $\top$-filters on X by $\mathsf{F}^{\top}_{\mathsf{L}}(X)$.

Example 2.3. For $x \in X$, $[x] = \{a \in L^X \,:\, a(x) = \top\}$ is a $\top$-filter.

Definition 2.4. [29, 10] A subset $\mathbb{B} \subseteq L^X$ is called a $\top$-*filter base* if

($\top$-B1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{B}$;

(⊤-B2) $a, b \in \mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a ⊤-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X \ : \ \bigvee_{b\in\mathbb{B}}[b, a] = \top\}$ is the ⊤-filter generated by $\mathbb{B}$.

It is well-known, that for a ⊤-filter $\mathbb{F} \in \mathsf{F}_L^\top(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) \ : \ a \in \mathbb{F}\}$ is a ⊤-filter base on Y and we denote $\varphi(\mathbb{F})$ the generated ⊤-filter on Y, the *image of* $\mathbb{F}$ *under* φ, see [10].

3 ⊤-nets and their relation to ⊤-filters

A *directed set* $(D, \prec)$ is a nonvoid set with a reflexive and transitive relation which satisfies moreover that for $d, e \in D$ there is $f \in D$ such that $d, e \prec f$. We will also often write $e \succ d$ for $d \prec e$.

We denote $L^* = L \setminus \{\bot\}$. Let $(D, \prec)$ be a directed set. We consider two mappings $s_X : D \longrightarrow X$ and $s_L : D \longrightarrow L^*$. If $\bigvee_{d\prec e} s_L(e) = \top$ for all $d \in D$, then we call the pair $s = (s_X, s_L) : D \longrightarrow X \times L^*$ a ⊤-*net in* X.

Example 3.1. A constant ⊤-net with value $x \in X$ is defined by $c^x : D \longrightarrow X \times L^*$, $c^x_X(d) = x$ and $c_L(d) = \top$ for all $d \in D$.

Theorem 3.2. *Let* $s = (s_X, s_L) : D \longrightarrow X \times L^*$ *be a ⊤-net in* X.

(i) The set $\mathbb{B}_s = \{b^s_d \ : \ d \in D\}$*, with* $b^s_d = \bigvee_{d\prec e} s_L(e) * \top_{s_X(e)}$ *a "tail" of the ⊤-net s, is a ⊤-filter basis.*

(ii) For the generated ⊤-filter $\mathbb{F}_s = [\mathbb{B}_s]$ *we have* $a \in \mathbb{F}_s$ *if and only if*

$$\bigvee_{d\in D}\bigwedge_{d\prec e}(s_L(e) \to a(s_X(e))) = \top.$$

Proof. We first show (1). We have $\bigvee_{z\in X} b_d(z) \geq \bigvee_{d\prec e} b_d(s_X(e)) = \bigvee_{d\prec e} s_L(e) = \top$ for each $d \in D$ and hence (⊤-B1) is satisfied. For (⊤-B2), let $b_d, b_e \in \mathbb{B}_s$. For $d, e \in D$ there is $f \in D$ with $d, e \prec f$. Then $b_f \leq b_d \wedge b_e$ and we conclude $\bigvee_{b\in\mathbb{B}_s}[b, b_d \wedge b_e] \geq [b_f, b_d \wedge b_e] = \top$.

To show (2), we note that for $d \in D$ and $a \in L^X$ we have

$$[b_d, a] = \bigwedge_{z\in X}(b_d(z) \to a(z)) = \bigwedge_{z\in X}\bigwedge_{d\prec e}(s_L(e) * \top_{s_X(e)}(z) \to a(z)) = \bigwedge_{d\prec e} s_L(e) \to a(s_X(e)).$$

□

It is a simple exercise to show that $\mathbb{F}_{c^x} = [x]$ for a constant ⊤-net.

Remark 3.3. For the special case that $(D, \prec) = (\mathbb{N}, \leq)$ we obtain the concept of a ⊤-*sequence.*

Proposition 3.4. *Let* $s = (s_X, s_L) : D \longrightarrow X \times L^*$ *be a ⊤-net and let* $\varphi : X \longrightarrow Y$ *be a mapping. We define the* image of s under φ *by* $\varphi(s) = (\varphi \circ s_X, s_L) : D \longrightarrow Y \times L^*$. *Then* $\mathbb{F}_{\varphi(s)} = \varphi(\mathbb{F}_s)$.

Proof. We have $a \in \varphi(\mathbb{F}_s)$ if and only if $\varphi^{\leftarrow}(a) \in \mathbb{F}_s$. This is equivalent to

$$\top = \bigvee_{d\in D}\bigwedge_{d\prec e}(s_L(e) \to \varphi^{\leftarrow}(a)(s_X(e))) = \bigvee_{d\in D}\bigwedge_{d\prec e}(s_L(e) \to a(\varphi \circ s_X(e))),$$

i.e. to $a \in \mathbb{F}_{\varphi(s)}$. □

Let now $\mathbb{F} \in \mathsf{F}_L^\top(X)$ be a ⊤-filter. We define

$$D_{\mathbb{F}} = \{((x, \alpha), f) \ : \ \bot \neq \alpha \lhd \top, f \in \mathbb{F}, f(x) \geq \alpha\}$$

and for $((x, \alpha), f), ((y, \beta), g) \in D_{\mathbb{F}}$ we define $((x, \alpha), f) \prec ((y, \beta), g)$ if and only if $g \leq f$.

Proposition 3.5. *Let $\mathbb{F} \in \mathsf{F}_L^\top(X)$. Then $(D_\mathbb{F}, \prec)$ is a directed set.*

Proof. We note that $D_\mathbb{F}$ is not empty because $\mathbb{F}$ is a $\top$-filter. The reflexivity and transitivity of $\prec$ are obvious. Let $d_1 = ((x,\alpha), f), d_2 = ((y,\beta), g) \in D_\mathbb{F}$. Then $f, g \in \mathbb{F}$ and $f(x) \geq \alpha \neq \bot$ and $g(y) \geq \beta \neq \bot$ and $\alpha, \beta \lhd \bot$. Then $f \wedge g \in \mathbb{F}$ and, by our assumption on the quantale, also $\alpha \vee \beta \lhd \top$. From $\alpha \vee \beta \lhd \top = \bigvee_{z\in X} f \wedge g(z)$ we conclude that there is $z \in X$ such that $\alpha \vee \beta \leq f \wedge g(z)$. Hence, $d_3 = ((z, \alpha \vee \beta), f \wedge g) \in D_\mathbb{F}$ and clearly $d_1, d_2 \prec d_3$. □

We define now the mapping $s_\mathbb{F} : D_\mathbb{F} \longrightarrow X \times L^*$ by $s_\mathbb{F}((x,\gamma), f) = (x, \gamma)$. For simplicity, we denote $s_\mathbb{F} = (s_X, s_L)$. We note that if $((x,\alpha), f) \in D_\mathbb{F}$, then, as $f \in \mathbb{F}$, for each $\beta \lhd \top$, we have $\bigvee_{z\in X} f(z) = \top \rhd \beta$ and thus there is $z_\beta \in X$ such that $f(z_\beta) \geq \beta$. Therefore $((z_\beta, \beta), f) \in D_\mathbb{F}$ and clearly $((x,\alpha), f) \prec ((z_\beta, \beta), f)$. We conclude

$$\bigvee_{((x,\alpha),f)\prec((Y,\beta),g)} s_L((y,\beta), g) \geq \bigvee_{\beta\lhd\top} \beta = \top$$

and $s_\mathbb{F}$ is a $\top$-net on X.

Proposition 3.6. *Let $\mathbb{F} \in \mathsf{F}_L^\top(X)$. Then $\mathbb{F}_{(s_\mathbb{F})} = \mathbb{F}$.*

Proof. Let first $a \in \mathbb{F}$. For $\bot \neq \alpha \leq a(x)$ with $\alpha \lhd \top$ then $d = ((x,\alpha), a) \in D_\mathbb{F}$. If $((x,\alpha), a) \prec ((y,\beta), g) \in D_\mathbb{F}$ then $\beta \leq g(y) \leq a(y)$ and hence

$$\bigwedge_{((x,\alpha),a)\prec((y,\beta),g)} s_L(((y,\beta), g)) \to a(s_X(((y,\beta), g))) = \bigwedge_{((x,\alpha),a)\prec((y,\beta),g)} \beta \to a(y) = \top.$$

Therefore

$$\bigvee_{d\in D_\mathbb{F}} \bigwedge_{d\prec e} s_L(e) \to a(s_X(e)) = \top$$

and we have $a \in \mathbb{F}_{s_\mathbb{F}}$.

Conversely, let $a \in \mathbb{F}_{s_\mathbb{F}}$. Then

$$\begin{aligned}
\top &= \bigvee_{d\in D_\mathbb{F}} \bigwedge_{d\prec e} s_L(e) \to a(s_X(e)) \\
&\leq \bigvee_{f\in\mathbb{F}} \bigwedge_{((x,\gamma),f)\prec((y,\delta),f)} (\delta \to a(y)) \\
&\leq \bigvee_{f\in\mathbb{F}} \bigwedge_{y\in X} \bigwedge_{\delta : f(y)\geq\delta} (\delta \to a(y)) \\
&= \bigvee_{f\in\mathbb{F}} \bigwedge_{y\in X} ((\bigvee_{\delta : f(y)\geq\delta} \delta) \to a(y)) \\
&= \bigvee_{f\in\mathbb{F}} \bigwedge_{y\in X} (f(y) \to a(y)) \;=\; \bigvee_{f\in\mathbb{F}} [f, a],
\end{aligned}$$

and hence $a \in \mathbb{F}$. □

Clearly, for a $\top$-net $s : D \longrightarrow X \times L^*$ we do not have that $s_{(\mathbb{F}_s)}$ equals s as $D_{\mathbb{F}_s}$ does not coincide with the original directed set D. This is similar to the classical relation between nets and filters. For the "equivalence" of both concepts with regards to theories and applications of convergence, we need the notion of a $\top$-subnet.

Let $s : D \longrightarrow X \times L^*$ and $t : E \longrightarrow X \times L^*$ be two $\top$-nets on X. We call t a $\top$-*subnet* of s if there is a mapping $\phi : E \longrightarrow D$ with $t_X = s_X \circ \phi$, $t_L \leq s_L \circ \phi$ and if for all $d \in D$ there is $e \in E$ such that $e \prec h$ implies $d \prec \phi(h)$.

Proposition 3.7. *Let $t = (t_X, t_L) : E \longrightarrow X \times L^*$ be a ⊤-subnet of $s = (s_X, s_L) : D \longrightarrow X \times L^*$. Then $\mathbb{F}_t \geq \mathbb{F}_s$.*

Proof. Let $d \in D$ and let $b^s_d = \bigvee_{d \prec f} s_L(f) * \top_{s_X(f)}$ be an element of the ⊤-basis of $\mathbb{F}_s$. We choose $e \in E$ such that $e \prec h$ implies $d \prec \phi(h)$. Then for the element b^t_e of the ⊤-basis of $\mathbb{F}_t$ we have

$$b^t_e = \bigvee_{e \prec h} t_L(h) * \top_{t_X(h)} \leq \bigvee_{d \prec \phi(h)} s_L(\phi(h)) * \top_{s_X(\phi(h))} \leq \bigvee_{d \prec f} s_L(f) * \top_{s_X(f)} = b^s_d.$$

Hence, $b^s_d \in \mathbb{F}_t$ and we have $\mathbb{F}_s \leq \mathbb{F}_t$. □

Crucial for us is the following result.

Theorem 3.8. *Let $s = (s_X, s_L) : D \longrightarrow X \times L^*$ be a ⊤-net and let $\mathbb{G} \geq \mathbb{F}_s$. Then there is a ⊤-subnet $t = (t_X, t_L) : E \longrightarrow X \times L^*$ of s such that $\mathbb{G} = \mathbb{F}_t$.*

Proof. We define the set

$$E = \{(e, d, g, \varepsilon) \ : \ d, e \in D, d \prec e, g \in \mathbb{G}, \varepsilon \lhd \top, g(s_X(e)) \wedge s_L(e) \geq \varepsilon\}.$$

We note that for $\varepsilon \lhd \top, d \in D$ we have $b^s_d \in \mathbb{F}_s \leq \mathbb{G}$ and hence $b^s_d \wedge g \in \mathbb{G}$. From $\varepsilon \lhd \top = \bigvee_{z \in X} b^s_d \wedge g(z)$ we conclude that there is $z \in X$ such that $\varepsilon \lhd b^s_d(z) = \bigvee_{d \prec e} s_L(e) * \top_{s_X(e)}(z)$ and $\varepsilon \lhd g(z)$. Hence there is $e \succ d$ such that $s_X(e) = z$, $s_L(e) \geq \varepsilon$ and we conclude $g(s_X(e)) \wedge s_L(e) \geq \epsilon$. Therefore, the set E is not empty and for each $d \in D, \varepsilon \lhd \top, g \in \mathbb{G}$ there is an element $(e, d, g, \varepsilon) \in E$.

We define an order on E as follows:

$$(e_1, d_1, g_1, \varepsilon_1) \prec (e_2, d_2, g_2, \varepsilon_2) \iff d_1 \prec d_2 \text{ and } g_1 \geq g_2.$$

It is not difficult to see that $\prec$ is a reflexive and transitive relation on E. We show that $(E, \prec)$ is directed. Let $(e_1, d_1, g_1, \varepsilon_1), (e_2, d_2, g_2, \varepsilon_2) \in E$. We choose $d_3 \succ d_1, d_2$, $\varepsilon_3 \leq \varepsilon_1 \wedge \varepsilon_2$ and $g_3 = g_1 \wedge g_2 \in \mathbb{G}$. As we have just seen, for $\varepsilon_3 \lhd \top$ there is $e_3 \succ d_3$ such that $g_3(s_X(e_3)) \wedge s_L(e_3) \geq \varepsilon_3$ and hence $(e_3, d_3, g_3, \varepsilon_3) \in E$ and $\succ (e_1, d_1, g_1, \varepsilon_1), (e_2, d_2, g_2, \varepsilon_2)$.

We define now $\phi : E \longrightarrow D$ by $\phi(e, d, g, \varepsilon) = e$ and we put $t_X(e, d, g, \varepsilon) = s_X(e)$, $t_L(e, d, g, \varepsilon) = \epsilon$. Then $t_X = s_X \circ \Phi$ and $t_L \leq s_L \circ \Phi$. For $d \in D$ we choose $(e, d, g, \varepsilon) \in E$. If $(e_1, d_1, g_1, \varepsilon_1) \succ (e, d, g, \varepsilon)$ then by the definition of E we have $e_1 \succ d_1$ and from the order we get moreover $d_1 \succ d$. Hence $\Phi(e_1, d_1, g_1, \varepsilon_1) = e_1 \succ d$. In order to conclude that $t : E \longrightarrow X \times L^*$ is a ⊤-subnet of s, we need only to show that t is a ⊤-net. To this end, let $(e_0, d_0, g_0, \varepsilon_0) \in E$. For $\varepsilon_1 \lhd \top$ we choose, as $b^s_{d_0} \wedge g_0 \in \mathbb{G}$, as before $e \succ d_0$ such that $s_X(e) = z, s_L(e) \geq \varepsilon_1, g_0(z) \geq \varepsilon_1$. Then $(e, d_0, g_0, \varepsilon_1) \in E$ and is $\succ (e_0, d_0, g_0, \varepsilon_0)$. Hence

$$\bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e) \geq \varepsilon_1.$$

This is true for all $\varepsilon_1 \lhd \top$ and hence $\bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e, d, g, \varepsilon) = \top$. Hence t is a ⊤-subnet of s.

We will now show that $\mathbb{G} = \mathbb{F}_t$. Consider a "tail" of $t = (t_X, t_L)$,

$$b^t_{(e_0, d_0, g_0, \varepsilon_0)} = \bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e, d, g, \varepsilon) * \top_{t_X(e, d, g, \varepsilon)} = \bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} \varepsilon * \top_{s_X(e)}.$$

If $(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)$ then $e \succ d$, $d \succ d_0$, $g \leq g_0$, $\varepsilon \lhd \top$ and $g(s_X(e)) \wedge s_L(e) \geq \varepsilon$ and we have

$$b^s_{d_0}(s_X(e)) = \bigvee_{\overline{e} \succ d} s_(\overline{e}) * \top_{s_X(\overline{e})}(s_X(e)) \geq s_L(e)$$

and

$$g_0(s_X(e)) \wedge s_L(e) \geq g(s_X(e)) \wedge s_L(e) \geq \varepsilon = \varepsilon * \top_{s_X(e)}(s_X(e)).$$

Hence we conclude $g_0 \wedge b^s_{d_0}(z) \geq \varepsilon * \top_{s_X(e)}(z)$ for all $z \in X$ and we have $b^t_{(e_0,d_0,g_0,\varepsilon_0)} \leq g_0 \wedge b_{d_0}$.

Conversely, let $\eta \lhd g_0 \wedge b^s_{d_0}(z) = g_0(z) \wedge \bigvee_{e \succ d_0} s_L(e) * \top_{s_X(e)}$. Then $g_0(z) \geq \eta$ and there is $e \succ d_0$ such that $z = s_X(e)$ and $s_L(e) \geq \eta$. We conclude $(e, d_0, g_0, \eta) \in E$ and $\succ (e_0, d_0, g_0, \varepsilon_0)$. Hence, $b^t_{(e_0,d_0,g_0,\varepsilon_0)}(z) \geq s_L(e) \wedge \eta \wedge \top_{s_X(e)}(z) = \eta$ and we have $g_0 \wedge b^s_{d_0} \leq b^t_{(e_0,d_0,g_0,\varepsilon_0)}$. Together, we have shown $g_0 \wedge b^s_{d_0} = b^t_{(e_0,d_0,g_0,\varepsilon_0)}$. As the "tails" $b^t_{(e_0,d_0,g_0,\varepsilon_0)}$ are a $\top$-basis of $\mathbb{F}_t$, we finally show that the set $\mathbb{B} = \{g \wedge b^s_d \ : \ g \in \mathbb{G}, d \in D\}$ is a $\top$-basis of $\mathbb{G}$. The property ($\top$-B1) follows, as $b^s_d \in \mathbb{F}_s \leq \mathbb{G}$ and therefore $g \wedge b^s_d \in \mathbb{G}$. The property ($\top$-B2) can be seen as follows. Let $g_1 \wedge b^s_{d_1}, g_2 \wedge b^s_{d_2} \in \mathbb{B}$. We choose $d_3 \succ d_1, d_2$. Then $b^s_{d_3} \leq b^s_{d_1} \wedge b^s_{d_2}$ and also $g_3 = g_1 \wedge g_2 \in \mathbb{G}$. Hence $g_3 \wedge b^s_{d_3} \leq (g_1 \wedge b^s_{d_1}) \wedge (g_2 \wedge b^s_{d_2})$ and we conclude $\bigvee_{g \in \mathbb{G}, d \in D}[g \wedge b^s_d, (g_1 \wedge b_{d^s_1}) \wedge (g_2 \wedge b^s_{d_2})] = \top$. Hence $\mathbb{B}$ is in fact a $\top$-basis. Let now $g \in \mathbb{G}$, then $g \wedge b^s_d \leq g$ and hence $\bigwedge_{h \in \mathbb{G}, d \in D}[h \wedge b^s_d, g] = \top$ and we have $g \in \mathbb{G}$. Conversely, if $\top = \bigvee_{h \in \mathbb{G}, d \in D}[h \wedge b^s_d, g]$, then, as $h \wedge b^s_d \in \mathbb{G}$, also $\bigvee_{h \in \mathbb{G}}[h, g] = \top$ which implies $g \in \mathbb{G}$. Therefore, $\mathbb{B}$ is a $\top$-basis of $\mathbb{G}$ and the proof is complete. □

4 The equivalence of ⊤-filter and ⊤-net convergence in L-topology

A subset $\tau \subseteq L^X$ is called a *strong L-topology* [32] (or a probabilistic topology [10]) if the following conditions are satisfied.

(ST1) $\bot_X, \top_X \in \tau$,

(ST2) $f \wedge g \in \tau$ whenever $f, g \in \tau$,

(ST3) $\bigvee_{j \in J} f_j \in \tau$ whenever $f_j \in \tau$ for all $j \in J$,

(ST4) $\alpha * f \in \tau$ whenever $f \in \tau$ and $\alpha \in L$,

(ST5) $\alpha \to f \in \tau$ whenever $f \in \tau$ and $\alpha \in L$.

The pair (X, τ) is called a *strong L-topological space.* For a strong L-topological space (X, τ) and $x \in X$ we define the $\top$*-neighbourhood filter of* x [10] by

$$\mathbb{U}^x_\tau = \{u \in L^X \ : \ \bigvee_{g \in \tau, g(x) = \top} [g, u] = \top\}$$

and we call a $\top$-filter $\mathbb{F} \in \mathsf{F}(X)$ *convergent to* x if $\mathbb{F} \geq \mathbb{U}^x_\tau$ and we write $\mathbb{F} \xrightarrow{\tau} x$ in this case. A mapping $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ between the strong L-topological spaces (X, τ) and (Y, σ) is called *continuous* if for all $x \in X$ we have $\mathbb{U}^{\varphi(x)}_\sigma \leq \varphi(\mathbb{U}^x_\tau)$.

We call a $\top$-net $s : (s_X, s_L) : D \longrightarrow X \times L^*$ *convergent to* x if for all $u \in \mathbb{U}^x_\tau$ we have $\top = \bigvee_{d \in D} \bigwedge_{e \succ d}(s_L(e) \to u(s_X(e)))$. This is equivalent to the fact that $\mathbb{F}_s$ is convergent to x and we write $s \xrightarrow{\tau} x$ in this case.

A strong L-topological space (X, τ) can be charaterized by an interior operator, $\mathrm{int}(a) = \bigvee_{g \in \tau}[g, a] * g$ for all $a \in L^X$, [32]. It is shown in [5] that $\mathrm{int}(a) = \bigvee_{g \in \tau, g \leq a} g$. The interior operator has the following properties [32, 5]. For $a, b \in L^X$ and $\alpha \in L$ we have

(I1) $[a, b] \leq [\mathrm{int}(a), \mathrm{int}(b)]$;

(I2) $\mathrm{int}(a) \leq a$;

(I3) $\mathrm{int}(\alpha \to a) = \alpha \to \mathrm{int}(a)$;

(I4) $\mathrm{int}(a \wedge b) = \mathrm{int}(a) \wedge \mathrm{int}(b)$;

(I5) $\mathrm{int}(int(a)) = \mathrm{int}(a)$.

The strong L-topology τ consists of the fixed-points of int, i.e. we have $g \in \tau \iff \mathrm{int}(g) = g$. For the $\top$-neighbourhood filter $\mathbb{U}_\tau^x$ we have $u \in \mathbb{U}_\tau^x$ if and only if $\mathrm{int}(u)(x) = \top$. For $u \in \mathbb{U}_\tau^x$ we have on the one hand $\mathrm{int}(u)(x) \geq \bigvee_{g\in\tau, g(x)=\top}[g,u] * g(x) = \bigvee_{g\in\tau,g(x)=\top}[g,u] = \top$ and if $\mathrm{int}(u)(x) = \top$ we have, on the other hand, $\bigvee_{g\in\tau,g(x)=\top}[g,u] \geq \bigvee_{g\in\tau,g(x)=\top} g(x) \to u(x) = u(x) \geq \mathrm{int}(u)(x) = \top$ by (I2) and hence $u \in \mathbb{U}_\tau^x$.

We first characterize the interior operator by convergence.

Proposition 4.1. *Let (X,τ) be a strong topological space and let $a \in L^X$. Then*

$$\mathrm{int}(a)(x) = \bigvee_{u\in\mathbb{U}^x}[u,a] = \bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a] = \bigwedge_{s\xrightarrow{\tau}x}\bigvee_{d\in D}[b_d^s,a].$$

In the last equality, the meet is taken over all convergent $\top$-nets $s : D \longrightarrow X \times L^*$.

Proof. We first show the first equality. We have on the one hand

$$\begin{aligned}\bigvee_{u\in\mathbb{U}_\tau^x}[u,a] &= \bigvee_{u\in\mathbb{U}_\tau^x}\ \bigvee_{g\in\tau,g(x)=\top}[g,u]*[u,a] \leq \bigvee_{g\in\tau,g(x)=\top}[g,a]\\ &\leq \bigvee_{g\in\tau}[g,a]*g(x) = \mathrm{int}(a)(x).\end{aligned}$$

On the other hand, we define an L-set $b \in L^X$ by $b(z) = \mathrm{int}(a)(x) \to a(z)$ for $z \in X$. Then, using (I3), $\mathrm{int}(b)(x) = \top$, i.e. $b \in \mathbb{U}_\tau^x$ and we conclude

$$\bigvee_{u\in\mathbb{U}_\tau^x}[u,a] \geq \bigwedge_{z\in X}((\mathrm{int}(a)(x) \to a(z)) \to a(z)) \geq \mathrm{int}(a)(x).$$

For the second equality, we get $\bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a] \leq \bigvee_{u\in\mathbb{U}^x}[u,a]$ as $\mathbb{U}_\tau^x \xrightarrow{\tau} x$. Let now $\eta \lhd \bigvee_{u\in\mathbb{U}_\tau^x}[u,a]$. Then there is $u \in \mathbb{U}_\tau^x$ such that $\eta \leq [u,a]$. If $\mathbb{F} \xrightarrow{\tau} x$, then $u \in \mathbb{F}$ and hence $\eta \leq \bigvee_{f\in\mathbb{F}}[f,a]$ and hence $\eta \leq \bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a]$. This shows $\bigvee_{u\in\mathbb{U}^x}[u,a] \leq \bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a]$.

The last equality can finally be shown as follows. If $s \xrightarrow{\tau} x$, then $\mathbb{F}_s \xrightarrow{\tau} x$ and the "tails" b_d^s form a $\top$-basis of $\mathbb{F}_s$. Hence $\bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a] \leq \bigwedge_{s\xrightarrow{\tau}x}\bigvee_{d\in D}[b_d^s,a]$. On the other hand, if $\mathbb{F} \xrightarrow{\tau} x$, then $s_\mathbb{F} \xrightarrow{\tau} x$ and we have $\mathbb{F} = \mathbb{F}_{(s_\mathbb{F})}$. Hence $\bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a] = \bigwedge_{s_\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}_{(s_\mathbb{F})}}[f,a] \geq \bigwedge_{s\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}_s}[f,a] \geq \bigwedge_{s\xrightarrow{\tau}x}\bigvee_{d\in D}[b_d^s,a]$. □

Corollary 4.2. *Let (X,τ) be a strong L-topological space. Then the following assertions are equivalent.*

1. *$g \in \tau$;*
2. *$g(x) \leq \bigwedge_{\mathbb{F}\xrightarrow{\tau}x}\bigvee_{f\in\mathbb{F}}[f,a]$ for all $x \in X$;*
3. *$g(x) \leq \bigwedge_{s\xrightarrow{\tau}x}\bigvee_{d\in D}[b_d^s,a]$ for all $x \in X$.*

We define the *closure* of an L-set $a \in L^X$ in a strong L-topological space in accordance with [27] by

$$\overline{a}(x) = \bigvee_{\mathbb{G}\geq\mathbb{U}_\tau^x}\bigvee_{g\in\mathbb{G}}[g,a], \qquad x \in X.$$

This is an L-valued interpretation of the closure of a subset A in a topological space X: A point $x \in X$ belongs to the closure of A if and only if there is a filter converging to x which contains A.

We can also characterize the closure of an L-set using $\top$-nets.

Proposition 4.3. *Let (X,τ) be a strong L-topological space and let $a \in L^X$. Then*

$$\overline{a}(x) = \bigvee_{s\to x} \bigvee_{d\in D} [b^s_d, a], \qquad x \in X.$$

Proof. We have $s \to x$ if and only if $\mathbb{F}_s \geq \mathbb{U}^x_\tau$. Hence

$$\overline{a}(x) = \bigvee_{\mathbb{F}\geq \mathbb{U}^x_\tau} \bigvee_{f\in\mathbb{F}} [f,a] \geq \bigvee_{s\to x} \bigvee_{f\in\mathbb{F}_s} [f,a] \geq \bigvee_{s\to x} \bigvee_{d\in D} [b^s_d, a].$$

On the other hand, for $f \in \mathbb{F}_s$ we have $\bigvee_{d\in D}[b^s_d, f] = \top$. Using $\mathbb{F} = \mathbb{F}_{(s_\mathbb{F})}$ we conclude

$$\overline{a}(x) = \bigvee_{\mathbb{F}\geq \mathbb{U}^x_\tau} \bigvee_{f\in\mathbb{F}} [f,a] = \bigvee_{s_\mathbb{F}\to x} \bigvee_{f\in\mathbb{F}_{(s_\mathbb{F})}} [f,a]$$

$$\leq \bigvee_{s\to x} \bigvee_{f\in\mathbb{F}_s} [f,a] = \bigvee_{s\to x} \bigvee_{f\in\mathbb{F}_s} \bigvee_{d\in D} [b^s_d, f] * [f,a] \leq \bigvee_{s\to x} \bigvee_{d\in D} [b^s_d, a]$$

and the proof is complete. □

Next we turn to the concept of a cluster point.

For a $\top$-filter $\mathbb{F} \in \mathsf{F}^L_\top(X)$ a point $x \in X$ is called a *cluster point* of $\mathbb{F}$ if $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists or, equivalently, if for all $f \in \mathbb{F}$ and all $u \in \mathbb{U}^x_\tau$ we have $\bigvee_{x\in X} f(x) \wedge u(x) = \top$. In [10] a cluster point of a $\top$-filter is called an *adherent point* of the $\top$-filter.

Lemma 4.4. *Let (X,τ) be a strong L-topological space and let $\mathbb{F}$ be a $\top$-filter in X and let $x \in X$. Then x is a cluster point of $\mathbb{F}$ if and only if there is a $\top$-filter $\mathbb{G} \geq \mathbb{F}$ which converges to x.*

Proof. If x is a cluster point of $\mathbb{F}$, then we can choose $\mathbb{G} = \mathbb{F} \vee \mathbb{U}^x_\tau$, which clearly converges to x. If there is $\mathbb{G} \geq \mathbb{F}$ converging to x, then $\mathbb{G} \geq \mathbb{U}^x_\tau$ and hence $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and x is a cluster point of $\mathbb{F}$. □

Similarly, for a $\top$-net $s = (s_X, s_L) : D \longrightarrow X$ a point $x \in X$ is called a *cluster point* of s if $\bigvee_{d\prec e} s_L(e) \wedge u(s_X(e)) = \top$ for all $d \in D$ and all $u \in \mathbb{U}^x_\tau$.

Proposition 4.5. *Let (X,τ) be a strong L-topological space and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X and let $x \in X$. Then x is a cluster point of s if and only if x is a cluster point of $\mathbb{F}_s$.*

Proof. Let first x be a cluster point of s and let $f \in \mathbb{F}_s$ and $u \in \mathbb{U}^x_\tau$. Then $\bigvee_{d\in D} \bigwedge_{d\prec e}(s_L(e) \to f(s_X(e))) = \top$, because $f \in \mathbb{F}_s$, and $\bigvee_{d\prec h} s_L(h) \wedge u(s_X(h)) = \top$. We conclude, using the inequality $(\alpha\wedge\beta)*\gamma \leq \alpha \wedge (\beta * \gamma)$,

$$\begin{aligned}
\top &= \bigvee_{d\in D} \left(\left[\bigwedge_{d\prec e} (s_L(e) \to f(s_X(e))) \right] * \left[\bigvee_{d\prec h} s_L(h) \wedge u(s_X(h)) \right] \right) \\
&= \bigvee_{d\in D} \bigvee_{d\prec h} \left((u(s_X(h)) \wedge s_L(h)) * \bigwedge_{d\prec e} (s_L(e) \to f(s_X(e))) \right) \\
&\leq \bigvee_{d\in D} \bigvee_{d\prec h} u(s_X(h)) \wedge (s_L(h) * (s_L(h) \to f(s_X(h)))) \\
&\leq \bigvee_{d\in D} \bigvee_{d\prec h} u(s_X(h)) \wedge f(s_X(h)) \\
&\leq \bigvee_{x\in X} u(x) \wedge f(x).
\end{aligned}$$

Hence x is a cluster point of $\mathbb{F}_s$.

For the converse, we choose $f = \bigvee_{d \prec e} s_L(e) * \top_{s_X(e)} \in \mathbb{F}_s$. Then, x being a cluster point of $\mathbb{F}_s$ we obtain

$$\top = \bigvee_{x \in X} (\bigvee_{d \prec e} s_L(e) * \top_{s_X(e)}(x) \wedge u(x) = \bigvee_{d \prec e} s_L(e) \wedge u(s_X(e))$$

which means that x is a cluster point of s. □

Corollary 4.6. *Let (X, τ) be a strong L-topological space and let $\mathbb{F}$ be a $\top$-filter in X and let $x \in X$. Then x is a cluster point of $\mathbb{F}$ if and only if x is a cluster point of $s_{\mathbb{F}}$.*

Proof. By Proposition 4.3, x is a cluster point of $s_{\mathbb{F}}$ if and only if x is a cluster point of $\mathbb{F}_{(s_{\mathbb{F}})} = \mathbb{F}$. □

Proposition 4.7. *Let (X, τ) be a strong L-topological space and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X and let $x \in X$. Then x is a cluster point of s if and only if there is a $\top$-subnet t of s which converges to x.*

Proof. Proposition 4.3 shows that x is a cluster point of s if and only if x is a cluster point of $\mathbb{F}_s$. This is by Lemma 4.2 equivalent to the existence of $\mathbb{G} \geq \mathbb{F}_s$, converging to x. Theorem 3.7 shows that this is equivalent to the existence of a $\top$-subnet t of s such that $\mathbb{G} = \mathbb{F}_t$, converging to x. But this means that the subnet t converges to x. □

We now characterize cluster points using the closure.

Proposition 4.8. *Let (X, τ) be a strong L-topological space, let $\mathbb{F}$ be a $\top$-filter on X and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X.*

1. *x is a cluster point of $\mathbb{F}$ if and only if $\overline{f}(x) = \top$ for all $f \in \mathbb{F}$;*
2. *x is a cluster point of s if and only if $\overline{b^s_d}(x) = \top$ for all $d \in D$.*

Proof. (1) Let first x be a cluster point of $\mathbb{F}$ and let $f \in \mathbb{F}$. Then $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and converges to x. Also $f \wedge u$ is in $\mathbb{F} \vee \mathbb{U}^x_\tau$ for all $u \in \mathbb{U}^x_\tau$. We conclude

$$\overline{f}(x) \geq \bigvee_{g \in \mathbb{F} \vee \mathbb{U}^x_\tau} [g, f] \geq \bigvee_{u \in \mathbb{U}^x_\tau} [f \wedge u, f] = \top.$$

Conversely, let $\overline{f}(x) = \top$ for all $f \in \mathbb{F}$. We fix $f \in \mathbb{F}$. Then

$$\top = \bigvee_{\mathbb{G} \geq \mathbb{U}^x_\tau} \bigvee_{g \in \mathbb{G}} [g, f] = \bigvee_{\mathbb{G} \geq \mathbb{U}^x_\tau} \bigvee_{g \in \mathbb{G}} \bigwedge_{z \in X} (g(z) \to f(z))$$

Let $\alpha \lhd \top$. Then there is $\mathbb{G} \geq \mathbb{U}^x_\tau$ and $g \in \mathbb{G}$ such that for all $z \in X$ we have $\alpha * g(z) \leq f(z)$. Let $u \in \mathbb{U}^x_\tau$. Then $g \wedge u \in \mathbb{G}$ and hence $\bigvee_{z \in X} g \wedge u(z) = \top$. We conclude $(g \wedge u(z)) * \alpha \leq f \wedge u(z)$ for all $z \in X$ and hence

$$\alpha = \alpha * \bigvee_{z \in X} g \wedge u(z) \leq \bigvee_{z \in X} f \wedge u(z).$$

The complete distributivity then yields $\top = \bigvee_{z \in X} f \wedge u(z)$. Hence $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and x is a cluster point of $\mathbb{F}$.

(2) A point x is a cluster point of s if and only if it is a cluster point of $\mathbb{F}_s$. According to (1) this is equivalent to $\overline{f}(x) = \top$ for all $f \in \mathbb{F}_s$ and this implies, the "tails" b^s_d being members of $\mathbb{F}_s$, that $\overline{b^s_d}(x) = \top$.

Conversely, if $\overline{b_d}(x) = \top$ for all $d \in D$, then for $f \in \mathbb{F}_s$ we conclude

$$\top = \bigvee_{d \in D} [b^s_d, f] \leq \bigvee_{d \in D} [\overline{b^s_d}, \overline{f}] \leq \bigvee_{d \in D} \overline{b^s_d}(x) \to \overline{f}(x) = \overline{f}(x).$$

Hence x is a cluster point of $\mathbb{F}_s$, which means that x is a cluster point of s. □

We can characterize continuity by convergence.

Proposition 4.9. *Let $(X,\tau),(Y,\sigma)$ be strong L-topological spaces and let $\varphi : X \longrightarrow Y$ be a mapping. The following assertions are equivalent.*

1. *φ is continuous;*
2. *for all $\mathbb{F} \in \mathsf{F}^L_\top(X)$, $\varphi(\mathbb{F})$ converges to $\varphi(x)$ whenever $\mathbb{F}$ converges to x;*
3. *for all $\top$-nets s on X, $\varphi(s)$ converges to $\varphi(x)$ whenever s converges to x.*

Proof. The equivalence of (1) and (2) is not difficult and not shown. We show the equivalence of (2) and (3). If the $\top$-net s converges to x, then $\mathbb{F}_s \geq \mathbb{U}^x_\tau$ and hence, using Proposition 3.3 and (2), $\mathbb{F}_{\varphi(s)} = \varphi(\mathbb{F}_s) \geq \mathbb{U}^{\varphi(x)}_\sigma$. This means that $\varphi(s)$ converges to $\varphi(x)$. Conversely, if (3) is valid and $\mathbb{F}$ converges to x, then with Proposition 3.5 we get $\mathbb{F}_{(s_\mathbb{F})} = \mathbb{F} \geq \mathbb{U}^x_\tau$, i.e. the $\top$-net $s_\mathbb{F}$ converges to x. With (3) then also $\varphi(s)$ converges to $\varphi(x)$ which means $\varphi(\mathbb{F}) = \varphi(\mathbb{F}_{(s_\mathbb{F})}) = \mathbb{F}_{\varphi(s_\mathbb{F})} \geq \mathbb{U}^{\varphi(x)}_\sigma$, i.e. $\varphi(\mathbb{F})$ converges to $\varphi(x)$. □

We now turn our attention to separation. A strong L-topological space (X,τ) is called $\top$-*Hausdorff separated* [10] if for $x,y \in X$, $x \neq y$ there are $u \in \mathbb{U}^x_\tau$, $v \in \mathbb{U}^y_\tau$ such that $\bigvee_{z\in X} u \wedge v(z) \neq \top$.

Proposition 4.10. *Let (X,τ) be a strong L-topological space. Then*

1. *(X,τ) is $\top$-Hausdorff separated if and only if each $\top$-filter converges to at most one point;*
2. *(X,τ) is $\top$-Hausdorff separated if and only if each $\top$-net converges to at most one point.*

Proof. We only prove (2). Let (X,τ) be $\top$-Hausdorff separated and assume that the $\top$-net converges to x and y. Then $\mathbb{F}_s \geq \mathbb{U}^x_\tau$ and $\mathbb{F}_s \geq \mathbb{U}^y_\tau$ and hence $\mathbb{U}^x_\tau \vee \mathbb{U}^y_\tau$ exists. Therefore, for all $u \in \mathbb{U}^x_\tau$ and all $v \in \mathbb{U}^y_\tau$ we have $\bigvee_{z\in X} u \wedge v(z) = \top$, a contradiction.

Conversely, let each $\top$-net converge to only one point and assume that $\bigvee_{z\in X} u \wedge v(z) = \top$ for all $u \in \mathbb{U}^x_\tau$ and all $v \in \mathbb{U}^y_\tau$. Then $\mathbb{F} = \mathbb{U}^x_\tau \vee \mathbb{U}^y_\tau$ exists and, as $\mathbb{F}_{(s_\mathbb{F})} = \mathbb{F}$, $s_\mathbb{F}$ converges to both x and y. Hence $x = y$. □

Without going into more details we have shown in this section that $\top$-nets, like $\top$-filters, are versatile tools for the theory of strong L-topological spaces. We would simply like to mention that compactness of a space can be defined by the requirement that each $\top$-net has a cluster point or, equivalently, that each $\top$-net has a convergent $\top$-subnet.

5 A diagonal principle

We first need some preparations, where we follow the work of Fang and Yue [7]. Let J be a set. For a "selection function" $\sigma : J \longrightarrow \mathsf{F}^\top_L(X)$ and $f \in L^X$ we define $\widehat{\sigma}(f) \in L^J$ by $\widehat{\sigma}(j) = \bigvee_{h\in\sigma(j)}[h,f]$ for $j \in J$. Then, for $\mathbb{G} \in \mathsf{F}^\top_L(J)$ we define $\kappa\sigma\mathbb{G} \in \mathsf{F}^\top_L(X)$ by $f \in \kappa\sigma\mathbb{G}$ if and only if $\widehat{\sigma}(f) \in \mathbb{G}$. The $\top$-filter $\kappa\sigma\mathbb{G}$ is called the $\top$-*diagonal filter* of $(\mathbb{G},\sigma)$.

The next property of the $\top$-neighborhood filters is well-known but we shall provide a proof because it is important for us later and to point out that the assumption of a complete MV-algebra, which is usually assumed in the corresponding papers, is not needed here.

Proposition 5.1. *Let (X,τ) be a strong L-topological space. We define a selection function $\sigma_N : X \longrightarrow \mathsf{F}^\top_L(X)$ by $\sigma_N(y) = \mathbb{U}^y_\tau$ for $y \in X$. Then we have $\mathbb{U}^x_\tau \leq \kappa\sigma_N\mathbb{U}^x_\tau$ for all $x \in X$.*

Proof. From Proposition 4.1 we know that for $u \in \mathbb{U}^x_\tau$ we have $\mathrm{int}(u) = \widehat{\sigma_N}(u)$. Hence, using (I5), we have for $u \in \mathbb{U}^x_\tau$ that $\mathrm{int}(\mathrm{int}(u))(x) = \top$, i.e. that $\mathrm{int}(u) = \widehat{\sigma_N}(u) \in \mathbb{U}^x_\tau$, which means that $u \in \kappa\sigma_N\mathbb{U}^x_\tau$. □

We note that the other inequality is always true [6], i.e. that we have $\mathbb{U}^x_\tau = \kappa\sigma_N\mathbb{U}^x_\tau$ for all $x \in X$. Fang and Yue [7] show that Proposition 5.1 implies the following result. Again an MV-algebra is not needed here.

Proposition 5.2 ([7]). *Let (X,τ) be a strong L-topological space. Then the following axiom (⊤-F) is true. For any selection function $\sigma : J \longrightarrow \mathsf{F}_L^\top(X)$, $\mathbb{G} \in \mathsf{F}_L^\top(J)$ and mapping $\varphi : J \longrightarrow X$ we have: if $\sigma(j) \xrightarrow{\tau} \varphi(j)$ for all $j \in J$ and if $\varphi(\mathbb{G}) \xrightarrow{\tau} x$ then $\kappa\sigma\mathbb{G} \xrightarrow{\tau} x$.*

We will now use this result and show a diagonal principle for ⊤-nets in a strong L-topological space (X,τ). Again, we first need some preparations.

If $s : D \longrightarrow X \times L^*$ is a ⊤-net and $d \in D$, then also $D^d = \{e \in D \ : \ e \succ d\}$ is directed and $s^d : D^d \longrightarrow X \times L^*$ defined by $s_X^d(e) = s_X(e), s_L^d(e) = s_L(e)$ for $e \in D^d$ is a ⊤-net. If $s \xrightarrow{\tau} x$, then we have $\bigvee_{d\in D} \bigwedge_{e\succ d}(s_(e) \to u(s_X(e))) = \top$ for all $u \in \mathbb{U}_\tau^x$. If $\eta \lhd \top$ there is $d_0 \in D$ such that for all $e \succ d_0$ we have $\eta \le s_L(e) \to u(s_X(e))$. We choose $d_1 \succ d, d_0$. Then $d_1 \in D^d$ and for all $e \succ d_1$ we have $\eta \le s_L(e) \to u(s_X(e))$. Hence

$$\eta \le \bigwedge_{e\succ d_1} (s_L(e) \to u(s_X(e))) \le \bigvee_{d_1\in D^d} \bigwedge_{e\succ d_1} (s_L(e) \to u(s_X(e))).$$

The complete distributivity then yields $\top = \bigvee_{d_1\in D^d} \bigwedge_{e\succ d_1}(s_L(e) \to u(s_X(e)))$ for all $u \in \mathbb{U}_\tau^x$ which means that also $s^d \xrightarrow{\tau} x$.

If $(D_j, \prec_j)$ are directed sets for all $j \in J$, then also the product $\prod_{j\in J} D_j$ becomes directed by the product order, i.e. $(d_j)_{j\in J} \prec (e_j)_{j\in J}$ if and only if for all $j \in J$ we have $d_j \prec_j e_j$. We will in the sequel, to simplify the notation, write $\prec$ for all orders and hope that the set, on which this order is defined, will be clear from the context.

Let D and E_d be directed sets for each $d \in D$ and denote $J = \bigcup_{d\in D}(\{d\} \times E_d)$. For $(d,e), (\overline{d}, \overline{e}) \in J$ we define $(\overline{d}, \overline{e}) \succ (d,e)$ if $\overline{d} \succ d$ or if $\overline{d} = d$ and $\overline{e} \succ e$. It is not difficult to show that $(J, \prec)$ is a directed set.

We consider now a ⊤-net $s : J \longrightarrow X \times L^*$, $(d,e) \longmapsto (s_X(d,e), s_L(d,e))$ such that for all $d \in D$, $s^d : E_d \longrightarrow X \times L$, $e \longmapsto (s_X^d(e) = s_X(d,e), s_L^d(e) = s_L(d,e))$ is a ⊤-net which converges to a point $y_d \in X$, i.e. $s^d \xrightarrow{\tau} y_d$. Furthermore, the ⊤-net $y : D \longrightarrow X \times L^*$, defined by $y_X(d) = y_d$, $y_L(d) = \top$ for $d \in D$ shall converge to $x \in X$, i.e. we have $y \xrightarrow{\tau} x$. We shall write $(y_d, \top)$ for y.

We denote $F = D \times \prod_{d\in D} E_d$ and define the ⊤-net $r : F \longrightarrow J \times L^*$ by $r_X(d,(e_j)) = (d, e_d)$ and $r_L(d,(e_j)) = \top$. This ⊤-net is used to select a "diagonal ⊤-net" from s, defined by

$$s \circ r : \begin{cases} F & \longrightarrow & X \times L^* \\ (d,(e_j)) & \longmapsto & (s_X(d,e_d), s_L(d,e_d)) \end{cases}.$$

We note that $s \circ r$ is a ⊤-net. We are now in the position to state the "diagonal principle".

Theorem 5.3. *Let (X,τ) be a strong L-topological space and define, as above, $J = \bigcup_{d\in D}(\{d\} \times E_d)$ and $F = D \times \prod_{d\in D} E_d$ and the ⊤-nets $s : J \longrightarrow X \times L^*$, $s^d : E_d \longrightarrow X \times L^*$, $r : F \longrightarrow J \times L^*$ and $s \circ r : F \longrightarrow X \times L^*$.*

If $s^d \xrightarrow{\tau} y_d$ for each $d \in D$ and $(y_d, \top) \xrightarrow{\tau} x$, then there is a ⊤-subnet t of $s \circ r$, a "diagonal ⊤-net", with $t \xrightarrow{\tau} x$.

Proof. For $e \in E_d$ we define $s^{de} : E_d^e = \{f \in E_d \ : \ f \succ e\} \longrightarrow X \times L^*$, $f \longmapsto (s_X^d(f), s_L^d(f))$. With this we define the selection mapping $\sigma : J \longrightarrow \mathsf{F}_L^\top(X)$ by $\sigma(d,e) = \mathbb{F}_{s^{de}}$. Furthermore we define $\varphi : J \longrightarrow X$ by $\varphi(d,e) = y_d$. Then $\sigma(d,e) \xrightarrow{\tau} \varphi(d,e)$ for all $(d,e) \in J$. For $\mathbb{F}_r \in \mathsf{F}_L^\top(J)$ we have $\varphi(\mathbb{F}_r) = \mathbb{F}_{\varphi(r)}$ with $\varphi(r) = (\varphi \circ r_X, r_L)$, i.e. $\varphi(r)(d,(e_j)) = (\varphi(d,e_d), \top) = (y_d, \top)$ for $(d,(e_j)) \in F$. Hence $\varphi(\mathbb{F}_r) = \mathbb{F}_y \xrightarrow{\tau} x$. The axiom (⊤-F) then yields $\kappa\sigma\mathbb{F}_r \xrightarrow{\tau} x$.

We now show $\mathbb{F}_{s\circ r} \le \kappa\sigma\mathbb{F}_r$. First, let $f \in L^X$. Then $\widehat{\sigma}(f) \in L^J$ is defined by

$$\widehat{\sigma}(f)(d,e) = \bigvee_{h\in\mathbb{F}_{s^{de}}} [h,f] = \bigvee_{\overline{e}\in E_d^e} [b_{\overline{e}}^{s^{de}}, f] = \bigvee_{\overline{e}\in E_d^e} \bigwedge_{\overline{\overline{e}}\succ\overline{e}} (s_L(d,\overline{\overline{e}}) \to f(s_X(d,\overline{\overline{e}}))).$$

Hence we have $f \in \kappa\sigma\mathbb{F}_r$ if and only if $\widehat{\sigma}(f) \in \mathbb{F}_r$ if and only if

$$\top = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \widehat{\sigma}(f)(\overline{d}, \overline{e}_{\overline{d}}) = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \bigvee_{\widetilde{e}\in E_{\overline{d}}^{\overline{e}_{\overline{d}}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\widetilde{e}} (s_L(d,\widetilde{\widetilde{e}}) \to f(s_X(d,\widetilde{\widetilde{e}}))).$$

Let now $f \in \mathbb{F}_{s\circ r}$. Then

$$\top = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} (s_L(\overline{d}, \overline{e}_{\overline{d}}) \to f(s_X(\overline{d}, \overline{e}_{\overline{d}}))).$$

Let $\eta \lhd \top$. Then there is $(d,(e_j)) \in F$ such that for all $(\overline{d},(\overline{e}_j)) \succ (d,(e_j))$ we have $\eta \le s_L(\overline{d}, \overline{e}_{\overline{d}}) \to f(s_X(\overline{d}, \overline{e}_{\overline{d}}))$. Then $\overline{e}_{\overline{d}} \in E_{\overline{d}}^{\overline{e}_{\overline{d}}}$ and for $\widetilde{\widetilde{e}} \succ \overline{e}_{\overline{d}}$ we define $(\overline{d},(e_j^*)) \in F$ by $e_j^* = e_j$ for $j \ne d$ and $e_d^* = \widetilde{\widetilde{e}}$. Then $(\overline{d},(e_j^*)) \succ (d,(e_j))$ and hence

$$\eta \le s_L(\overline{d}, e_{\overline{d}}^*) \to f(s_X(\overline{d}, e_{\overline{d}}^*)) = s_L(\overline{d},\widetilde{\widetilde{e}}) \to f(s_X(\overline{d},\widetilde{\widetilde{e}})).$$

Therefore we obtain

$$\eta \le \bigwedge_{\widetilde{\widetilde{e}}\succ\overline{e}_{\overline{d}}} (s_L(\overline{d},\widetilde{\widetilde{e}}) \to f(s_X(\overline{d},\widetilde{\widetilde{e}}))) \le \bigvee_{\overline{e}\in E_{\overline{d}}^{\overline{e}_{\overline{d}}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\overline{e}_{\overline{d}}} (s_L(\overline{d},\widetilde{\widetilde{e}}) \to f(s_X(\overline{d},\widetilde{\widetilde{e}}))).$$

This holds for all $(\overline{d},(\overline{e}_j)) \succ (d,(e_j))$ and we get

$$\eta \le_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \bigvee_{\widetilde{e}\in E_{\overline{d}}^{\overline{e}_{\overline{d}}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\widetilde{e}} (s_L(d,\widetilde{\widetilde{e}}) \to f(s_X(d,\widetilde{\widetilde{e}}))).$$

This is true for all $\eta \lhd \top$ and the complete distributivity then yields $f \in \kappa\sigma\mathbb{F}_r$.

Hence we have shown $\mathbb{F}_r \le \kappa\sigma\mathbb{F}_r$ and we conclude from Theorem 3.7 that there is a $\top$-subnet t of $s \circ r$ with $\mathbb{F}_t = \kappa\sigma\mathbb{F}_r$, i.e. $t \xrightarrow{\tau} x$. $\square$

6 First countable spaces and $\top$-sequences

We call a strong L-topological space *first countable* if for each $x \in X$ the $\top$-neighborhood filter $\mathbb{U}_\tau^x$ has a countable $\top$-basis.

In first countable spaces, $\top$-sequences suffice for the definition and study of most concepts. We shall illustrate this with one example.

Proposition 6.1. *Let the lattice L have a sequence $\bot \ne \alpha_1 \le \alpha_2 \le \alpha_3 \le ...$ with $\alpha_k \lhd \top$ for all $k = 1,2,3,...$ and $\bigvee_{k=1}^\infty \alpha_k = \top$ and let (X,τ) be a first countable, strong L-topological space. Then for $a \in L^X$ and $x \in X$ we have*

$$\overline{a}(x) = \bigvee_{t\to x,t\ \top\text{-sequence}} \bigvee_{n\in\mathbb{N}} [b_n^t, a],$$

where $b_n^t = \bigvee_{k\ge n} t_L(k) * \top_{t_X(k)}$ is a "tail" of the $\top$-sequence $t = (t_X, t_L) : \mathbb{N} \longrightarrow X \times L^*$.

Proof. As $\top$-sequences are $\top$-nets, we obtain from Proposition 4.7 that $\bigvee_{t\to x}\bigvee_{n\in\mathbb{N}}[b_n^t, a] \le \overline{a}(x)$, where the first join extends of all $\top$-sequences t that converge to x. For the converse, let $\eta \lhd \overline{a}(x)$. Then there is a $\top$-net $s = (s_X, s_L) : D \longrightarrow X \times L^*$ converging to x and a $d \in D$ such that $[b_d^s, a] \ge \eta$. We consider a countable $\top$-basis $v_1, v_2, v_3, ...$ of $\mathbb{U}^x_\tau$. Then $b^s_d \in \mathbb{F}_s \geq \mathbb{U}^x_\tau$ and hence we have $b^s_d \wedge v_1, b^s_d \wedge v_2, ... \in \mathbb{F}_s$. For $\alpha_k \lhd \top = \bigvee_{x\in X} b^s_d \wedge v_k(x)$ we choose $x_k \in X$ such that $b^s_d(x_k) \wedge v_k(x_k) \geq \alpha_k$ for $k = 1, 2, 3, ...$ and we consider the $\top$-sequence $t = (x_k, \alpha_k)$. As $\alpha_1, \alpha_2, ... \leq \alpha_n$ for each $n \in \mathbb{N}$, we have $\bigvee_{k\geq n} \alpha_k = \bigvee_{k=1}^\infty \alpha_k = \top$, i.e. t is in fact a $\top$-sequence. For a "tail" $b^t_k = \bigvee_{n\geq k} \alpha_k * \top_{x_k}$ we have

$$\begin{aligned} b^t_k(x) &\leq \bigvee_{n\geq k} (b^s_d(x_n) \wedge v_k(x_n)) * \top_{x_n}(x) \\ &= \begin{cases} \bot & \text{if } x \neq x_n \text{ for all } n \geq k \\ \bigvee_{n\geq k, x=x_n} b^s_d(x_n) \wedge v_k(x_n) & \text{if } x = x_n \text{ for some } n \geq k \end{cases} \\ &= \begin{cases} \bot & \text{if } x \neq x_n \text{ for all } n \geq k \\ b^s_d(x) \wedge v_k(x) & \text{if } x = x_n \text{ for some } n \geq k \end{cases} \\ &\leq b^s_d(x) \wedge v_k(x). \end{aligned}$$

Hence $b^t_k \leq b^s_d \wedge v_k$ and we therefore conclude that $v_k \in \mathbb{F}_t$ for all $k = 1, 2, ...,$ i.e. $\mathbb{U}^x_\tau \leq \mathbb{F}_t$ and $t \to x$. Moreover we have $\bigvee_{n=1}^\infty [b^t_k, a] \geq [b^s_d, a] \geq \eta$. This is true for all $\eta \lhd \overline{a}(x)$ and the missing inequality follows. □

7 Conclusions

We have shown in this paper that besides a convergence theory based on $\top$-filters, also a convergence theory based on $\top$-nets is available in strong L-topological spaces. Both concepts seem equivalent to one another in the sense that definitions and proofs that are given using one concept can also be given using the other. This was demonstrated with some examples like interior and closure of an L-set, cluster points of $\top$-filters or $\top$-nets, continuity, and Hausdorff separation.

It was shown in [7] that $\top$-filters can be used to develop an abstract theory of $\top$-convergence spaces and, similarly, for a theory of $\top$-uniform convergence spaces [16]. It seems that also $\top$-nets could be used for such a purpose. This research question is left open at this stage.

Important for the theory may be the concept of a $\top$-sequence as a special case of a $\top$-net. This concept will allow to naturally extend and study notions like countable compactness or countable completeness and so on. We will postpone this, however, to future work.

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NeutroAlgebra & AntiAlgebra vs. Classical Algebra

Florentin Smarandache

Abstract. NeutroAlgebra & AntiAlgebra vs. Classical Algebra is a like Realism vs. Idealism. Classical Algebra does not leave room for partially true axioms nor partially well-defined operations. Our world is full of indeterminate (unclear, conflicting, unknown, etc.) data.

This paper is a review of the emerging, development, and applications of the NeutroAlgebra and AntiAlgebra [2019-2022] as generalizations and alternatives of classical algebras.

AMS Subject Classification 2020: 08A72
Keywords and Phrases: Classical Algebra, NeutroAlgebra, AntiAlgebra, NeutroOperation, AntiOperation, NeutroAxiom, AntiAxiom

1 Introduction

The Classical Algebraic Structures were generalized in 2019 by Smarandache [16] to NeutroAlgebraic Structures (or NeutroAlgebras) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false} and on 2020 he continued to develop them [18, 20, 17].

The NeutroAlgebras & AntiAlgebras form a *new field of research*, which is inspired by our real world. Many researchers from various countries around the world have contributed to this new field, such as F. Smarandache, A.A.A. Agboola, A. Rezaei, M. Hamidi, M.A. Ibrahim, E.O. Adeleke, H.S. Kim, E. Mohammadzadeh, P.K. Singh, D.S. Jimenez, J.A. Valenzuela Mayorga, M.E. Roja Ubilla, N.B. Hernandez, A. Salama, M. Al-Tahan, B. Davvaz, Y.B. Jun, R.A. Borzooei, S. Broumi, M. Akram, A. Broumand Saeid, S. Mirvakilii, O. Anis, S. Mirvakili, etc (See [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

2 Distinctions between Classical Algebraic Structures vs. NeutroAlgebras & AntiAlgebras

In classical algebraic structures, all operations are 100% well-defined, and all axioms are 100% true, but in real life, in many cases, these restrictions are too harsh since in our world we have things that only partially verify some operations or some laws.

Using the process of *NeutroSophication* of a classical algebraic structure we produce a NeutroAlgebra, while the process of *AntiSophication* of a classical algebraic structure produces an AntiAlgebra.

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3 The neutrosophic triplet (Operation, NeutroOperation, AntiOperation)

When we define an operation on a given set, it does not automatically mean that the operation is well-defined. There are three possibilities:

(i) The operation is well-defined (also called inner-defined) for all set's elements [degree of truth $T = 1$] (as in classical algebraic structures; this is a classical **Operation**). Neutrosophically we write: Operation$(1, 0, 0)$.

(ii) The operation if well-defined for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and outer-defined for the other elements [degree of falsehood F], where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$ (this is a **NeutroOperation**). Neutrosophically we write: NeutroOperation(T, I, F).

(iii) The operation is outer-defined for all set's elements [degree of falsehood $F = 1$] (this is an **AntiOperation**). Neutrosophically we write: AntiOperation$(0, 0, 1)$.

4 The neutrosophic triplet (Axiom, NeutroAxiom, AntiAxiom)

Similarly for an axiom, defined on a given set, endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set elements. We have three possibilities again:

(i) The axiom is true for all set's elements (totally true) [degree of truth $T = 1$] (as in classical algebraic structures; this is a classical **Axiom**). Neutrosophically we write: Axiom$(1, 0, 0)$.

(ii) The axiom is true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for other elements [degree of falsehood F], where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$ (this is **NeutroAxiom**). Neutrosophically we write NeutroAxiom(T, I, F).

(iii) The axiom is false for all set's elements [degree of falsehood $F = 1$](this is **AntiAxiom**). Neutrosophically we write AntiAxiom$(0, 0, 1)$.

5 The neutrosophic triplet (Theorem, NeutroTheorem, AntiTheorem)

In any science, a classical Theorem, defined on a given space, is a statement that is 100% true (i.e. true for all elements of the space). To prove that a classical theorem is false, it is sufficient to get a single counter-example where the statement is false.

Therefore, the classical sciences do not leave room for the *partial truth* of a theorem (or a statement). But, in our world and our everyday life, we have many more examples of statements that are only partially true, than statements that are totally true. The NeutroTheorem and AntiTheorem are generalizations and alternatives of the classical Theorem in any science.

Let's consider a theorem, stated on a given set, endowed with some operation(s). When we construct the theorem on a given set, it does not automatically mean that the theorem is true for all set elements. We have three possibilities again:

(i) The theorem is true for all set's elements [totally true] (as in classical algebraic structures; this is a classical **Theorem**). Neutrosophically we write Theorem$(1, 0, 0)$.

(ii) The theorem is true for some elements [degree of truth T], indeterminate for other elements [degree of indeterminacy I], and false for the other elements [degree of falsehood F], where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$ (this is a **NeutroTheorem**). Neutrosophically we write NeutroTheorem(T, I, F).

(iii) The theorem is false for all set's elements (this is an **AntiTheorem**). Neutrosophically we write AntiTheorem$(0, 0, 1)$.

And similarly, for (Lemma, NeutroLemma, AntiLemma), (Consequence, NeutroConsequence, AntiConsequence), (Algorithm, NeutroAlgorithm, AntiAlgorithm), (Property, NeutroProperty, AntiProperty), etc.

6 The neutrosophic triplet (Algebra, NeutroAlgebra, AntiAlgebra)

(i) An algebraic structure whose all operations are well-defined and all axioms are totally true is called a classical Algebraic Structure (or **Algebra**).

(ii) An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called a NeutroAlgebraic Structure (or **NeutroAlgebra**).

(iii) An algebraic structure that has at least one AntiOperation or one Anti Axiom is called an AntiAlgebraic Structure (or **AntiAlgebra**).

Therefore, a neutrosophic triplet is formed: ¡Algebra, NeutroAlgebra, AntiAlgebra¿, where Algebra can be any classical algebraic structure, such as a groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, etc.

7 Theorems and Examples

Theorem 7.1. *If a Classical Statement (theorem, lemma, congruence, property, proposition, equality, inequality, formula, algorithm, etc.) is totally true in a classical Algebra, then the same Statement in a NeutroAlgebra maybe be:*

- *totally true (degree of truth $T = 1$, degree of indeterminacy $I = 0$, and degree of falsehood $F = 0$);*
- *partially true (degree of truth T), if partial indeterminate (degree of indeterminacy I), and partial falsehood (degree of falsehood F), where $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.*
- *totally false (degree of falsehood $F = 1$, degree of truth $T = 0$, and degree of indeterminacy $I = 0$).*

Example 7.2. (**Examples of Classical Algebra, NeutroAlgebra, and AntiAlgebra**)

Let $S = \{a, b, c\}$ be a set, and a binary law (operation) $*$ defined on S:

$$* : S^2 \to S.$$

As in the below Cayley Table:

$*$	a	b	c
a	a	c	a
b	a	b	a
c	b	c	a

Then:

1. $(S, *)$ is a Classical Grupoid since the law $*$ is totally (100%) well-defined (classical law), or $\forall\ x, y \in S,\ x * y \in S$.

2. $(S, *)$ is a NeutroSemigroup, since:

(i) the law $*$ is totally well-defined (classical law);

(ii) the associativity law is a NeutroAssociativity, i.e.

- partially true, because $\exists\ a, b, c \in S$ such that
$$(a * b) * c = c * c = a = a * (b * c) = a * a = a,$$
the degree of truth $T > 0$,
- degree of indeterminacy $I = 0$ since no indeterminacy exists;
- and partially false, because $\exists\ c, c, c \in S$ such that
$$(c * c) * c = a * c = a \neq c * (c * c) = a * a = b,$$
so degree of falsehood $F > 0$.

3. $(S, *)$ is an AntiCommutative NeutroSemigroup, since:
(i) the law $*$ is totally well-defined (classical law);
(ii) the associativity is a NeutroAssociativity (as proven above);
(iii) the commutativity is an AntiCommutativity, since:
$$\forall\ x, y \in S, \quad x * y \neq y * x.$$

Proof.
$$a * b = c \neq a = b * a,$$
$$a * c = a \neq b = c * a,$$
$$b * c = a \neq c = c * b.$$
□

Theorem 7.3. *If a Classical Statement is false in a classical Algebra, then in a NeutroAlgebra it may be:*
(i) *either a NeutroStatement, i.e. true* (T) *for some elements, indeterminate* (I) *for other elements, and false* (F) *for the others, where* (T, I, F) *is different from* $(1, 0, 0)$ *and from* $(0, 0, 1)$*;*
(ii) *or an AntiStatement, i.e. false for the elements.*

Theorem 7.4. *A Classical Group can be:*
(i) *either Commutative (the commutative law is true for all elements);*
(ii) *or NeutroCommutative (the commutative law is true* (T) *for some elements, indeterminate* (I) *for others, and false* (F) *for the other elements where* (T, I, F) *is different from* $(1, 0, 0)$ *and from* $(0, 0, 1)$*;*
(iii) *or AntiCommutative (the commutative law is false for all the elements).*

Corollary 7.5. *The Classical Non-Commutative Group is either NeutroCommutative or AntiCommutative.*

Corollary 7.6. *The Classical Non-Associative Groupoid is either NeutroAssociative or AntiAssociative.*

8 Conclusion

The Classical Structures in science mostly exist in theoretical, abstract, perfect, homogeneous, idealistic spaces - because in our everyday life almost all structures are NeutroStructures, since they are neither perfect nor applying to the whole population, and not all elements of the space have the same relations and same attributes in the same degree (not all elements behave in the same way).

The indeterminacy and partiality, with respect to the space, to their elements, to their relations or their attributes are not taken into consideration in the Classical Structures. But our Real World is full of structures with indeterminate (vague, unclear, conflicting, unknown, etc.) data and partialities.

There are exceptions to almost all laws, and the laws are perceived in different degrees by different people in our every-day life.

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Rough Convergence of Bernstein Fuzzy Triple Sequences

Ayhan Esi*, **Subramanian Nagarajan**

Abstract. The aim of this paper is to introduce and study a new concept of convergence almost surely (a.s.), convergence in probability, convergence in mean, and convergence in distribution are four important convergence concepts of random sequence and also discusses some convergence concepts of the fuzzy sequence: convergence almost surely, convergence in credibility, convergence in mean, and convergence in distribution.

AMS Subject Classification 2020: 40F05; 40J05; 40G05
Keywords and Phrases: Triple sequences, Rough convergence, Convergence almost surely, Convergence in probability, Convergence in mean, Convergence in distribution.

1 Introduction

The idea of rough convergence was first introduced by Phu [13, 14, 15] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$imit set of the sequence is equal to the intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. Dundar and Cakan [4] investigated of rough ideal convergence and defined the set the rough ideal limit points of a sequence The notion of $I-$convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all-natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set

$$K_{ij\ell}=\{(m,n,k)\in K: m\leq i, n\leq j, k\leq \ell\}.$$

Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

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where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

The Bernstein operator of order rst is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x)) \in \mathbb{R}, m,n,k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st-\lim\ x = 0$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{r,s,t\to\infty} \frac{1}{rst}\left|\{m \leq r, n \leq s, k \leq t : |B_{mnk}(f,x) - (f,x)| \geq \epsilon\}\right| = 0.$$

In this case, we write $\delta - \lim B_{mnk}(f,x) = f(x)$ or $B_{mnk}(f,x) \to^{S_B} f(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, $\chi_A-$the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{i,j,\ell\to\infty} \frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} \chi_A(K).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers, and complex numbers respectively. The different types of notions of triple sequence were introduced and investigated at the initial by Sahiner et al. [16, 17], Esi et al. [6, 8, 9, 7, 10, 11, 12], Dutta et al. [5], Subramanian et al. [18], Debnath et al. [3] and many others.

The set of fuzzy real numbers is denoted by $f(x)(\mathbb{R})$, and d denotes the supremum metric on $f(X)\left(\mathbb{R}^3\right)$. Now let r be nonnegative real number. A triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ of fuzzy numbers is $r-$convergent to a fuzzy number $f(X)$ and we write

$$B_{mnk}(f,X) \to^r f(X) \text{ as } m,n,k \to \infty,$$

provided that for every $\epsilon > 0$ there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d(B_{mnk}(f,X), f(X)) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

The set $LIM^r B_{mnk}(f,X) := \left\{f(X) \in f(X)\left(\mathbb{R}^3\right) : B_{mnk}(f,X) \to^r f(X),\ \text{as } m,n,k \to \infty\right\}$ is called the $r-$limit set of the triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$.

A triple sequence space of Bernstein polynomials of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$B_{mnk}(f,X) = \begin{cases} \eta(X), & \text{if } m,n,k \text{ are odd } integers, \\ \mu(X), & \text{otherwise} \end{cases},$$

where

$$\eta(X) = \begin{cases} X, & \text{if } X \in [0,1], \\ -X+2, & \text{if} X \in [1,2], \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu(X) = \begin{cases} X-3, & \text{if } X \in [3,4], \\ -X+5, & \text{if} X \in [4,5], \\ 0, & \text{otherwise} \end{cases}.$$

Then we have where

$$LIM^r B_{mnk}(f,X) = \begin{cases} \phi, & \text{if } r < \frac{3}{2}, \\ [\mu - r_1, \eta + r_1], & \text{otherwise} \end{cases},$$

where r_1 is a nonnegative real number with

$$[\mu - r_1, \eta + r_1] := \left\{ B_{mnk}(f,X) \in f(X)\left(\mathbb{R}^3\right) : \mu - r_1 \leq B_{mnk}(f,X) \leq \eta + r_1 \right\}.$$

The ideal of rough convergence of a triple sequence space of Bernstein polynomials can be interpreted as follows:

Let $(B_{mnk}(f,Y))$ be a convergent triple sequence space of Bernstein polynomials of fuzzy numbers. Assume that $(B_{mnk}(f,Y))$ can not be determined exactly for every $(m,n,k) \in \mathbb{N}^3$. That is, $(B_{mnk}(f,Y))$ cannot be calculated so we can use approximate value of $(B_{mnk}(f,Y))$ for simplicity of calculation. We only know that $(B_{mnk}(f,Y)) \in [\mu_{mnk}, \lambda_{mnk}]$, where $d(\mu_{mnk}, \lambda_{mnk}) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. The triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ satisfying $(B_{mnk}(f,X)) \in [\mu_{mnk}, \lambda_{mnk}]$, for all m,n,k. Then the triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ may not be convergent, but the inequality

$$d(B_{mnk}(f,X), f(X)) \leq d(B_{mnk}(f,X), B_{mnk}(f,Y)) + d(B_{mnk}(f,Y), f(Y)) \leq r + d(B_{mnk}(f,Y), f(Y))$$

implies that the triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ is $r-$convergent.

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where $X^0; = cl\left\{x \in \mathbb{R}^3 : X(x) > 0\right\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$level set X^α defined by

$$X^\alpha = \left\{x \in \mathbb{R}^3 : X(x) \geq \alpha\right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$$

is a non-empty compact convex subset of $\mathbb{R}^3$.

The supremum metric d on the set $L\left(\mathbb{R}^3\right)$ is defined by

$$d(X,Y) = \sup_{\alpha \in [0,1]} \max\left(\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right).$$

Now, given $X, Y \in L\left(\mathbb{R}^3\right)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0,1]$.

We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^3\right)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu^{'}$ for all upper bounds $\mu^{'}$.

A lower bound and the greatest lower bound is defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L\left(\mathbb{R}^3\right)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence space (X_{mnk}) is defined by

$$\lim_{m,n,k \to \infty} \inf\ X_{mnk} := \inf\ A_X.$$

$$\lim_{m,n,k \to \infty} \sup\ X_{mnk} := \inf\ B_X.$$

where

$$A_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{mnk} < \mu\right\} \text{is infinite}\right\}$$
$$B_X := \left\{\mu \in L\left(\mathbb{R}^3\right) : \text{The set } \left\{(m,n,k) \in \mathbb{N}^3 : X_{mnk} > \mu\right\} \text{is infinite}\right\}.$$

Now, given two fuzzy numbers $X, Y \in L\left(\mathbb{R}^3\right)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0,1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L\left(\mathbb{R}^3\right)$, which is defied by

$$a_1(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{cases}.$$

An order interval in $L\left(\mathbb{R}^3\right)$ is defined by $[X, Y] := \left\{Z \in L\left(\mathbb{R}^3\right) : X \leq Z \leq Y\right\}$, where $X, Y \in L\left(\mathbb{R}^3\right)$.

A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in E$.

2 Main Results

Definition 2.1. A rough triple sequence of fuzzy variables of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be rough convergent almost surely to the fuzzy variables of real number $B_{mnk}(f,X)$ if and only if there exists a set A with $Cr(A) = 1$ such that

$$\lim_{m,n,k\to\infty} |B_{mnk}(f, X(\theta), f(X))| = 0 \tag{1}$$

for every $\theta \in A$. In that case we write $B_{mnk}(f,X) \to f(X)$ almost surely.

Definition 2.2. A rough triple sequence of fuzzy variables of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be rough converges in credibility to the fuzzy variable of Bernstein polynomials if

$$\lim_{m,n,k\to\infty} Cr\left\{|B_{mnk}((f,X), f(X))| \geq \beta + \epsilon\right\} = 0 \tag{2}$$

for every $\epsilon > 0$.

Definition 2.3. A rough triple sequence of fuzzy variables of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be convergent in mean to the fuzzy variables $f(X)$ if

$$\lim_{m,n,k\to\infty} E\left[|B_{mnk}((f,X), f(X))|\right] = 0.$$

Example 2.4. Rough convergent almost surely does not imply rough convergence in credibility.

Let us consider $\theta = \{\theta_{111}.\theta_{222}, \ldots\}$, $\text{Pos}\{\theta_{111}\} = 1$ and $\text{Pos}\{\theta_{uvw}\} = \frac{(u-1)(v-1)(w-1)}{uvw}$ for $u, v, w = 2, 3, 4, \ldots$, and the rough triple sequence of Bernstein polynomials of fuzzy variables are defined by

$$B_{mnk}(f, X(\theta)) = \begin{cases} mnk & \text{if } m = u, n = v, k = w \\ 0 & \text{otherwise} \end{cases}$$

for $m, n, k = 1, 2, 3, \ldots$.

Then the triple sequence of Bernstein polynomials of $B_{mnk}(f,X)$ rough converges almost surely to $f(X)$, we have

$$Cr\left\{|B_{mnk}((f,X), f(X))| \geq \beta + \epsilon\right\} = \frac{(m-1)(n-1)(k-1)}{3(mnk)} \not\to 0.$$

That is, the rough triple sequence of Bernstein polynomials of $B_{mnk}(f,X)$ does not rough converges in credibility to $f(X)$.

Example 2.5. Rough convergence incredibility does not imply rough convergence almost surely.

Let us consider $\theta = \{\theta_{111}.\theta_{222}, \ldots\}$, $\text{Pos}\{\theta_{uvw}\} = \frac{1}{uvw}$ for $u, v, w = 1, 2, 3, \ldots$, and the rough triple sequence of Bernstein polynomials of fuzzy variables are defined by

$$B_{mnk}\left(f, X\left(\theta_{uvw}\right)\right) = \begin{cases} \frac{(u+1)(v+1)(w+1)}{uvw} & \text{if } u = m, m+1, m+2, \cdots;\ v = n, n+1, n+2, \cdots; \\ & w = k, k+1, k+2, \cdots. \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for $m, n, k = 1, 2, 3, \ldots$, and $m, n, k = 0$.

We have

$$Cr\left\{\left|B_{mnk}\left(\left(f, X\right), f\left(X\right)\right)\right| \geq \beta + \epsilon\right\} = \frac{1}{2\left(mnk\right)} \longrightarrow 0.$$

Thus the triple sequence of Bernstein polynomials of $B_{mnk}\left(f, X\right)$ rough converges in incredibility to $f\left(X\right)$. Hence $B_{mnk}\left(f, X\right) \not\to f\left(X\right)$ almost surely.

Example 2.6. Rough convergence in mean does not imply convergence almost surely.

Let us consider the rough triple sequence of Bernstein polynomials of fuzzy variables defined by the equation (3) which does not rough converge almost surely to $f\left(X\right)$. Hence

$$E\left[\left|B_{mnk}\left(\left(f, X\right), f\left(X\right)\right)\right|\right] = \frac{(m+1)(n+1)(k+1)}{3\left(m^2n^2k^2\right)} \to 0.$$
$$\Longrightarrow B_{mnk}\left(f, X\right) \text{ rough converges in mean to } f\left(X\right).$$

Example 2.7. Rough convergence almost surely does not imply rough convergence in mean.

Let us consider $\theta = \{\theta_{111}.\theta_{222}, \ldots\}$, $\text{Pos}\{\theta_{uvw}\} = \frac{1}{uvw}$ for $u, v, w = 1, 2, 3, \ldots$, and the rough triple sequence of Bernstein polynomials of fuzzy variables are defined by

$$B_{mnk}\left(f, X\left(\theta_{uvw}\right), f\left(X\right)\right) = \begin{cases} mnk & \text{if } u = m,\ v = n,\ w = k \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

for $m, n, k = 1, 2, 3, \ldots$, and $f\left(X\right) = 0$. Then the rough triple sequence of Bernstein polynomials of $B_{mnk}\left(f, X\right)$ converges almost surely. Thus

$$E\left[\left|B_{mnk}\left(\left(f, X\right), f\left(X\right)\right)\right|\right] \cong \frac{1}{3} \not\to 0.$$

Hence the rough triple sequence of Bernstein polynomials of $B_{mnk}\left(f, X\right)$ does not rough converge in mean to $f\left(X\right)$.

Theorem 2.8. *Let $\left(x_{mnk}\right)$ be a triple sequence of rough variables and f be a nonnegative Borel measurable function. If f is even increasing on $[0, \infty)$, then for any number $t > 0$, we have*

$$Tr\left\{|x| \geq t\right\} \leq \frac{E\left[f\left(x\right)\right]}{f\left(t\right)} \tag{5}$$

Proof. It is clear that $Tr\left\{|x| \geq f^{-1}\left(\eta\right)\right\}$ is a monotone decreasing function from η on $[0, \infty)$. It follows from the nonnegativity of $f(x)$ that

$$
\begin{aligned}
E[f(x)] &= \int_0^{\infty} Tr\{f(x) \geq \eta\}\, d\eta \\
&= \int_0^{\infty} Tr\left\{|x| \geq f^{-1}\eta\right\} d\eta \\
&\geq \int_0^{f(t)} Tr\left\{|x| \geq f^{-1}(\eta)\right\} d\eta \\
&\geq \int_0^{f(t)} d\eta \,\cdot\, Tr\left\{|x| \geq f^{-1}(f(t))\right\} \\
&= f(t) \,\cdot\, Tr\{|x| \geq t\}.
\end{aligned}
$$

□

Theorem 2.9. *Let (x_{mnk}) be a triple sequence of rough variables. Then for any given numbers $t > 0$ and $p > 0$, we have*

$$Tr\{|x| \geq t\} \leq \frac{E[|x^p|]}{t^p} \tag{6}$$

Proof. It is follows from Theorem 2.8 when $f(x) = |x|^p$. □

Theorem 2.10. *Rough triple sequence of Bernstein polynomials of $B_{mnk}(f, X)$ of fuzzy variables of a real number. If it is rough convergence in mean then it is rough convergence in credibility.*

Proof. It follows from Theorem 2.9 that,

$$Cr\{|B_{mnk}((f, X), f(X))| \geq \beta + \epsilon\} \leq \frac{E[|B_{mnk}((f, X), f(X))|]}{\beta + \epsilon} \to 0 \text{ as } m, n, k \to \infty.$$

Thus $B_{mnk}(f, X)$ converges in credibility to $f(X)$. □

3 Conclusion

In this paper, we introduced and studied a new concept of convergence almost surely (a.s.), convergence in probability, convergence in mean, and convergence in distribution are four important convergence concepts of random sequence and also discusses some convergence concepts of the fuzzy sequence, convergence almost surely, convergence in credibility, convergence in mean, and convergence in distribution for triple sequence space of Bernstein polynomials of rough convergence of fuzzy numbers. For the reference sections, consider the following introduction described the main results are motivating the research.

Conflict of Interest: The authors declare no conflict of interest.

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Representations on Raised Very Thin H_v-fields

Thomas Vougiouklis

Abstract. The hyperstructures have applications in mathematics and other sciences such as biology, physics, linguistics, sociology, to mention but a few. For this, mainly, the largest class of the hyperstructures, the H_v-structures, is used, which satisfy the *weak axioms* where the non-empty intersection replaces the equality and they are straightly related to fuzzy set theory. The *fundamental relations* connect the H_v-structures with the classical ones, moreover, they reveal new concepts as the *H_v-fields*. H_v-numbers are called the elements of an H_v-field and they are used in representation theory. We introduce the *raised finite H_v-fields*, and present some results and examples on 2×2 representations on them.

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Keywords and Phrases: Hyperstructure, Hope, H_v-structure, H_v-group, H_v-ring, H_v-field

1 Introduction

The hyperstructures called H_v-structures, introduced in 1990 [14] and [15] by Vougiouklis, satisfy the *weak axioms* where the non-empty intersection replaces the equality. The h/v-structures are a generalization of H_v-structures, where a *reproductivity of classes*, is valid instead of the reproductivity of elements [18] and [21]. Some basic definitions:

Algebraic hyperstructure $(H, \cdot)$, is a set H equipped with a *hyperoperation* (abbreviated by ***hope***):

$$\cdot : H \times H \to P(H) - \{\varnothing\}.$$

Denote
WASS the weak associativity: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$
and
COW the weak commutativity: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$.
The $(H, \cdot)$ is called H_v-semigroup if it is WASS, it is called ***H_v*-group** if it is reproductive H_v-semigroup: $xH = Hx = H,\ \forall x \in H$.
Motivation. The quotient of a group by any invariant subgroup, is a group. The quotient of a group by any subgroup is a hypergroup, Marty 1934. The quotient of a group by any partition H_v-group, Vougiouklis 1990.

In an H_v-semigroup $(H, \cdot)$, the powers are defined by

$$h^1 = \{h\},\ h^2 = h \cdot h,\ \ldots,\ h^n = h^\circ h^\circ \ldots h^\circ,$$

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where $(^\circ)$ is the n-ary circle hope: take the union of hyperproducts n times, with all possible patterns of parentheses on them. An $(H,\cdot)$ is cyclic of period s if there is a generator h and the minimum s, such that

$$H = h^1 \cup h^2 \cup \ldots \cup h^s.$$

Analogously, the cyclicity for the infinite period is defined. If there are h and s, the minimum one, such that $H = h^s$, then we say that the $(H,\cdot)$, is a single-power cyclic of period s.

A hyperstructure $(R,+,\cdot)$ is called ***H_v*-ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is weak distributive to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing\ ,\ (x+y)z\cap(xz+yz)\neq\varnothing\ ,\ \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be an H_v-ring, a COW H_v-group $(M,+)$ is called ***H_v*-module** over R, if there is an external hope

$$\cdot : R\times M \to P(M)-\{\varnothing\} : (a,x)\mapsto ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing.$$

In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the H_v-vector space is defined.

For more definitions and applications on H_v-structures one can see in books and papers as [1], [3], [6], [15] and [16].

Let $(H,\cdot)$ and $(H,*)$ be H_v-semigroups, then the hope $(\cdot)$ is **smaller** than $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an automorphism

$$f\in \mathrm{Aut}(H,*) \text{ such that } \ xy\subset f(x*y),\ \forall x,y\in H.$$

We say that $(H,*)$ contains $(H,\cdot)$. If $(H,\cdot)$ is a classical structure then it is the basic structure, and $(H,*)$ is H_b-structure.

Minimal is called an H_v-group if it contains no other H_v-group on the same set. We extend this definition to any H_v-structures with more hopes.

The little theorem. *Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively.*

The little theorem leads to a partial order on H_v-structures and posets. Therefore, we can obtain an extremely large number of H_v-structures just putting more elements on any result.

The problem of enumeration and classification of H_v-structures is complicated because we have very great numbers. For example, the number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian; the 1.013.598 are cyclic.

A class of H_v-structures, introduced in [13] and [15], is the following:

Definition 1.1. An H_v-structure is called ***very thin*** iff all hopes are operations except one, which has all results singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $\mathrm{card}A>1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

Some large classes of H_v-structures are the following [19]:

Definition 1.2. Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f: G\to G$ be any map. We define a hope (∂), called *theta-hope*, we write *∂-hope*, on G as follows:

$$x\partial y=\{f(x)\cdot y, x\cdot f(y)\},\quad \forall x,y\in G\quad (\text{resp. } \ x\partial y=(f(x)\cdot y)\cup(x\cdot f(y)),\quad \forall x,y\in G$$

If $(\cdot)$ is commutative, then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

The motivation for this definition is the map derivative where only the product of functions can be used. The basic property is that if $(G, \cdot)$ is a semigroup then $\forall f,$ the (∂) is WASS.

Definition 1.3. (See [12], [15]) Let $(G, \cdot)$ be a groupoid, then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called *P-hopes*: $\forall x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \quad \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called *P-hyperstructures*. The usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup. In some cases, a depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes is the following [4]:

Let $(G, \cdot)$ be abelian group, P any subset of G with more than one element. We define the hope $\times_P$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & ; \text{ if } \ x \neq e \ \ and \ \ y \neq e \\ \\ x \cdot y & ; \text{ if } \ x = e \ \text{ or } \ y = e \end{cases}$$

We call this hope P_e-*hope*. The hyperstructure $(G, \times_P)$ is an abelian $\mathrm{H_v}$-group.

Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we take the restriction of $(\cdot)$ in $H - \{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace h by $\underline{h}$. $\underline{h} \in H$ merges with $h \in H$, if we take as the product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$ and consider them in the same class with representative $\underline{h}$.

2 Fundamental Relations

The main tool to study the hyperstructures is the fundamental relation. In 1970 [8] M. Koskas defined in hypergroups the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [14], [15], [16] and [22] introduced the γ^* and ε^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations β^*, γ^* and ε^*, fundamental relations because they play a very important role in the study of hyperstructures, espicially in their representation theory of them. In 1991, D. Freni [7], proved an open problem that for the classical hypergroups, where the equality is valid, we have $\beta^* = \beta$. However, this problem is open for H_v-groups, therefore, some special classes of them are investigated for which the $\beta^* = \beta$, is valid.

Definition 2.1. The ***fundamental relations*** β^*, γ^*, and ε^* are defined in H_v-groups, H_v-rings, and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring, and vector spaces, respectively.

Remark 2.2. *Let $(G, \cdot)$ be a group and R be any partition in G, then $(G/R, \cdot)$ is an H_v-*group*, so the quotient $(G/R, \cdot)/\beta^*$ is a group, the fundamental one. The classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the R-classes.*

The main theorem together with a way to find the fundamental classes is the following:

Theorem 2.3. *Let $(H, \cdot)$ be H_v-*group *and denote by U the set of all finite products of elements of H. Define the relation β in H by $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β.*

We present a proof for the analogous to the above theorem in the case of an H_v-ring [14], [15], [16] and [6]:

Theorem 2.4. *Let $(R,+,\cdot)$ be an H_v-ring. Denote by U the set of all finite polynomials of elements of R. We define the relation γ in R as follows:*

$$x \ \gamma \ y \quad \textit{iff} \ \ \{x,y\} \subset u, \quad \textit{where} \ \ u \in U.$$

Then, the relation γ^* is the transitive closure of the relation γ.

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of the element a. First, we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) \ : \ c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \quad \forall a,b \in R.$$

Take $a' \in \underline{\gamma}(a)$ and $b' \in \underline{\gamma}(b)$. Then we have $a'\underline{\gamma}\ a$ iff $\exists\ x_1,\dots,x_{m+1}$ with $x_1 = a',\ x_{m+1} = a$ and $u_1,\dots,u_m \in U$ such that $\{x_i,x_{i+1}\} \subset u_i,\ i = 1,\dots,m$ and $b'\underline{\gamma}\ b$ iff $\exists\ y_1,\dots,y_{n+1}$ with $y_1 = b',\ y_{n+1} = b$ and $v_1,\dots,v_n \in U$ such that $\{y_j,y_{j+1}\} \subset v_j,\ j = 1,\dots,n.$

From the above we obtain

$$\{x_i,x_{i+1}\} + y_1 \subset u_i + v_1,\ i = 1,\dots,m-1 \quad \text{and} \quad x_{m+1} + \{y_j,y_{j+1}\} \subset u_m + v_j,\ j = 1,\dots,n.$$

The sums

$$u_i + v_1 = t_i,\ i = 1,\dots,m-1 \quad \text{and} \quad u_m + v_j = t_{m+j-1},\ j = 1,\dots,n,$$

are also polynomials, therefore $t_k \in U$ for all $k \in \{1,\dots,m+n-1\}$.

Now, pick up elements $z_1,\dots,z_{m+n}$ such that

$$z_i \in x_i + y_1,\ i = 1,\dots,n \quad \text{and} \quad z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1,\dots,n,$$

therefore, using the above relations we obtain $\{z_k,z_{k+1}\} \subset t_k,\ k = 1,\dots,m+n-1.$

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b.$ Thus $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c), \quad \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b).$$

In a similar way, we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d), \quad \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b).$$

The WASS and the weak distributivity on R guarantee that the associativity and the distributivity are valid for the quotient R/γ^*. Therefore, R/γ^* is a ring.

Now let σ be an equivalence relation in R such that R/σ is a ring. Denote $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons, i.e. $\forall a,b \in R$, we have

$$\sigma(a) \oplus \sigma(b) = \sigma(c),\ \forall c \in \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(a) \otimes \sigma(b) = \sigma(d),\ \forall d \in \sigma(a) \cdot \sigma(b).$$

Thus we can write, $\forall a,b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B) \quad \text{and} \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

By induction, we extend these relations on finite sums and products. Thus, $\forall u \in U$, we have $\sigma(x) = \sigma(u)$, $\forall x \in u$. Consequently,

$$x \in \gamma(a) \quad \text{implies} \quad x \in \sigma(a),\ \forall x \in R.$$

But σ is transitively closed, so we obtain:

$$x \in \underline{\gamma}(x) \quad \text{implies} \quad x \in \sigma(a).$$

That means that $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. □

An element is called ***single*** if its fundamental class is singleton [15].

Fundamental relations are used for general definitions. Thus we have [14]:

Definition 2.5. An H_v-ring $(R, +, \cdot)$ is called ***H_v-field*** if R/γ^* is a field.

The analogous to Theorem 2.4 on H_v-vector spaces, can be proved:

Let $(V, +)$ be H_v-vector space over the H_v-field F. Denote U the set of all expressions of finite hopes on finite sets of elements of F and V. Define the relation ε, in V, as follows: $x\varepsilon y$ iff $\{x, y\} \subset u$ where $u \in U$. Then ε^* is the transitive closure of ε.

Definition 2.6. Let $(L, +)$ be H_v-vector space over an H_v-field $(F, +, \cdot)$; $\varphi : F \to F/\gamma^*$ the canonical map; $\omega_F = \{x \in F \ : \ \varphi(x) = 0\}$, the core, 0 is the zero of F/γ^*. Let ω_L be the core of $\varphi' : L \to L/\varepsilon^*$ and denote by 0 the zero of L/ε^*, as well. Take the *bracket (commutator) hope*:

$$[\ ,\ ] : L \times L \to P(L) : (x, y) \mapsto [x, y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing, \ \forall x, x_1, x_2, y, y_1, y_2 \in L$ and $\forall \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \ \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \ \forall x, y, z \in L$

Definition 2.7. (See [18] and [21]) The H_v-semigroup $(H, \cdot)$ is called ***h/v-group*** if H/β^* is a group.

The H_v-group is a generalization of H_v-group, where a reproductive of classes, is valid: if $\sigma(x)$, $\forall x \in H$, equivalence classes, then $x\sigma(y) = \sigma(xy) = \sigma(x)y$, $\forall x, y \in H$. Similarly, h/v-rings, h/v-fields, h/v-vector spaces etc, are defined.

The ***uniting elements*** method, introduced by Corsini & Vougiouklis in 1989, is the following [2]: Let $\boldsymbol{G}$ be a structure and a not valid property d, described by a set of equations. Take the partition in $\boldsymbol{G}$ for which put in the same class, all pairs of elements that cause the non-validity of d. The quotient by this partition $\boldsymbol{G}/d$ is an H_v-structure. Then, quotient out $\boldsymbol{G}/d$ by β^*, is a stricter structure $(\boldsymbol{G}/d)/\beta^*$ for which the property d is valid.

Theorem 2.8. (See [15]) *Let $(\boldsymbol{R}, +, \cdot)$ be a ring, and $F = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ be system of equations on $\boldsymbol{R}$ consisting of subsystems $F_m = \{f_1, \ldots, f_m\}$ and $F_n = \{f_{m+1}, \ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements using F and F_m respectively, and σ_n the equivalence defined on F_n on the ring $\boldsymbol{R}_m = (\boldsymbol{R}/\sigma_m)/\gamma^*$. Then*

$$(\boldsymbol{R}/\sigma)/\gamma^* \cong (\boldsymbol{R}_m/\sigma_n)/\gamma^*.$$

Theorem 2.9. *Let $(\boldsymbol{H}, \cdot)$ be an H_v-group and $\boldsymbol{H}/\beta^*$ its fundamental group. Suppose that $\boldsymbol{H}/\beta^*$ is not commutative or it is not cyclic, then $(\boldsymbol{H}, \cdot)$ is not COW or cyclic, respectively.*

Proof. Straightforward since if $(\boldsymbol{H}, \cdot)$ is COW or cyclic then its fundamental group $\boldsymbol{H}/\beta^*$ is commutative or cyclic, respectively. □

3 H_v-fields

Definition 3.1. We call ***Raised Very Thin H_v-fields*** the ones obtained from classical rings by enlarging only one result adding only one element, of the underline set, such that the fundamental structure is a field.

Combining the uniting elements procedure with the raise theory we can obtain stricter structures or hyperstructures. So, raising operations or hopes we can obtain more complicated structures as we can see in the following.

Theorem 3.2. *In the ring of integers $(\boldsymbol{Z},+,\cdot)$, we fix a number $m>1$. We raise in the product the special result $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then $(\boldsymbol{Z},+,\otimes)$ becomes an* H_v-ring, *with a finite fundamental ring:*

$$(\boldsymbol{Z},+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_m,+,\cdot).$$

If $m=p$, prime, then $(\boldsymbol{Z},+,\otimes)$ is a raised very thin H_v-field, *with the finite fundamental field.*
Raising only the result $a\cdot b$ of two fixed elements $a,b\in\boldsymbol{Z}-\{0,1\}$, by setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, then we have the same results and $(\boldsymbol{Z},+,\otimes)$ is a raised very thin H_v-field, *where the elements* 0 *and* 1 *are scalars.*

Proof. Remark that the expressions of sums and products which contain more than one element are the ones that have at least one time the $0\otimes m$. Adding to $0\otimes m$ the element 1, several times we have the modm equivalence classes. On the other side, by adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, the γ^*-classes form a ring isomorphic to $(\boldsymbol{Z}_m,+,\cdot)$.

The rest of the proof is straightforward. Notice only that we can transfer the generalized raised case if we consider the expression $a\otimes b-a\cdot b=\{0,m\}$. □

Theorem 3.3. *In the ring $(\boldsymbol{Z}_n,+,\cdot)$, with $n=ms$ we raise in the product only the result $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then*

$$(\boldsymbol{Z}_n,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_m,+,\cdot).$$

If $m=p$, prime, then $(\boldsymbol{Z}_n,+,\otimes)$ is a raised very thin H_v-field.
Raising only the result $a\cdot b$ of two fixed elements $a,b\in\boldsymbol{Z}_n-\{0,1\}$, by setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, then we have the same results but $(\boldsymbol{Z}_n,+,\otimes)$ is a raised very thin H_v-field, *where, moreover, the elements* 0 *and* 1 *are scalars.*

Proof. Analogous to the above Theorem. □

Now, we focus on raised very thin minimal H_v-fields obtained by a classical field.

Theorem 3.4. *In a field $(\boldsymbol{F},+,\cdot)$, we raise only the product of two elements $a\cdot b$, by $a\otimes b=\{a\cdot b,c\}$, where $c\neq a\cdot b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v-field *$(\boldsymbol{F},+,\otimes)/\gamma^*\cong\{0\}$.*
Thus, there is no non-degenerate H_v-field *obtained by a field by raising any product.*

Proof. Take any $x\in\boldsymbol{F}-\{0\}$, then from $a\otimes b=\{ab,c\}$ we obtain $(a\otimes b)-ab=\{0,c-ab\}$ and then $(x(c-ab)^{-1})\otimes((a\otimes b)-ab)=\{0,x\}$. thus, $0\gamma x, x\in\boldsymbol{F}-\{0\}$. Which means that every x is in the same fundamental class with 0. Thus, $(\boldsymbol{F},+,\otimes)/\gamma^*\cong\{0\}$. □

Theorem 3.5. *In a field $(\boldsymbol{F},+,\cdot)$, we raise only the sum of two elements $a+b$, by setting $a\oplus b=\{a+b,c\}$, where $c\neq a+b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v-field *$(\boldsymbol{F},\oplus,\cdot)/\gamma^*\cong\{0\}$.*
Thus, there is no non-degenerate H_v-field *obtained by a field by raising any sum.*

Proof. Take any $x \in \boldsymbol{F} - \{0\}$, then from $a \oplus b = \{a+b, c\}$ we obtain $(a \oplus b) - (a+b) = \{0, c-(a+b)\}$ and then $[x(c-(a+b))^{-1}] \cdot [(a \oplus b) - (a+b)] = \{0, x\}$. Thus, $0 \ \gamma \ x, x \in \boldsymbol{F} - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus, $(\boldsymbol{F}, \oplus, \cdot)/\gamma^* \cong \{0\}$. □

The above two theorems state that all H_v-fields obtained from a field by raising any sum or product, are degenerate.

Several results can be obtained by using ∂-hopes [19]: For example, consider the group of integers $(\boldsymbol{Z}, +)$ and $n \neq 0$ be natural number. Take the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{0\}$, then $(\boldsymbol{Z}, \partial)/\beta^* \cong (\boldsymbol{Z}_n, +)$.

Theorem 3.6. *Take the ring of integers $(\boldsymbol{Z}, +, \cdot)$ and fix $n \neq 0$ a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{0\}$. Then $(\boldsymbol{Z}, \partial_+, \partial.)$, where ∂_+ and $\partial.$ are the ∂-*hopes *refereed to the sum and the product, respectively, is an H_v*-near-ring, *with*

$$(\boldsymbol{Z}, \partial_+, \partial.)/\gamma^* \cong \boldsymbol{Z}_n.$$

We have the same result if we consider the map f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in \boldsymbol{Z} - \{n\}$.

A special case of the above is for $n = p$, prime, then $(\boldsymbol{Z}, \partial_+, \partial.)$ is an H_v-field.

From the very thin hopes the Attach Construction is obtained [20]:

Definition 3.7. ***(a)*** Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \ \forall x \in H \ \text{ and } \ v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is called *attach h/v-group* of $(H, \cdot)$, where $(\underline{H}, \cdot)/\beta^* \cong \boldsymbol{Z}_2$ and v is single. Scalars and units of $(H, \cdot)$ are scalars and units in $(\underline{H}, \cdot)$. If $(H, \cdot)$ is *COW* then $(\underline{H}, \cdot)$ is *COW*.
(b) $(H, \cdot)$ H_v-semigroup, $v \notin H$, $(\underline{H}, \cdot)$ its attached h/v-group. Take $0 \notin \underline{H}$ and define in $\underline{H}_\circ = H \cup \{v, 0\}$ two hopes:

$$\text{hypersum}(+) : 0+0 = x+v = v+x = 0, \ 0+v = v+0 = x+y = v, \ 0+x = x+0 = v+v = H, \ \forall x, y \in H$$

$$\text{hyperproduct } (\cdot) : \text{remains the same as in } \underline{H}, \ \text{moreover}, \ 0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \ \forall x \in \underline{H}.$$

Then $(\underline{H}_\circ, +, \cdot)$ is an h/v-field with $(\underline{H}_\circ, +, \cdot)/\gamma^* \cong \boldsymbol{Z}_3$. $(+)$ is associative, $(\cdot)$ is WASS and weak distributive to $(+)$. 0 is zero absorbing in $(+)$. $(\underline{H}_\circ, +, \cdot)$ is the *attached h/v-field* of $(H, \cdot)$.

Let $(G, \cdot)$ be semigroup and $v \notin G$ be an element appearing in a product ab, where $a, b \in G$, thus the result becomes $a \otimes b = \{ab, v\}$. Then the minimal hope $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G', \otimes)$ is a minimal H_v-semigroup which has a fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy, \quad \forall (x, y) \in G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \quad \text{and} \quad v \otimes x = abx, \quad \forall x \in G.$$

$(G', \otimes)$ is very thin H_v-semigroup. If $(G, \cdot)$ is commutative then $(G', \otimes)$ is strong commutative.

4 Representations and applications

H_v-structures used in Representation Theory (abbreviate ***rep***) of H_v-groups can be achieved by generalized permutations or by H_v-matrices [6], [15], [17].

$\boldsymbol{H_v}$**-matrix** is a matrix with entries of an H_v-ring. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the n-ary circle hope on the hyper-sum.

Notation. In a set of matrices or H_v-matrices, we denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries.

The problem of the H_v-matrix reps is the following:

Definition 4.1. Let $(H, \cdot)$ be H_v-group. Find an H_v-ring $(R, +, \cdot)$, a set $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$ and a map $T : H \to M_R : h \mapsto T(h)$, called $\boldsymbol{H_v}$***-matrix rep***, such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing,\ \forall h_1, h_2 \in H.$$

If $T(h_1h_2) \subset T(h_1)T(h_2)$, then T is an *inclusion rep.*
If $T(h_1h_2) = T(h_1)T(h_2) = \{T(h) \mid h \in h_1h_2\}$, then T is a *good rep.*
If T is a *good rep* and one to one then it is a *faithful rep.*

The rep problem is simplified in cases such as if the H_v-rings have scalars 0 and 1.

The main theorem of the theory of reps is the following:

Theorem 4.2. *A necessary condition to have an inclusion rep T of an H_v-group $(H, \cdot)$ by $n \times n$, H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*

$\forall \beta^*(x),\ x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i, j \in \{1, \dots, n\}\}$$

The inclusion rep $T : H \longrightarrow M_R : a \mapsto T(a) = (a_{ij})$ induces a homomorphic

$$T^* : H/\beta^* \longrightarrow R/\gamma^* : T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \quad \forall \beta^*(a) \in H/\beta^*,$$

where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$.

An important hope on non-square matrices is defined [5] and [6]:

Definition 4.3. Let $A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, $1 \le s \le m$, $1 \le t \le n$. Define a mod-like map, called *helix-projection* of type $\underline{st}$, $\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij})$, where A has entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \le i \le s,\ \ 1 \le j \le t \quad \text{and} \quad \kappa, \lambda \in N,\ \ i + \kappa s \le m,\ \ j + \lambda t \le n\}.$$

$A\underline{st}$ is a set of s × t-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}$, $\forall i, j$. Obviously, $A\underline{mn} = A$.
Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be matrices.
Denote $s = \min(m, u)$, $t = \min(n, u)$, then we define the ***helix-sum*** by

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A, B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \quad \text{and} \quad b_{ij} \in \underline{b}_{ij}\}$.
Denote $s = \min(n, u)$, then we define the ***helix-product*** by

$$\otimes : M_{m\times n} \times M_{u\times v} \to P(M_{m\times v}) : (A, B) \to A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m\times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it}b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \quad \text{and} \quad b_{ij} \in \underline{b}_{ij}\}$.

The helix-sum is commutative and WASS. The helix-product is WASS.

The definition of a Lie-bracket is immediate, so, the *helix-Lie Algebra* is defined.

Using several classes of H_v-structures one can face several representations [15]:

Let $\boldsymbol{M} = \boldsymbol{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring $\boldsymbol{R}$ and $\boldsymbol{P} = \{P_i \ : \ i \in I\} \subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on $\boldsymbol{M}$ as follows

$$\underline{P} : \boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B \ : \ i \in I\} \subseteq \boldsymbol{M}$$

where P^t denotes the transpose of the matrix P.

In last decades the hyperstructures had a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics - conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and papers [1], [3], [4], [6] and [22], one can find numerous applications.

The Lie-Santilli theory on isotopies was born in the 1960s to solve Hadronic Mechanics problems. Santilli proposed a lifting of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix [9], [10], [11]. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields, needed in this theory correspond to the hyperstructures, were introduced by Santilli & Vougiouklis in 1999 [4], [6], [11].

Definition 4.4. $(F, +, \cdot)$, where $(+)$ is operation and $(\cdot)$ hope, is an ***e-hyperfield*** if the following are valid: $(F, +)$ is an abelian group with unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$, there exist a scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there is a unique inverse $x^{-1} : 1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. If the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we have a *strong e-hyperfield.*

The Main e-Construction: Given a group $(G, \cdot)$, e unit, define hopes $(\otimes)$ by:

$$x \otimes y = \{xy, g_1, g_2, \ldots\}, \ \forall x, y \in G - \{e\} \quad \text{and} \quad g_1, g_2, \ldots \in G - \{e\}$$

$(G, \otimes)$ is H_b-group which contains $(G, \cdot)$. $(G, \otimes)$ is e-hypergroup. Moreover, if $\forall x, y$ such that $xy = e$, so $x \otimes y = e$, then $(G, \otimes)$ becomes a strong e-hypergroup.

Example 4.5. In the set of quaternions $\boldsymbol{Q} = \{1, -1, i, -i, j, -j, k, -k\}$, with $i^2 = j^2 = -1$, $ij = -ji = k$, we denote $\underline{i} = \{i, -i\}$, $\underline{j} = \{j, -j\}$, $\underline{k} = \{k, -k\}$ and we define hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}$, $k * i = \underline{j}$ and $i * j = \underline{k}$. Then $(\boldsymbol{Q}, *)$ is strong e-hypergroup.

5 On 2×2 Very Thin H_v-matrix representations

From now to the end we focus on the small non-degenerate H_v-fields on $(\boldsymbol{Z}_n, +, \cdot)$, which in isotheory, satisfy the following conditions:

1. very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. if an element has an inverse element, this is unique.

Therefore, we cannot raise the result if it is 1 and we cannot put 1 in enlargement.

We present some known results and examples on the topic [20], [21] and [23], along with some new ones.

Theorem 5.1. *The multiplicative H_v-fields on $(\boldsymbol{Z}_4, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic ones:*
The only product which is set is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$.
Fundamental classes: $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ and we have $(\boldsymbol{Z}_4, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$.

Example 5.2. Take the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:

$$I = E_{11} + E_{22}, \quad a = E_{11} + E_{12} + E_{22}, \quad b = E_{11} + 2E_{12} + E_{22}, \quad c = E_{11} + 3E_{12} + E_{22},$$

$$d = E_{11} + 3E_{22}, \quad e = E_{11} + E_{12} + 3E_{22}, \quad f = E_{11} + 2E_{12} + 3E_{22}, \quad g = E_{11} + 3E_{12} + 3E_{22},$$

then, for $\boldsymbol{X} = \{I, a, b, c, d, e, f, g\}$, we obtain the following multiplicative table:

$\otimes$	$\boldsymbol{I}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$
$\boldsymbol{I}$	I	a	b	c	d	e	f	g
$\boldsymbol{a}$	a	b	c	I	g	d	e	f
$\boldsymbol{b}$	b	c	I	a	d, f	e, g	d, f	e, g
$\boldsymbol{c}$	c	I	a	b	e	f	g	d
$\boldsymbol{d}$	d	e	f	g	I	a	b	c
$\boldsymbol{e}$	e	f	g	d	c	I	a	b
$\boldsymbol{f}$	f	g	d	e	I, b	a, c	I, b	a, c
$\boldsymbol{g}$	g	d	e	f	a	b	c	b

The $(\boldsymbol{X}, \otimes)$ is COW H_v-group where the fundamental classes are $\underline{I} = \{I, b\}$, $\underline{a} = \{a, c\}$, $\underline{d} = \{d, f\}$, $\underline{e} = \{e, g\}$ and the fundamental group is isomorphic to $(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2, +)$. There is only one unit and every element has a unique double inverse. Only f has one more right inverse element d, since $f \otimes d = \{I, b\}$. $(\boldsymbol{X}, \otimes)$ is not cyclic.

Example 5.3. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:

$$a = E_{11} + E_{22}, \quad a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22}, \quad a_3 = E_{11} + 3E_{12} + E_{22},$$

$$b = E_{11} + 3E_{22}, \quad b_1 = E_{11} + E_{12} + 3E_{22}, \quad b_2 = E_{11} + 2E_{12} + 3E_{22}, \quad b3 = E_{11} + 3E_{12} + 3E_{22},$$

$$c = 3E_{11} + E_{22}, \quad c_1 = 3E_{11} + E_{12} + E_{22}, \quad c_2 = 3E_{11} + 2E_{12} + E_{22}, \quad c3 = 3E_{11} + 3E_{12} + E_{22},$$

$$d = 3E_{11} + 3E_{22}, \quad d_1 = 3E_{11} + E_{12} + 3E_{22}, \quad d_2 = 3E_{11} + 2E_{12} + 3E_{22}, \quad d_3 = 3E_{11} + 3E_{12} + 3E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, a_2, a_3, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d, d_1, d_2, d_3\}$, we obtain the following multiplicative table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{d}$	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	$\boldsymbol{d_3}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	b	b_1	b_2	b_3	c	c_1	c_2	c_3	d	d_1	d_2	d_3
$\boldsymbol{a_1}$	a_1	a_2	a_3	a	b_3	b	b_1	b_2	c_1	c_2	c_3	c	d_3	d	d_1	d_2
$\boldsymbol{a_2}$	a_2	a_3	a	a_1	b, b_2	b_1, b_3	b, b_2	b_1, b_3	c_2	c_3	c	c_1	d, d_2	d_1, d_3	d, d_2	d_1, d_3
$\boldsymbol{a_3}$	a_3	a	a_1	a_2	b_1	b_2	b_3	b	c_3	c	c_1	c_2	d_1	d_2	d_3	d
$\boldsymbol{b}$	b	b_1	b_2	b_3	a	a_1	a_2	a_3	d	d_1	d_2	d_3	c	c_1	c_2	c_3
$\boldsymbol{b_1}$	b_1	b_2	b_3	b	a_3	a	a_1	a_2	d_1	d_2	d_3	d	c_3	c	c_1	c_2
$\boldsymbol{b_2}$	b_2	b_3	b	b_1	a, a_2	a_1, a_3	a, a_2	a_1, a_3	d_2	d_3	d	d_1	c, c_2	c_1, c_3	c, c_2	c_1, c_3
$\boldsymbol{b_3}$	b_3	b	b_1	b_2	a_1	a_2	a_3	a	d_3	d	d_1	d_2	c_1	c_2	c_3	c
$\boldsymbol{c}$	c	c_3	c_2	c_1	d	d_3	d_2	d_1	a	a_3	a_2	a_1	b	b_3	b_2	b_1
$\boldsymbol{c_1}$	c_1	c	c_3	c_2	d_3	d_2	d_1	d	a_1	a	a_3	a_2	b_3	b_2	b_1	b
$\boldsymbol{c_2}$	c_2	c_1	c	c_3	d, d_2	d_1, d_3	d, d_2	d_1, d_3	a_2	a_1	a	a_3	b, b_2	b_1, b_3	b, b_2	b_1, b_3
$\boldsymbol{c_3}$	c_3	c_2	c_1	c	d_1	d	d_3	d_2	a_3	a_2	a_1	a	b_1	b	b_3	b_2
$\boldsymbol{d}$	d	d_3	d_2	d_1	c	c_3	c_2	c_1	b	b_3	b_2	b_1	a	a_3	a_2	a_1
$\boldsymbol{d_1}$	d_1	d	d_3	d_2	c_3	c_2	c_1	c	b_1	b	b_3	b_2	a_3	a_2	a_1	a
$\boldsymbol{d_2}$	d_2	d_1	d	d_3	c, c_2	c_1, c_3	c, c_2	c_1, c_3	b_2	b_1	b	b_3	a, a_2	a_1, a_3	a, a_2	a_1, a_3
$\boldsymbol{d_3}$	d_3	d_2	d_1	d	c_1	c	c_3	c_2	b_3	b_2	b_1	b	a_1	a	a_3	a_2

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group where the fundamental classes are $\underline{a} = \{a, a_2\}$, $\underline{a}_1 = \{a_1, a_3\}$, $\underline{b} = \{b, b_2\}$, $\underline{b}_1 = \{b_1, b_3\}$, $\underline{c} = \{c, c_2\}$, $\underline{c}_1 = \{c_1, c_3\}$, $\underline{d} = \{d, d_2\}$, $\underline{d}_1 = \{d_1, d_3\}$, with multiplicative table the following:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c_1}}$	$\underline{\boldsymbol{d}}$	$\underline{\boldsymbol{d_1}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_1$	$\underline{c}$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_1$
$\underline{\boldsymbol{a_1}}$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_1$	$\underline{d}$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b_1}}$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_1$	$\underline{a}$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_1$
$\underline{\boldsymbol{c_1}}$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{d}}$	$\underline{d}$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_1$	$\underline{b}$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{d_1}}$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$

Moreover, in $(\boldsymbol{X}, \otimes)$ there is only one unit and every element has unique double inverse. The element b_2 is left inverse to b and b_2 because $a \in b_2 b$ and $a \in b_2 b_2$. The element d_2 is left inverse to d and d_2 because $a \in d_2 d$, $a \in d_2 d_2$. $(\boldsymbol{X}, \otimes)$ is not cyclic, since, from Theorem 2.9, the $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is not cyclic.

Example 5.4. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4, +, \otimes)$ of the case that only $3 \otimes 2 = \{0, 2\}$ is a hyperproduct:

$$a = E_{11} + E_{22}, \quad a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22}, \quad a_3 = E_{11} + 3E_{12} + E_{22},$$

$$b = E_{11} + 3E_{22}, \quad b_1 = E_{11} + E_{12} + 3_{E_{22}}, \quad b_2 = E_{11} + 2E_{12} + 3E_{22}, \quad b_3 = E_{11} + 3E_{12} + 3E_{22},$$

$$c = 3E_{11} + E_{22}, \quad c_1 = 3E_{11} + E_{12} + E_{22}, \quad c_2 = 3E_{11} + 2E_{12} + E_{22}, \quad c_3 = 3E_{11} + 3E_{12} + E_{22},$$

$$d = 3E_{11} + 3E_{22}, \quad d_1 = 3E_{11} + E_{12} + 3E_{22}, \quad d_2 = 3E_{11} + 2E_{12} + 3E_{22}, \quad d_3 = 3E_{11} + 3E_{12} + 3E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, a_2, a_3, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d, d_1, d_2, d_3\}$, we obtain the following table:

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_2\}$, $\underline{a}_1 = \{a_1, a_3\}$, $\underline{b} = \{b, b_2\}$, $\underline{b}_1 = \{b_1, b_3\}$, $\underline{c} = \{c, c_2\}$, $\underline{c}_1 = \{c_1, c_3\}$, $\underline{d} = \{d, d_2\}$, $\underline{d}_1 = \{d_1, d_3\}$, with table as the above example.

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{d}$	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	$\boldsymbol{d_3}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	b	b_1	b_2	b_3	c	c_1	c_2	c_3	d	d_1	d_2	d_3
$\boldsymbol{a_1}$	a_1	a_2	a_3	a	b_3	b	b_1	b_2	c_1	c_2	c_3	c	d_3	d	d_1	d_2
$\boldsymbol{a_2}$	a_2	a_3	a	a_1	b_2	b_3	b	b_1	c_2	c_3	c	c_1	d_2	d_3	d	d_1
$\boldsymbol{a_3}$	a_3	a	a_1	a_2	b_1	b_2	b_3	b	c_3	c	c_1	c_2	d_1	d_2	d_3	d
$\boldsymbol{b}$	b	b_1	b_2	b_3	a	a_1	a_2	a_3	d	d_1	d_2	d_3	c	c_1	c_2	c_3
$\boldsymbol{b_1}$	b_1	b_2	b_3	b	a_3	a	a_1	a_2	d_1	d_2	d_3	d	c_3	c	c_1	c_2
$\boldsymbol{b_2}$	b_2	b_3	b	b_1	a_2	a_3	a	a_1	d_2	d_3	d	d_1	c_2	c_3	c	c_1
$\boldsymbol{b_3}$	b_3	b	b_1	b_2	a_1	a_2	a_3	a	d_3	d	d_1	d_2	c_1	c_2	c_3	c
$\boldsymbol{c}$	c	c_3	c, c_2	c_1	d	d_3	d, d_2	d_1	a	a_3	a, a_2	a_1	b	b_3	b, b_2	b_1
$\boldsymbol{c_1}$	c_1	c	c_1, c_3	c_2	d_3	d_2	d_1, d_3	d	a_1	a	a_1, a_3	a_2	b_3	b_2	b_1, b_3	b
$\boldsymbol{c_2}$	c_2	c_1	c, c_2	c_3	d_2	d_1	d, d_2	d_3	a_2	a_1	a, a_2	a_3	b_2	b_1	b, b_2	b_3
$\boldsymbol{c_3}$	c_3	c_2	c_1, c_3	c	d_1	d	d_1, d_3	d_2	a_3	a_2	a_1, a_3	a	b_1	b	b_1, b_3	b_2
$\boldsymbol{d}$	d	d_3	d, d_2	d_1	c	c_3	c, c_2	c_1	b	b_3	b, b_2	b_1	a	a_3	a, a_2	a_1
$\boldsymbol{d_1}$	d_1	d	d_1, d_3	d_2	c_3	c_2	c_1, c_3	c	b_1	b	b_1, b_3	b_2	a_3	a_2	a_1, a_3	a
$\boldsymbol{d_2}$	d_2	d_1	d, d_2	d_3	c_2	c_1	c, c_2	c_3	b_2	b_1	b, b_2	b_3	a_2	a_1	a, a_2	a_3
$\boldsymbol{d_3}$	d_3	d_2	d_1, d_3	d	c_1	c	c_1, c_3	c_2	b_3	b_2	b_1, b_3	b	a_1	a	a_1, a_3	a_2

Moreover, in $(\boldsymbol{X}, \otimes)$ there is only one unit a, and every element has unique double inverse. The element c_2 is right inverse to c and c_2 because $a \in cc_2$, $a \in c_2c_2$. The element d_2 is right inverse to d and d_2 because $a \in dd_2$, $a \in d_2d_2$. $(\boldsymbol{X}, \otimes)$ is not cyclic, since, from Theorem 2.9, the $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is not cyclic.

Theorem 5.5. *All multiplicative H_v-fields on $(\boldsymbol{Z}_6, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, with one hyperproduct, are the following isomorphic cases:*

(I) $2 \otimes 3 = \{0, 3\}$, $2 \otimes 4 = \{2, 5\}$, $3 \otimes 4 = \{0, 3\}$, $3 \otimes 5 = \{0, 3\}$, $4 \otimes 5 = \{2, 5\}$

Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ *and* $(\boldsymbol{Z}_6, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$, $2 \otimes 4 = \{0, 2\}$ *or* $\{2, 4\}$, $2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$, $3 \otimes 4 = \{0, 2\}$ *or* $\{0, 4\}$, $3 \otimes 5 = \{3, 5\}$, $4 \otimes 5 = \{0, 2\}$ *or* $\{2, 4\}$.

In all cases, fundamental classes are $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ *and* $(\boldsymbol{Z}_6, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$.

Example. In the H_v-field $(\boldsymbol{Z}_6, +, \otimes)$ where only the hyperproduct is $2 \otimes 4 = \{2, 5\}$, take the H_v-matrices of type $\underline{i} = E_{11} + iE_{12} + 4E_{22}$, where $i = 0, 1, \ldots, 5$, then the multiplicative table of the hyperproduct of those H_v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}, \underline{5}$	$\underline{0}, \underline{3}$	$\underline{1}, \underline{4}$	$\underline{2}, \underline{5}$	$\underline{0}, \underline{3}$	$\underline{1}, \underline{4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

Classes: $[\underline{0}] = \{\underline{0}, \underline{3}\}$, $[\underline{1}] = \{\underline{1}, \underline{4}\}$, $[\underline{2}] = \{\underline{2}, \underline{5}\}$ and fundamental group isomorphic to $(\boldsymbol{Z}_3, +)$. $(\boldsymbol{Z}_6, \otimes)$ is h/v-group which is cyclic where $\underline{2}$ is generator of period 4 and $\underline{4}$ is generator of period 5.

Example 5.6. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_6, +, \otimes)$ of the case that only $4 \otimes 5 = \{2, 5\}$ is a hyperproduct. We set

$$a = E_{11} + E_{22}, \quad a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22},$$

$$a_3 = E_{11} + 3E_{12} + E_{22}, \quad a_4 = E_{11} + 4E_{12} + E_{22}, \quad a_5 = E_{11} + 5E_{12} + E_{22},$$

$$b = E_{11} + 5E_{22}, \quad b_1 = E_{11} + E_{12} + 5E_{22}, \quad b_2 = E_{11} + 2E_{12} + 5E_{22},$$

$$b_3 = E_{11} + 3E_{12} + 5E_{22}, \quad b_4 = E_{11} + 4E_{12} + 5E_{22}, \quad b_5 = E_{11} + 5E_{12} + 5E_{22},$$

$$c = 5E_{11} + E_{22}, \quad c_1 = 5E_{11} + E_{12} + E_{22}, \quad c_2 = 5E_{11} + 2E_{12} + E_{22},$$

$$c_3 = 5E_{11} + 3E_{12} + E_{22}, \quad c_4 = 5E_{11} + 4E_{12} + E_{22}, \quad c_5 = 5E_{11} + 5E_{12} + E_{22},$$

$$d = 5E_{11} + 5E_{22}, \quad d_1 = 5E_{11} + E_{12} + 5E_{22}, \quad d_2 = 5E_{11} + 2E_{12} + 5E_{22},$$

$$d_3 = 5E_{11} + 3E_{12} + 5E_{22}, \quad d_4 = 5E_{11} + 4E_{12} + 5E_{22}, \quad d_5 = 5E_{11} + 5E_{12} + 5E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, a_2, a_3, a_4, a_5, b, b_1, b_2, b_3, b_4, b_5, c, c_1, c_2, c_3, c_4, c_5, d, d_1, d_2, d_3, d_4, d_5\}$, we obtain the table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{b_5}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{c_4}$	$\boldsymbol{c_5}$	$\boldsymbol{d}$	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	$\boldsymbol{d_3}$	$\boldsymbol{d_4}$	$\boldsymbol{d_5}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	a_4	a_5	b	b_1	b_2	b_3	b_4	b_5	c	c_1	c_2	c_3	c_4	c_5	d	d_1	d_2	d_3	d_4	d_5
$\boldsymbol{a_1}$	a_1	a_2	a_3	a_4	a_5	a	b_5	b	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4	c_5	c	d_5	d	d_1	d_2	d_3	d_4
$\boldsymbol{a_2}$	a_2	a_3	a_4	a_5	a	a_1	b_4	b_5	b	b_1	b_2	b_3	c_2	c_3	c_4	c_5	c	c_1	d_4	d_5	d	d_1	d_2	d_3
$\boldsymbol{a_3}$	a_3	a_4	a_5	a	a_1	a_2	b_3	b_4	b_5	b	b_1	b_2	c_3	c_4	c_5	c	c_1	c_2	d_3	d_4	d_5	d	d_1	d_2
$\boldsymbol{a_4}$	a_4	a_5	a	a_1	a_2	a_3	b_2, b_5	b, b_3	b_1, b_4	b_2, b_5	b, b_3	b_1, b_4	c_4	c_5	c	c_1	c_2	c_3	d_2, d_5	d, d_3	d_1, d_4	d_2, d_5	d, d_3	d_1, d_4
$\boldsymbol{a_5}$	a_5	a	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	b_5	b	c_5	c	c_1	c_2	c_3	c_4	d_1	d_2	d_3	d_4	d_5	d
$\boldsymbol{b}$	b	b_1	b_2	b_3	b_4	b_5	a	a_1	a_2	a_3	a_4	a_5	d	d_1	d_2	d_3	d_4	d_5	c	c_1	c_2	c_3	c_4	c_5
$\boldsymbol{b_1}$	b_1	b_2	b_3	b_4	b_5	b	a_5	a	a_1	a_2	a_3	a_4	d_1	d_2	d_3	d_4	d_5	d	c_5	c	c_1	c_2	c_3	c_4
$\boldsymbol{b_2}$	b_2	b_3	b_4	b_5	b	b_1	a_4	a_5	a	a_1	a_2	a_3	d_2	d_3	d_4	d_5	d	d_1	c_4	c_5	c	c_1	c_2	c_3
$\boldsymbol{b_3}$	b_3	b_4	b_5	b	b_1	b_2	a_3	a_4	a_5	a	a_1	a_2	d_3	d_4	d_5	d	d_1	d_2	c_3	c_4	c_5	c	c_1	c_2
$\boldsymbol{b_4}$	b_4	b_5	b	b_1	b_2	b_3	a_2, a_5	a, a_3	a_1, a_4	a_2, a_5	a, a_3	a_1, a_4	d_4	d_5	d	d_1	d_2	d_3	c_2, c_5	c, c_3	c_1, c_4	c_2, c_5	c, c_3	c_1, c_4
$\boldsymbol{b_5}$	b_5	b	b_1	b_2	b_3	b_4	a_1	a_2	a_3	a_4	a_5	a	d_5	d	d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c
$\boldsymbol{c}$	c	c_5	c_4	c_3	c_2	c_1	d	d_5	d_4	d_3	d_2	d_1	a	a_5	a_4	a_3	a_2	a_1	b	b_5	b_4	b_3	b_2	b_1
$\boldsymbol{c_1}$	c_1	c	c_5	c_4	c_3	c_2	d_5	d_4	d_3	d_2	d_1	d	a_1	a	a_5	a_4	a_3	a_2	b_5	b_4	b_3	b_2	b_1	b
$\boldsymbol{c_2}$	c_2	c_1	c	c_5	c_4	c_3	d_4	d_3	d_2	d_1	d	d_5	a_2	a_1	a	a_5	a_4	a_3	b_4	b_3	b_2	b_1	b	b_5
$\boldsymbol{c_3}$	c_3	c_2	c_1	c	c_5	c_4	d_3	d_2	d_1	d	d_5	d_4	a_3	a_2	a_1	a	a_5	a_4	b_3	b_2	b_1	b	b_5	b_4
$\boldsymbol{c_4}$	c_4	c_3	c_2	c_1	c	c_5	d_2, d_5	d_1, d_4	d, d_3	d_2, d_5	d_1, d_4	d, d_3	a_4	a_3	a_2	a_1	a	a_5	b_2, b_5	b_1, b_4	b, b_3	b_2, b_5	b_1, b_4	b, b_3
$\boldsymbol{c_5}$	c_5	c_4	c_3	c_2	c_1	c	d_1	d	d_5	d_4	d_3	d_2	a_5	a_4	a_3	a_2	a_1	a	b_1	b	b_5	b_4	b_3	b_2
$\boldsymbol{d}$	d	d_5	d_4	d_3	d_2	d_1	c	c_5	c_4	c_3	c_2	c_1	b	b_5	b_4	b_3	b_2	b_1	a	a_5	a_4	a_3	a_2	a_1
$\boldsymbol{d_1}$	d_1	d	d_5	d_4	d_3	d_2	c_5	c_4	c_3	c_2	c_1	c	b_1	b	b_5	b_4	b_3	b_2	a_5	a_4	a_3	a_2	a_1	a
$\boldsymbol{d_2}$	d_2	d_1	d	d_5	d_4	d_3	c_4	c_3	c_2	c_1	c	c_5	b_2	b_1	b	b_5	b_4	b_3	a_4	a_3	a_2	a_1	a	a_5
$\boldsymbol{d_3}$	d_3	d_2	d_1	d	d_5	d_4	c_3	c_2	c_1	c	c_5	c_4	b_3	b_2	b_1	b	b_5	b_4	a_3	a_2	a_1	a	a_5	a_4
$\boldsymbol{d_4}$	d_4	d_3	d_2	d_1	d	d_5	c_2, c_5	c_1, c_4	c, c_3	c_2, c_5	c_1, c_4	c, c_3	b_4	b_3	b_2	b_1	b	b_5	a_2, a_5	a_1, a_4	a, a_3	a_2, a_5	a_1, a_4	a, a_3
$\boldsymbol{d_5}$	d_5	d_4	d_3	d_2	d_1	d	c_1	c	c_5	c_4	c_3	c_2	b_5	b_4	b_3	b_2	b_1	b	a_1	a	a_5	a_4	a_3	a_2

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes:

$$\underline{a} = \{a, a_3\}, \quad \underline{a}_1 = \{a_1, a_4\} \quad , \underline{a}_2 = \{a_2, a_5\}, \quad \underline{b} = \{b, b_3\}, \quad \underline{b}_1 = \{b_1, b_4\} \quad , \underline{b}_2 = \{b_2, b_5\},$$

$$\underline{c} = \{c, c_3\}, \quad \underline{c}_1 = \{c_1, c_4\}, \quad \underline{c}_2 = \{c_2, c_5\}, \quad \underline{d} = \{d, d_3\}, \quad \underline{d}_1 = \{d_1, d_4\}, \quad \underline{d}_2 = \{d_2, d_5\},$$

and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

Theorem 5.7. *All multiplicative H_v-fields defined on $(\boldsymbol{Z}_9, +, \cdot)$, which have a non-degenerate fundamental field and satisfy the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,*

$$2 \otimes 3 = \{0, 6\} \quad or \ \{3, 6\}, \quad 2 \otimes 4 = \{2, 8\} \quad or \ \{5, 8\}, \quad 2 \otimes 6 = \{0, 3\} \quad or \ \{3, 6\}, \quad 2 \otimes 7 = \{2, 5\} \quad or \ \{5, 8\},$$

$$2 \otimes 8 = \{1, 7\} \quad or \ \{4, 7\}, \quad 3 \otimes 4 = \{0, 3\} \quad or \ \{3, 6\}, \quad 3 \otimes 5 = \{0, 6\} \quad or \ \{3, 6\}, \quad 3 \otimes 6 = \{0, 3\} \ or \ \{0, 6\},$$

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a}_1}$	$\underline{\boldsymbol{a}_2}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b}_1}$	$\underline{\boldsymbol{b}_2}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c}_1}$	$\underline{\boldsymbol{c}_2}$	$\underline{\boldsymbol{d}}$	$\underline{\boldsymbol{d}_1}$	$\underline{\boldsymbol{d}_2}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{d}_2$
$\underline{\boldsymbol{a}}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{d}_2$	$\underline{d}$	$\underline{d}_1$
$\underline{\boldsymbol{a}}_2$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{d}_1$	$\underline{d}_2$	$\underline{d}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{d}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$
$\underline{\boldsymbol{b}}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{d}_1$	$\underline{d}_2$	$\underline{d}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b}}_2$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{d}_2$	$\underline{d}$	$\underline{d}_1$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_2$	$\underline{c}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{d}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{a}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{b}_1$
$\underline{\boldsymbol{c}}_1$	$\underline{c}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{d}_2$	$\underline{d}_1$	$\underline{d}$	$\underline{a}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{c}}_2$	$\underline{c}_2$	$\underline{c}_1$	$\underline{c}$	$\underline{d}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{a}_2$	$\underline{a}_1$	$\underline{a}$	$\underline{b}_1$	$\underline{b}$	$\underline{b}_2$
$\underline{\boldsymbol{d}}$	$\underline{d}$	$\underline{d}_2$	$\underline{d}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{c}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{b}_1$	$\underline{a}$	$\underline{a}_2$	$\underline{a}_1$
$\underline{\boldsymbol{d}}_1$	$\underline{d}_1$	$\underline{d}$	$\underline{d}_2$	$\underline{c}_2$	$\underline{c}_1$	$\underline{c}$	$\underline{b}_1$	$\underline{b}$	$\underline{b}_2$	$\underline{a}_2$	$\underline{a}_1$	$\underline{a}$
$\underline{\boldsymbol{d}}_2$	$\underline{d}_2$	$\underline{d}_1$	$\underline{d}$	$\underline{c}_1$	$\underline{c}$	$\underline{c}_2$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$	$\underline{a}_1$	$\underline{a}$	$\underline{a}_2$

$3\otimes 7=\{0,3\}$ *or* $\{3,6\}$, $3\otimes 8=\{0,6\}$ *or* $\{3,6\}$, $4\otimes 5=\{2,5\}$ *or* $\{2,8\}$, $4\otimes 6=\{0,6\}$ *or* $\{3,6\}$,
$4\otimes 8=\{2,5\}$ *or* $\{5,8\}$, $5\otimes 6=\{0,3\}$ *or* $\{3,6\}$, $5\otimes 7=\{2,8\}$ *or* $\{5,8\}$, $5\otimes 8=\{1,4\}$ *or* $\{4,7\}$,
$6\otimes 7=\{0,6\}$ *or* $\{3,6\}$, $6\otimes 8=\{0,3\}$ *or* $\{3,6\}$, $7\otimes 8=\{2,5\}$ *or* $\{2,8\}$

In all the above cases the fundamental classes are $[0]=\{0,3,6\}$, $[1]=\{1,4,7\}$, $[2]=\{2,5,8\}$, *and we have* $(\boldsymbol{Z}_9,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot)$.

Example 5.8. 8 Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_9,+,\otimes)$ of the case that only $2\otimes 8=\{4,7\}$ is a hyperproduct. We set, for $i=1,2,\ldots,8$,

$$a=E_{11}+E_{22},\quad a_i=E_{11}+iE_{12}+E_{22},$$

$$b=E_{11}+8E_{22},\quad b_i=E_{11}+iE_{12}+8E_{22},$$

then, for $\boldsymbol{X}=\{a,a_1,\ldots,a_8,b,b_1,\ldots,b_8\}$, we obtain the following table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{a_8}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{b_5}$	$\boldsymbol{b_6}$	$\boldsymbol{b_7}$	$\boldsymbol{b_8}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$\boldsymbol{a_1}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$\boldsymbol{a_2}$	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_4, b_7	b_5, b_8	b, b_6	b_1, b_7	b_2, b_8	b, b_3	b_1, b_4	b_2, b_5	b_3, b_6
$\boldsymbol{a_3}$	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5
$\boldsymbol{a_4}$	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4
$\boldsymbol{a_5}$	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3
$\boldsymbol{a_6}$	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2
$\boldsymbol{a_7}$	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1
$\boldsymbol{a_8}$	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b
$\boldsymbol{b}$	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$\boldsymbol{b_1}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\boldsymbol{b_2}$	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	a_4, a_7	a_5, a_8	a, a_6	a_1, a_7	a_2, a_8	a, a_3	a_1, a_4	a_2, a_5	a_3, a_6
$\boldsymbol{b_3}$	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5
$\boldsymbol{b_4}$	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4
$\boldsymbol{b_5}$	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3
$\boldsymbol{b_6}$	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2
$\boldsymbol{b_7}$	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1
$\boldsymbol{b_8}$	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_3, a_6\}$, $\underline{a}_1 = \{a_1, a_4, a_7\}$, $\underline{a}_2 = \{a_2, a_5, a_8\}$, $\underline{b} = \{b, b_3, b_6\}$, $\underline{b}_1 = \{b_1, b_4, b_7\}$, $\underline{b}_2 = \{b_2, b_5, a_b\}$, and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{a_2}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{b_2}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$
$\underline{\boldsymbol{a_1}}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_2$	$\underline{b}_1$	$\underline{b}$
$\underline{\boldsymbol{a_2}}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$
$\underline{\boldsymbol{b_1}}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{b_2}}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$

Example 5.9. Consider the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_9, +, \otimes)$ of the case that only $2 \otimes 8 = \{4, 7\}$ is a hyperproduct. We set $i = 1, 2, \ldots, 8$,

$$a = E_{11} + E_{22}, \quad a_i = E_{11} + iE_{12} + E_{22},$$

$$b = E_{11} + 4E_{22}, \quad b_i = E_{11} + iE_{12} + 4E_{22},$$

$$c = E_{11} + 7E_{22}, \quad c_i = E_{11} + iE_{12} + 7E_{22},$$

then, for $\boldsymbol{X} = \{a, a_1, \ldots, a_8, b, b_1, \ldots, b_8, c, c_1, \ldots, c_8\}$, we obtain the following table:

$\otimes$	$\boldsymbol{a}$	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{a_8}$	$\boldsymbol{b}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	$\boldsymbol{b_5}$	$\boldsymbol{b_6}$	$\boldsymbol{b_7}$	$\boldsymbol{b_8}$	$\boldsymbol{c}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{c_3}$	$\boldsymbol{c_4}$	$\boldsymbol{c_5}$	$\boldsymbol{c_6}$	$\boldsymbol{c_7}$	$\boldsymbol{c_8}$
$\boldsymbol{a}$	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$\boldsymbol{a_1}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6
$\boldsymbol{a_2}$	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_2, b_8	b, b_3	b_1, b_4	b_2, b_5	b_3, b_6	b_4, b_7	b, b_6	b_1, b_7	b_4, b_7	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4
$\boldsymbol{a_3}$	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2
$\boldsymbol{a_4}$	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c
$\boldsymbol{a_5}$	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$\boldsymbol{a_6}$	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5
$\boldsymbol{a_7}$	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3
$\boldsymbol{a_8}$	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1
$\boldsymbol{b}$	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$\boldsymbol{b_1}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6
$\boldsymbol{b_2}$	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	c_2, c_8	c, c_3	c_1, c_4	c_2, c_5	c_3, c_6	c_4, c_7	c_5, c_8	c, c_6	c_1, c_7	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4
$\boldsymbol{b_3}$	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2
$\boldsymbol{b_4}$	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a
$\boldsymbol{b_5}$	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\boldsymbol{b_6}$	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5
$\boldsymbol{b_7}$	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3
$\boldsymbol{b_8}$	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1
$\boldsymbol{c}$	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	c_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$\boldsymbol{c_1}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6
$\boldsymbol{c_2}$	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	a_2, a_8	a, a_3	a_1, a_4	a_2, a_5	a_3, a_6	a_4, a_7	a_5, a_8	a, a_6	a_1, a_7	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4
$\boldsymbol{c_3}$	c_3	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	a_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2
$\boldsymbol{c_4}$	c_4	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b
$\boldsymbol{c_5}$	c_5	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	a_1	b_8	b	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$\boldsymbol{c_6}$	c_6	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	a_5	b_6	b_7	b_8	b	b_1	b_2	b_3	b_4	b_5
$\boldsymbol{c_7}$	c_7	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a	b_4	b_5	b_6	b_7	b_8	b	b_1	b_2	b_3
$\boldsymbol{c_8}$	c_8	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	a_5	a_6	a_7	a_8	a	a_1	a_2	a_3	a_4	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b	b_1

The $(\boldsymbol{X}, \otimes)$ is a COW H_v-group with fundamental classes: $\underline{a} = \{a, a_3, a_6\}$, $\underline{a}_1 = \{a_1, a_4, a_7\}$, $\underline{a}_2 = \{a_2, a_5, a_8\}$, $\underline{b} = \{b, b_3, b_6\}$, $\underline{b}_1 = \{b_1, b_4, b_7\}$, $\underline{b}_2 = \{b_2, b_5, a_b\}$, $\underline{c} = \{c, c_3, c_6\}$, $\underline{c}_1 = \{c_1, c_4, c_7\}$, $\underline{c}_2 = \{c_2, c_5, c_8\}$, and the fundamental group $(\underline{\boldsymbol{X}}, \underline{\otimes})$ is defined with the table:

$\underline{\otimes}$	$\underline{\boldsymbol{a}}$	$\underline{\boldsymbol{a_1}}$	$\underline{\boldsymbol{a_2}}$	$\underline{\boldsymbol{b}}$	$\underline{\boldsymbol{b_1}}$	$\underline{\boldsymbol{b_2}}$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c_1}}$	$\underline{\boldsymbol{c_2}}$
$\underline{\boldsymbol{a}}$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{\boldsymbol{c}}$	$\underline{\boldsymbol{c_1}}$	$\underline{\boldsymbol{c_2}}$
$\underline{\boldsymbol{a_1}}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$
$\underline{\boldsymbol{a_2}}$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$
$\underline{\boldsymbol{b}}$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$
$\underline{\boldsymbol{b_1}}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$
$\underline{\boldsymbol{b_2}}$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$
$\underline{\boldsymbol{c}}$	$\underline{c}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{b}$	$\underline{b}_1$	$\underline{b}_2$
$\underline{\boldsymbol{c_1}}$	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}$	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{b}_1$	$\underline{b}_2$	$\underline{b}$
$\underline{\boldsymbol{c_2}}$	$\underline{c}_2$	$\underline{c}$	$\underline{c}_1$	$\underline{a}_2$	$\underline{a}$	$\underline{a}_1$	$\underline{b}_2$	$\underline{b}$	$\underline{b}_1$

Theorem 5.10. *All multiplicative H_v-fields on $(\boldsymbol{Z}_{10}, +, \cdot)$, with a non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*
(I) *We have the only one hyperproduct,*

$$2 \otimes 4 = \{3,8\},\quad 2 \otimes 5 = \{0,5\},\quad 2 \otimes 6 = \{2,7\},\quad 2 \otimes 7 = \{4,9\},\quad 2 \otimes 9 = \{3,8\},$$

$$3 \otimes 4 = \{2,7\},\quad 3 \otimes 5 = \{0,5\},\quad 3 \otimes 6 = \{3,8\},\quad 3 \otimes 8 = \{4,9\},\quad 3 \otimes 9 = \{2,7\},$$

$$4 \otimes 5 = \{0,5\},\quad 4 \otimes 6 = \{4,9\},\quad 4 \otimes 7 = \{3,8\},\quad 4 \otimes 8 = \{2,7\},$$

$$5 \otimes 6 = \{0,5\},\quad 5 \otimes 7 = \{0,5\},\quad 5 \otimes 8 = \{0,5\},\quad 5 \otimes 9 = \{0,5\},$$

$$6 \otimes 7 = \{2,7\},\quad 6 \otimes 8 = \{3,8\},\quad 6 \otimes 9 = \{4,9\},\quad 7 \otimes 9 = \{3,8\},\quad 8 \otimes 9 = \{2,7\}.$$

In all these cases the fundamental classes are
$[0] = \{0,5\}$, $[1] = \{1,6\}$, $[2] = \{2,7\}$, $[3] = \{3,8\}$, $[4] = \{4,9\}$ *and* $(\boldsymbol{Z}_{10}, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_5, +, \cdot)$.
(II) *The cases with classes* $[0] = \{0,2,4,6,8\}$ *and* $[1] = \{1,3,5,7,9\}$, *and with fundamental field* $(\boldsymbol{Z}_{10}, +, \otimes)/\gamma^* \cong (\boldsymbol{Z}_2, +, \cdot)$, *are described as follows: In the multiplicative table only the results above the diagonal, we raise each of the products by putting one element of the same class of the results. We do not raise setting* 1, *and we cannot raise only the* $3 \otimes 7 = 1$. *The number of those H_v-fields is* 103.

Conflict of Interest: The author declares no conflict of interest.

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Use of Soft Sets and the Blooms Taxonomy for Assessing Learning Skills

Michael Gr. Voskoglou

Abstract. Learning, a universal process that all individuals experience, is a fundamental component of human cognition. It combines cognitive, emotional and environmental influences for acquiring or enhancing ones knowledge and skills. Volumes of research have been written about learning and many theories have been developed for the description of its mechanisms. The goal was to understand objectively how people learn and then develop teaching approaches accordingly. In this paper soft sets, a generalization of fuzzy sets introduced in 1999 by D. Molodstov as a new mathematical tool for dealing with the uncertainty in a parametric manner, are used for assessing student learning skills with the help of the Blooms taxonomy. Blooms taxonomy has been applied and is still applied by generations of teachers as a teaching tool to help balance assessment by ensuring that all orders of thinking are exercised in student learning. The innovative assessment method introduced in this paper is very useful when the assessment has qualitative rather than quantitative characteristics. A classroom application is also presented illustrating its applicability under real conditions.

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1 Introduction

Learning, a universal process that all individuals experience, is a fundamental component of human cognition. It combines cognitive, emotional and environmental influences for acquiring or enhancing ones knowledge or skills.

Curiosity about how humans learn dates back to the ancient Greek philosophers Socrates, Plato and Aristotle, who explored whether knowledge and truth mostly come from intellectual reasoning, i.e. they could be found within oneself (*rationalism*) or through external observation (*empiricism*). Thousands of years later, during the 17th and 18th century, the same question was the reason for a historical confrontation of two academic schools of European philosophy: The rationalists Descartes, Spinoza, Leibniz (European continent), versus the U.K. empirists Bacon, Locke, Hume.

By the 19th century, psychologists began to answer this question with systematic scientific studies. Volumes of research have been written about learning and many theories have been developed for the description of its mechanisms. The goal was to understand objectively how people learn and then develop teaching approaches accordingly.

In 20th century, the debate among the educational specialists centred on whether people learn by responding to external stimuli (*behaviorism* [3]) or by using their brains to construct knowledge from external data (*cognitivism* [19]).

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Constructivism, a philosophical framework based on Piagets theory for learning and formally introduced by von Clasersfeld during the 1970s, suggests that knowledge is not passively received from the environment, but is actively constructed by the learner through a process of adaptation based on and constantly modified by the learners experience of the world [13]. This is usually referred as *cognitive constructivism.*

The synthesis of the ideas of constructivism with Vygoskys social development theory [4] created the issue of *social constructivism* [9]. According to Vygosky learning takes place within some socio-cultural setting. Shared meanings are formed through negotiation in the learning environment, leading to the development of common knowledge. The basic difference between cognitive and social constructivism is that the former argues that thinking precedes language, whereas the latter supports the exactly inverse approach.

In addition to the primary learning theories outlined above, i.e. behaviorism, cognitivism, constructivism and social constructivism, there are still more options [6]. *Humanism,* for example, focuses on creating an environment leading to self-actualization, where learners are free to determine their own goals while the teacher assists in meeting those goals. The *experiential* theory suggests to combine both learning about something and experiencing it, so that learners be able to apply the new knowledge to real-world situations. Also, the *transformative* theory, which is particularly relevant to adult learners, considers that the new information can change our world views when paired with critical reflection, etc.

The increasing use of technology as an educational tool has changed during the last years the learning landscape. Strongly influenced by technology, *connectivism,* focuses on a learners ability to frequently source and update accurate information. Knowing how and where to find the best information is as important as the information itself [5].

The target of the present paper is to use the Blooms taxonomy for teaching and learning and soft sets as tools for obtaining an assessment method of student learning skills in a parametric manner.

The motivation for writing this paper came from the fact that frequently the student assessment is attempted using not numerical, but linguistic grades, like A, B, C, D, E, F and sometimes $B-$, $B+$, etc. Also, it is important and useful to assess the student learning skills at each level of the learning process, as those levels are described by the Blooms taxonomy (see next section).

The rest of the paper is formulated as follows: A brief account of the Blooms taxonomy is exposed in the next section. The definition of soft set and its connection to fuzzy sets are presented in the third section. The assessment method is developed in fourth section with a classroom application and the main text closes with the final conclusion and some hints for future research contained in fifth section. An Appendix is presented also at the end of the paper, after the list of references, containing the questionnaire used in the classroom application.

2 The Blooms Taxonomy for Teaching and Learning

In 1956 Benjamin Bloom with collaborators Max Englehart, Edward Furst, Walter Hill, and David Krathwohl published a framework for categorizing educational goals, the *Taxonomy of Educational Objectives* [2]. The publication of the taxonomy followed a series of conferences from 1949 to 1953, which took place in order to improve communication between educators on the design of curricula and examinations. A revised version of the Blooms taxonomy was created in 2000 by Lorin Anderson [1], former student of Bloom. The six major levels of the revised taxonomy, moving through the lowest order processes to the highest, can be described as follows:

• L_1: *Knowing-Remembering:* Retrieving, recognizing, and recalling relevant knowledge from long-term memory.

• L_2: *Organizing-Understanding:* Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining. Understand uses and implications of terms, facts, methods, procedures, concepts.

• L_3: *Applying:* Make use of theory, solve problems and use information in new situations.

• L_4: *Analyzing:* Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing.

• L_5: *Generating-Evaluating:* Making judgements based on criteria and standards through checking and critiquing. Accept or reject on basis of criteria.

• L_6: *Integrating-Creating:* Put things together, bring together various parts, write theme, present speech, plan experiments and put information together in a new and creative way.

Most researchers and educators consider the last three levels L_4, L_5 and L_6 as being parallel, i.e. as happening simultaneously. For teaching a topic, the instructor should arrange his/her class work in the order to synchronize it with the six levels of Blooms taxonomy. The typical questions for evaluating the student achievement at the corresponding level must focus:

For Knowing-Remembering, on clarifying, recalling, naming, and listing. For Organizing-Understanding, on arranging information, comparing similarities and differences, classifying, and sequencing. For Applying, on prior knowledge to solve a problem. For Analyzing, on examining parts, identifying attributes, relationships, patterns, and main idea. For Generating-Evaluating, on producing new information, inferring, predicting, and elaborating with details. For Integrating-Creating, on connecting, combining, summarizing information and restructuring existing information to incorporate new information. For Evaluating, on reasonableness and quality of ideas, criteria for making judgments and confirming accuracy of claims.

Blooms taxonomy has been used and is still used by generations of teachers as a teaching tool to help balance assessment by ensuring that all orders of thinking are exercised in student's learning.

3 Fuzzy and soft sets

Probability theory used to be until the middle of the 1960's the unique tool in hands of the experts for dealing with the existing in real life and science situations of uncertainty. Probability, however, based on the principles of the bivalent logic, has been proved sufficient for tackling only problems of uncertainty connected to randomness, but not those connected to imprecision or incomplete information of the given data.

The *fuzzy* set theory, introduced by Zadeh in 1965 [20], and the connected to it infinite-valued in the interval $[0, 1]$ *fuzzy logic* [8] gave to scientists the opportunity to model under conditions of uncertainty which are vague or not precisely defined, thus succeeding to mathematically solve problems whose statements are expressed in the natural language. Through fuzzy logic the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner.

Fuzzy systems are considered to be part of the wider class of *Soft Computing,* also including *probabilistic reasoning* and *neural networks,* which are based on the function of biological networks [11]. One may say that neural networks and fuzzy systems try to emulate the operation of the human brain. The former concentrate on the structure of the human mind, i.e. the hardware, and the latter concentrate on the software emulating human reasoning.

Let U be the universal set of the discourse. It is recalled that a fuzzy set on U is defined with the help of its *membership function* $m : U \to [0, 1]$ as the set of the ordered pairs

$$A = \{(x, m(x)) \ : \ x \in U\}. \tag{1}$$

The real number $m(x)$ is called the *membership degree* of x in . The greater is $m(x)$, the more x satisfies the characteristic property of . Many authors, for reasons of simplicity, identify a fuzzy set with its membership function.

A crisp subset A of U is a fuzzy set on U with membership function taking the values $m(x) = 1$ if x belongs to A and 0 otherwise. In other words, the concept of fuzzy set is an extension of the concept of the ordinary sets.

It is of worth noting that there is not any exact rule for defining the membership function of a fuzzy set. The methods used for this purpose are usually empirical or statistical and the definition is not unique depending on the personal goals of the observer. The only restriction about it is to be compatible to the common logic; otherwise the resulting fuzzy set does not give a reliable description of the corresponding real situation.

For example, defining the fuzzy set of the young people of a country one could consider as young all those being less than 30 years old and another all those being less than 40 years old. As a result they assign different membership degrees to people with ages below those two upper bounds.

For general facts on fuzzy sets, fuzzy logic and the connected to them uncertainty we refer to the chapters $4-7$ of the book [15].

A lot of research has been carried out during the last 60 years for improving and extending the fuzzy set theory on the purpose of tackling more effectively the existing uncertainty in problems of science, technology and everyday life. Various generalizations of the concept of fuzzy set and relative theories have been developed like the type-2 fuzzy set, the intuitionistic fuzzy set, the neutrosophic set, the rough set, the grey system theory, etc. [17]. In 1999, Dmtri Molodstov, Professor of the Computing Center of the Russian Academy of Sciences in Moscow, proposed the notion of *soft set* as a new mathematical tool for dealing with the uncertainty in a parametric manner [10].

Let E be a set of parameters, let A be a subset of E and let f be a mapping of A into the set $\Delta(U)$ of all subsets of U. Then the soft set on U connected to A, denoted by (f, A), is defined as the set of the ordered pairs

$$(f, A) = \{(e, f(e) \ : \ e \in A\}. \tag{2}$$

In other words, a soft set is a parametrized family of subsets of U. Intuitively, it is "soft" because the boundary of the set depends on the parameters.

For example, let $U = \{H_1, H_2, H_3\}$ be a set of houses and let $E = \{e_1, e_2, e_3\}$ be the set of the parameters $e_1 = cheap$, $e_2 = expensive$ and $e_3 = beautiful$. Let us further assume that H_1, H_2 are the cheap and H_2, H_3 are the beautiful houses. Set $A = \{e_1, e_3\}$, then a mapping $f : A \to \Delta(U)$ is defined by $f(e_1) = \{H_1, H_2\}$, $f(e_3) = \{H_2, H_3\}$. Therefore, the soft set (f, A) representing the cheap and beautiful houses of U is the set of the ordered pairs

$$(f, A) = \{(e_1, \{H_1, H_2\}), (e_3, \{H_2, H_3\})\}. \tag{3}$$

A fuzzy set on U with membership function $y = m(x)$ is a soft set on U of the form $(f, [0, 1])$, where $f(\alpha) = \{x \in U \ : \ m(x) \geq \alpha\}$ is the corresponding α-cut of the fuzzy set, for each α in $[0, 1]$. The concept of soft set is, therefore, a generalization of the concept of fuzzy set.

An important advantage of soft sets is that, by using the set of parameters E, they pass through the existing difficulty of defining properly the membership function of a fuzzy set.

The theory of soft sets has found many and important applications to several sectors of the human activity like decision making, parameter reduction, data clustering and data dealing with incompleteness, etc. [14]. One of the most important steps for the theory of soft sets was to define mappings on soft sets, which was achieved by A. Kharal and B. Ahmad and was applied to the problem of medical diagnosis in medical expert systems [7]. But fuzzy mathematics has also significantly developed at the theoretical level providing important insights even into branches of classical mathematics like algebra, analysis, geometry, topology etc. For example, one can extend the concept of topological space, the most general category of mathematical space, to fuzzy structures and in particular can define soft topological spaces and generalize the concepts of convergence, continuity and compactness within such kind of spaces [12].

4 The Soft Set Assessment Method

In earlier works the present author has developed various methods for assessing human-machine performance under fuzzy conditions, including the measurement of uncertainty in fuzzy systems, the use of the Center of Gravity (COG) defuzzification technique, the use of fuzzy or grey numbers, etc. [16]. Recently he also constructed a soft set model for assessment in a parametric manner and provided examples to illustrate its applicability to real situations [18].

In this model the set of the discourse U is the set of all objects which are under assessment. Consider the set $E = \{e_1, e_2, e_3, e_4, e_5\}$ of the parameters $e_1 = excellent$, $e_2 = verygood$, $e_3 = good$, $e_4 = mediocre$ and $e_5 = failed$ and the mapping $f : E \to \Delta(U)$ assigning to each parameter of E the subset of U consisting of all elements whose performance is described by this parameter. Then the soft set

$$(f, U) = \{(e_i, f(e_i)), i = 1, 2, 3, 4, 5\}, \tag{4}$$

represents mathematically a qualitative assessment of the elements of U.

Here this model will be adapted for assessing student learning skills in terms of the Blooms taxonomy.

The student assessment will be materialized through the following classroom application, which was performed with subjects 30 students of the School of Technological Applications (prospective engineers) of the Graduate Technological Educational Institute (T. E. I.) of Western Greece attending the course Mathematics I of their first term of studies.

This course involved an introductory module repeating and extending the students knowledge from secondary education about real numbers. After the module was taught, the instructor wanted to investigate the students progress according to the principles of the Blooms taxonomy. For this, he asked them to answer in the classroom the written test given in the Appendix at the end of this paper, which is divided to six different parts, one for each level of the taxonomy.

The students answers were assessed separately for each level with respect to the parameters of the set E outlined above. The tests results are depicted in the following table, where L_i, $i = 1, 2, 3, 4, 5, 6$ denote the levels of the Blooms taxonomy and P denotes the student overall performance.

Table 1: The results of the test

Parameter	L_1	L_2	L_3	L_4	L_5	L_6	P
e_1	8	6	5	3	2	3	3
e_2	9	11	10	8	7	8	8
e_3	10	9	10	12	10	8	12
e_4	3	3	3	5	7	8	5
e_5	0	1	2	2	4	3	2

The instructor numbered the students with respect to their overall performance in the test moving from the best one to the worst by $S_1, S_2, \ldots, S_{30}$.

Let $U = \{S_1, S_2, \ldots, S_{30}\}$ be the set of the discourse and let $f : E \to \Delta(U)$ be the mapping assigning to each parameter of E the subset of U consisting of the students whose overall performance was assessed by this parameter. Then the soft set

$$\begin{aligned}(f, U) = \{&(e_1, \{S_1, S_2, S_3\}), (e_2, \{S_4, S_5, \ldots, S_{11}\}), (e_3, \{S_{12}, S_{13}, \ldots, S_{23}\}),\\ &(e_4, \{S_{24}, S_{25}, \ldots, S_{28}\}), (e_5, \{S_{29}, S_{30}\})\}.\end{aligned} \tag{5}$$

represents mathematically the student overall performance in the test.

In an analogous way one can represent by a soft set the student performance at each level of the Blooms taxonomy. In those cases, however, an additional search is required, because the data of Table 1 is not enough for finding the students whose performance was assessed by the corresponding parameter.

For example, for level L_5 the instructor found that the student performance can be represented by the soft set

$$(f,U) = \{(e_1, \{S_1, S_3\}), (e_2, \{S_2, S_4, S_5, S_6, S_8, S_9, S_{12}\}), (e_3, \{S_7, S_{10}, S_{11}, S_{13}, S_{14}, S_{15}, S_{16}, S_{18}, S_{19}, S_{22}\}), (e_4, \{S_{17}, S_{20}, S_{21}, S_{23}, S_{24}, S_{25}, S_{28}\}), (e_5, \{S_{26}, S_{27}, S_{29}, S_{30}\})\}, \tag{6}$$

etc.

This method gives also the opportunity to represent with a soft set each students individual profile with respect to his/her performance at the levels of the Blooms taxonomy. For this, consider $U = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ as the set of the discourse and let $g : E \to \Delta(V)$ be the mapping assigning to each parameter of E the subset of V consisting of the levels of the Blooms taxonomy in which the corresponding students performance was assessed by this parameter. For example the soft set

$$(g,V) = \{(e_1, \{L_1, L_2\}), (e_2, \{L_3\}), (e_3, \{L_4\}, (e_4, \{L_5, L_6\}), (e_5, \varnothing)\}, \tag{7}$$

corresponds to the profile of a student who demonstrated excellent performance at levels L_1 and L_2, very good at level L_3, good at level L_4 and mediocre performance at levels L_5 and L_6.

5 Conclusion

The discussion performed in this study leads to the conclusion that soft sets offer a potential tool for a qualitative assessment of student learning skills in a parametric manner with the help of the Blooms taxonomy.

Due to the generality of the assessment method used, a promising area for future research is the application of this method for assessing other student skills, like problem solving, mathematical modelling, analogical reasoning, etc. It could be also an interesting idea the development of alternative assessment methods under fuzzy conditions by using other types of generalizations of fuzzy sets or related theories, as they have been mentioned in the third section of this work.

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Appendix

The questionnaire used in our classroom application (Topic: Real numbers)

1. *Knowing-Remembering*

• Give the definitions and examples of a periodic decimal and of an irrational number (in the form of an infinite decimal).

2. *Organizing*

• Compare the set of all fractions with the set of periodic decimals. Compare the set of irrational numbers with the set of all roots (of any order) that have no exact values.

3. *Applying*

• Which of the following numbers are natural, integers, rational, irrational and real numbers?

$$2, \quad -\frac{5}{3}, \quad 0, \quad 9 \cdot 08, \quad 5, \quad 7 \cdot 333\ldots, \quad \pi = 3 \cdot 14159\ldots, \quad -\sqrt{4}, \quad \frac{22}{11}, \quad 5\sqrt{3},$$

$$-\frac{\sqrt{5}}{\sqrt{20}}, \quad (\sqrt{3}+2)(\sqrt{3}-2), \quad -\frac{\sqrt{5}}{2}, \quad \sqrt{7}-2, \quad \sqrt{\left(\frac{5}{3}\right)^2}.$$

• Write the number $0 \cdot 345345345\ldots$ in its fractional form.

4. *Analyzing*

• Find the digit which is in the 1005th place of the decimal $2 \cdot 825342342\ldots$.

• Compare the numbers 5 and $4 \cdot 9999\ldots$.

• Construct the line segment of length $\sqrt{3}$ with the help of the Pythagorean Theorem. Give a geometric interpretation.

5. *Generating - Evaluating*

• Justify why the decimals $2 \cdot 001313113111311111\ldots$ and $0 \cdot 1234567891011\ldots$ are irrational numbers.

• Construct the line segment of length $\sqrt[3]{2}$ by using the graph of the function $f(x) = \sqrt[3]{x}$.

6. *Integrating - Creating*

• Define the set of the real numbers in terms of their decimal representations (this definition was not given by the instructor to the class before the test).

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Fuzzy Subgroups and Digraphs Induced by Fuzzy Subgroups

Paul J. Allen, Joseph Neggers, Hee Sik Kim*

Abstract. Given a fuzzy subgroup μ of a group G, $x \rhd_u y$ if and only if $\mu(xy) < \mu(yx)$ defines a directed relation with an associated digraph $(G, \rhd_u)$. We consider (μ, ν)-homomorphisms $\varphi : (G, \mu) \to (H, \nu)$ where μ and ν are fuzzy subgroups of G and H respectively and the preservation of properties of the digraphs $(G, \rhd_u)$ several of which are also noted here, e.g., $(G, \rhd_u)$ is an anti-chain if and only if μ is a fuzzy normal subgroup of the group G.

AMS Subject Classification 2020: 20N25; 06A06
Keywords and Phrases: Fuzzy subgroup, μ-product relation, Fuzzy normal, Digraph, (μ, ν)-homomorphism.

1 Introduction

In this paper, we show that given a fuzzy subgroup μ of a group G, letting $x \rhd_u y$ if and only if $\mu(xy) < \mu(yx)$ defines a directed relation with an associated digraph $(G, \rhd_u)$ whose properties are related to both μ and the underlying group G. The associated digraph has a multitude of natural invariants associated with it, e.g., the adjacency matrix and its eigenvalues, the adjacency algebra and its dimension over the field of rationals, the radius, the diameter, and any other of the "standard" structures derived from such graphs. One can thus proceed to make a deeper study of the subject than we do here, where we mostly indicate some elementary properties of $(G, \rhd_u)$ as they relate to μ itself. Included in the fact that $(G, \rhd_u)$ is an anti-chain if and only if μ is a fuzzy normal subgroup of the group G. Furthermore we explore the consequences of homomorphisms induced on the digraphs $(G, \rhd_u)$ and $(H, \rhd_v)$ by (μ, ν)-homomorphisms $\varphi : (G, \mu) \to (H, \nu)$ to some extent including the effects on the (shortest) distance functions for these graphs, noting that distances shrink in general. For general references on fuzzy group theory we refer to [3, 5, 6].

2 Preliminaries

Rosenfeld [12] has defined fuzzy subgroupoid and fuzzy subgroups in the following way.

Definition 2.1. ([3]) *Let G be a group. A fuzzy set μ of G is said to be a fuzzy subgroup of G, if for all x, y in G,*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) \geq \mu(x)$.

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The following properties of fuzzy subgroups of a group G have been noted by many authors [2, 4, 7].

Proposition 2.2. *Let μ be any fuzzy subgroup of a group G with identity e. Then the following statements are true:*

(i) $\mu(x^{-1}) = \mu(x) \leq \mu(e)$ *for all* $x \in G$,

(ii) $\mu(xy) = \mu(y)$ *for all* $y \in G \iff \mu(x) = \mu(e)$, *where* $x \in G$,

(iii) *if* $\mu(x) < \mu(y)$ *for some* $x, y \in G$, *then* $\mu(xy) = \mu(x) = \mu(yx)$.

Proposition 2.3. ([3]) *Let G be a group and $A \subseteq G$. Then A is a subgroup of G if and only if the characteristic function χ_A of A is a fuzzy subgroup of G.*

Neggers and Kim in [8, 9, 10, 11] studied some relations between posets and several algebraic structures, e.g., semigroups, BCK-algebras, and associative algebras.

3 Fuzzy subgroups and digraphs

Given a fuzzy subgroup of a group $(G, \cdot)$, let

$$x \rhd_u y \iff \mu(x \cdot y) < \mu(y \cdot x)$$

denote the *μ-product relation* associated with fuzzy subgroup μ of G. This relation can be viewed as a digraph on G induced by the fuzzy subgroup μ.

Proposition 3.1. *Let G be a group and H be a subgroup of G. If χ_H is the characteristic function of H, then H is a normal subgroup of G if and only if the relation $\rhd_{\chi_H}$ is trivial.*

Proof. Assume that H is not a normal subgroup of G and let $x \in G$. Then $xyx^{-1} \notin H$ for some $y \in H$. If $u := yx^{-1}$ then $ux = (yx^{-1})x = y \in H$ and $xu = x(yx^{-1}) \notin H$ and hence $\chi_H(xu) = 0 < 1 = \chi_H(ux)$, i.e., $x \rhd_{\chi_H} u$. This means that $\rhd_{\chi_H}$ is not a trivial relation, a contradiction. Conversely, assume $\rhd_{\chi_H}$ is not a trivial relation. Then $x \rhd_{\chi_H} y$ for some $x, y \in G$, and hence $\chi_H(xy) = 0, \chi_H(yx) = 1$. Thus $xy \notin H$, $yx \in H$. Since $H \rhd G$, $xy = x(yx)x^{-1} \in H$, a contradiction. □

Notice that a fuzzy subgroup μ of a group G is said to be *fuzzy normal* ([4]), if $\mu(xy) = \mu(yx)$ for all $x, y \in G$. This means precisely that *the fuzzy subgroup μ of G is fuzzy normal provided the relation $\rhd_u$ is trivial.* Thus we may consider the digraph naturally associated with $(G, \rhd_u)$ as a "measure" of the "amount" the fuzzy subgroup μ of G strays from being a fuzzy normal subgroup. If $x \rhd_u y$ then $\mu(xy) < \mu(yx)$, and thus by Proposition 2.2 (*iii*) it follows that $\mu(x) < \mu(y)$ and $\mu(y) < \mu(x)$ are both impossible, so that $\mu(x) = \mu(y)$. We conclude that:

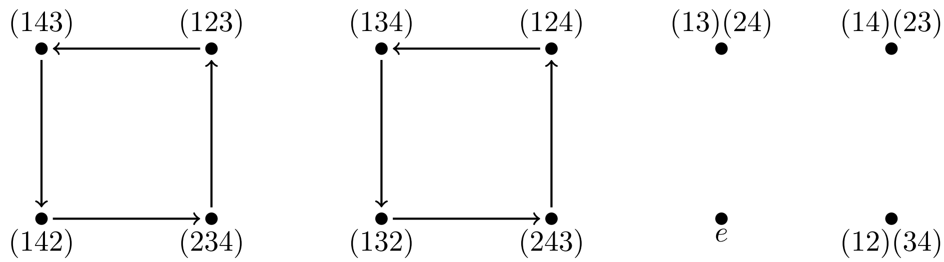
Proposition 3.2. *If μ is a fuzzy subgroup of a group G, then μ is constant on each component of the digraph $(G, \rhd_u)$.*

Example 3.3. Let $G := \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the octic group, where $a^4 = e = b^2$ and $ba = a^{-1}b$. If we define a fuzzy subset $\mu : G \to [0, 1]$ by $\mu(e) > \mu(a^2) > \mu(a) = \mu(a^3) > \mu(b) = \mu(ab) = \mu(a^2b) = \mu(a^3b)$, then μ is a fuzzy subgroup of G ([3]). Since there are no $x, y \in G$ such that $\mu(xy) < \mu(yx)$, the digraph $(G, \rhd_u)$ is an anti-chain.

Example 3.4. Consider the alternating group

$A_4 := \{e, (12)(34), (13)(24), (14)(23), (123), (132), (142), (124), (234), (243), (134), (143)\}.$

Define a fuzzy subset μ on A_4 by $\mu(e) = 1$, $\mu((12)(34)) = 1/2$, $\mu((14)(23)) = \mu((13)(24)) = 1/3$, $\mu((ijk)) = 0$, where $i, j, k \in \{1, 2, 3, 4\}$. Then μ is a fuzzy subgroup of A_4 ([1]). It is easy to check that $(234) \rhd_u (123), (123) \rhd_u (143)$, $(142) \rhd_u (234)$, $(143) \rhd_u (142)$, $(132) \rhd_u (243), (134) \rhd_u (132), (243) \rhd_u (124)$ and $(124) \rhd_u (134)$. From this relation we get the following diagram:



$(A_4, \rhd_u)$

If x is an isolated point of the digraph $(G, \rhd_u)$, then $d^-(x) = d^+(x) = 0$, i.e., the in-degree and the out-degree are both equal to 0. Thus, $\mu(xy) = \mu(yx)$ for all $y \in G$, and although this does not mean that x is in the center $Z(G)$ of G, it follows that x has properties "somewhat like those in the center". Thus, let $Z_\mu(G)$ denote the collection of all isolated points of the digraph $(G, \rhd_u)$. Then it follows that $Z_\mu(G)$ contains $Z(G)$ and also that:

Theorem 3.5. *Let G be a group with identity e. If μ is a fuzzy subgroup of G, then $Z_\mu(G)$ is a subgroup of G.*

Proof. Clearly, $e \in Z_\mu(G)$. Let $x, y \in Z_\mu(G)$. For any $z \in G$, $\mu(z(xy)) = \mu((zx)y) = \mu(y(zx)) = \mu((yz)x)$. Since $x \in Z_\mu(G)$, $\mu((yz)x) = \mu(x(yz)) = \mu((xy)z)$. It follows that $xy \in Z_\mu(G)$.

Let $x \in Z_\mu(G)$. Given $y \in G$, by Proposition 2.2(i), we obtain $\mu(x^{-1}y) = \mu((x^{-1}y)^{-1}) = \mu(y^{-1}x) = \mu(xy^{-1}) = \mu((xy^{-1})^{-1}) = \mu(yx^{-1})$. Hence $x^{-1} \in Z_\mu(G)$. This proves the theorem. □

Theorem 3.6. *A fuzzy subgroup μ of a group G is fuzzy normal if and only if $G = Z_\mu(G)$.*

Proof. If μ is a fuzzy normal subgroup of G, then $\mu(xy) = \mu(yx)$ for all x, y, whence $\rhd_u$ is trivial and $(G, \rhd_u)$ is an anti-chain. Since $Z_\mu(G)$ is precisely the collection of all isolated points of $(G, \rhd_u)$, we obtain that if $Z_\mu(G) = G$. Assume $G = Z_\mu(G)$. Then every element x of G is an isolated point of $(G, \rhd_u)$, i.e., $x \rhd_u y$ does not hold for any $y \in G$. It follows that $\mu(xy) = \mu(yx)$ for all $y \in G$. Hence μ is fuzzy normal. □

Let G be a group, and let $F(G)$ be the set of all fuzzy subgroups of G. Then we pose the following conjecture:

Conjecture. $Z(G) = \cap_{\mu \in F(G)} Z_\mu(G)$.

Given a digraph $(G, \rhd_\mu)$, let $|G| = n < \infty$. Define a polynomial $P((G, \rhd_\mu); z) = \sum_{i=0}^{n-1} |G|_i z^i$, where $|G|_i = |\{x_0 \rhd x_1 \rhd \cdots \rhd x_i\}|$ is the number of vertices of length $i \geq 1$ and $|G|_0 = |Z_\mu(G)|$. We call $P((G, \rhd_u); z)$ the *directed polynomial* of the directed graph $(G, \rhd_u)$.

Example 3.7. The directed graph $(A_4, \rhd_u)$ of Example 3.4 has the directed polynomial $2z^4 + 4$.

4 (μ,ν)-homomorphisms for fuzzy subgroups

We denote (G,μ) the group G and a fuzzy subgroup $\mu : G \to [0,1]$. Let (G,μ) and (H,ν) be fuzzy subgroups μ and ν of G and H respectively. A map $\varphi : G \to H$ is said to be a (μ,ν)-*homomorphism* if, for all $x,y \in G$,

(i) $\mu(x) < \mu(y)$ implies $\nu(\varphi(x)) < \nu(\varphi(y))$,

(ii) $\mu(x) = \mu(y)$ implies $\nu(\varphi(x)) = \nu(\varphi(y))$,

(iii) $\nu(\varphi(xy)) = \nu(\varphi(x)\varphi(y))$.

Proposition 4.1. *Let G,H be groups and let $\mu := \chi_S$ be a characteristic function of $S(\subseteq G)$ and let $\nu := \chi_T$ be a characteristic function of $T(\subseteq H)$. If $\varphi : G \to H$ is a (μ,ν)-homomorphism, then (i) $\varphi(S) \subseteq T$; (ii) $\varphi(G \setminus S) \subseteq H \setminus T$.*

Proof. If $\mu(x) < \mu(y)$, then $\mu(x) = 0$ and $\mu(y) = 1$, i.e., $x \notin S$ and $y \in S$. Since φ is a (μ,ν)-homomorphism, we obtain $\nu(\varphi(x)) < \nu(\varphi(y))$. It follows that $\nu(\varphi(x)) = 0$ and $\nu(\varphi((y)) = 1$, i.e., $\varphi(x) \notin T, \varphi(y) \in T$, which proves the proposition. □

Let μ be a fuzzy subset of a group G and let $\rhd_u$ be the μ-product relation on G and let $x,y \in G$. We denote an edge $x \to y$ if $x \rhd_u y$. Then $(G,\to) = (G,\rhd_u)$ is a digraph.

Given a digraph D, we denote the set of all vertices of D by $V(D)$, and denote the set of all edges of D by $A(D)$. Let D,H be digraphs. A map $\varphi : V(D) \to V(H)$ is called a *graph homomorphism* if it preserves edges, i.e., if $x \to y \in A(D)$ then $\varphi(x) \to \varphi(y) \in A(H)$.

Proposition 4.2. *If $\varphi : G \to H$ is a (μ,ν)-homomorphism, then $x \rhd_u y$ implies $\varphi(x) \rhd_v \varphi(y)$, i.e., φ induces a graph homomorphism $\widetilde{\varphi} : (G,\rhd_u) \to (H,\rhd_v)$.*

Proof. If $x \rhd_u y$, then $\mu(xy) < \mu(yx)$. Since φ is a (μ,ν)-homomorphism, we obtain $\nu(\varphi(x)\varphi(y)) = \nu(\varphi(xy)) < \nu(\varphi(yx)) = \nu(\varphi(y)\varphi(x))$ and therefore $\varphi(x) \rhd_v \varphi(y)$. □

Proposition 4.3. *If $\varphi : (G,\mu) \to (H,\nu)$ is both a (μ,ν)-homomorphism and a group homomorphism, and $\psi : (H,\nu) \to (K,\gamma)$ is a (ν,γ)-homomorphism, then $\psi \circ \varphi : (G,\mu) \to (K,\gamma)$ is a (μ,ψ)-homomorphism.*

Proof. Straightforward. □

If $d(x,y)$ represents the shortest distance from vertices x to y in $(G,\rhd_u)$ and if $\varphi : (G,\mu) \to (H,\nu)$ is an onto (μ,ν)-homomorphism, then the shortest path in $(G,\rhd_u)$ from x to y maps to a path in (H,ν) from $\varphi(x)$ to $\varphi(y)$ which map or may not be shortest. As a consequence, we find that $d(\varphi(x),\varphi(y)) \le d(x,y)$. Thus, various "distance-related parameters", diameter, radius, etc. are shrunk by this process.

A (μ,ν)-homomorphism $\varphi : (G,\mu) \to (H,\nu)$ is said to be a *d-isometry* if for all $x,y \in G$, $d(x,y) = d(\varphi(x),\varphi(y))$. A (μ,ν)-homomorphism $\varphi : (G,\mu) \to (H,\nu)$ is said to be an (μ,ν)-*isomorphism* if φ is a bijective function.

Theorem 4.4. *If $\varphi : (G,\mu) \to (H,\nu)$ is a (μ,ν)-isomorphism, then $\varphi^{-1} : (H,\nu) \to (G,\mu)$ is a (ν,μ)-isomorphism.*

Proof. (i) Let $\nu(\alpha) < \nu(\beta)$. Since φ is a bijective function, there are $a,b \in G$ such that $\varphi(a) = \alpha, \varphi(b) = \beta$. Assume $\mu(\varphi^{-1}(\alpha)) \ge \mu(\varphi^{-1}(\beta))$. If $\mu(\varphi^{-1}(\alpha)) = \mu(\varphi^{-1}(\beta))$, then $\nu(\varphi(\varphi^{-1}(\alpha))) = \nu(\varphi(\varphi^{-1}(\beta)))$, i.e., $\nu(\alpha) = \nu(\beta)$, a contradiction. If $\mu(\varphi^{-1}(\alpha)) > \mu(\varphi^{-1}(\beta))$, then $\nu(\varphi(\varphi^{-1}(\alpha))) > \nu(\varphi(\varphi^{-1}(\beta)))$, i.e., $\nu(\beta) < \nu(\alpha)$, a contradiction. Hence we obtain $\mu(\varphi^{-1}(\alpha)) < \mu(\varphi^{-1}(\beta))$.

(ii) Assume that there are $\alpha,\beta \in H$ such that $\nu(\alpha) = \nu(\beta), \mu(\varphi^{-1}(\alpha)) \ne \mu(\varphi^{-1}(\beta))$. If we let $\alpha := \varphi(a), \beta := \varphi(b)$, then $\mu(a) \ne \mu(b)$. If $\mu(a) < \mu(b)$, then $\nu(\varphi(a)) < \nu(\varphi(b))$, since φ is a (μ,ν)-homomorphism. It follows

that $\nu(\varphi(a)) < \nu(\varphi(b))$, i.e., $\nu(\alpha) < \nu(\beta)$, a contradiction. If $\mu(b) < \mu(a)$, then it leads to a contraction that $\nu(\beta) < \nu(\alpha)$, a contradiction.
(iii) Assume that there are $p, q \in H$ such that $\mu(\varphi^{-1}(pq)) \neq \mu(\varphi^{-1}(p)\varphi^{-1}(q))$. If we let $\varphi(a) = p, \varphi(b) = q$, then $\mu(\varphi^{-1}(pq)) \neq \mu(ab)$. If $\mu(\varphi^{-1}(pq)) \not< \mu(ab)$, then $\nu(\varphi(\varphi^{-1}(pq)) < \nu(\varphi(ab)) = \nu(\varphi(a)\varphi(b))$, since φ is a (μ, ν)-homomorphism. It follows that $\nu(pq) < \nu(\varphi(a)\varphi(b)) = \nu(pq)$, a contradiction. The case $\nu(\varphi(\varphi^{-1}(pq)) > \nu(\varphi(ab))$ also leads to a contradiction. This proves that $\varphi^{-1} : (H, \nu) \to (G, \mu)$ is a (ν, μ)-homomorphism. □

Corollary 4.5. *If $\varphi : (G, \mu) \to (H, \nu)$ is a (μ, ν)-isomorphism, then φ is a d-isometry.*

Proof. It follows from Theorem 4.4 that

$$d(x, y) \geq d(\varphi(x), \varphi(y)) \geq d(\varphi^{-1}(\varphi(x)), \varphi^{-1}(\varphi(y)) = d(x, y),$$

for all $x, y \in G$. □

5 Conclusions

In this paper we defined a directed relation with an associated digraph $(G, \rhd_u)$ for any fuzzy subgroup μ of a group G, and obtained that if μ is a fuzzy subgroup of G, then the collection of all isolated points of the digraph $(G, \rhd_u)$ forms a subgroup of G. By introducing the notion of (μ, ν)-homomorphism, we discussed graph homomorphisms of digraphs. In the consequence of research, intuitionistic fuzzy theory, hesitant fuzzy theory and soft set theory can be applied to the fuzzy subgroups and digraphs also.

Conflict of Interest: The authors declare no conflict of interest.

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Lifting Elements in Coherent Quantales

George Georgescu

Abstract. An ideal I of a ring R is a lifting ideal if the idempotents of R can be lifted modulo I. A rich literature has been dedicated to lifting ideals. Recently, new algebraic and topological results on lifting ideals have been discovered. This paper aims to generalize some of these results to coherent quantales. We introduce the notion of lifting elements in a quantale and a lot of results about them are proven. Some properties and characterizations of a coherent quantale in which any element is a lifting element are obtained. The formulations and the proofs of our results use the transfer properties of reticulation, a construction that assigns to each coherent quantale a bounded distributive lattice. The abstract results on lifting elements can be applied to study some Boolean lifting properties in concrete algebraic structures: commutative rings, bounded distributive lattices, residuated lattices, MV-algebras, BL-algebras, abelian l-groups, some classes of universal algebras, etc.

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1 Introduction

The lifting idempotent property (LIP) is a condition that achieves important classes of rings and ideals. LIP appears whenever we study clean and exchange rings, local and semilocal rings, maximal rings, Gelfand rings, mp–rings, purified rings, etc. (See [1], [36], [40], [43], etc.).

A lifting ideal of a unital ring R is an ideal I such that the idempotents of R can be lifted modulo I: if f is an idempotent of the quotient ring R/I then $f = e/I$, for some idempotent e of R. We say that the ring R has LIP if any ideal of R is a lifting ideal. A remarkable Nicholson's theorem [36] asserts that a commutative ring R has LIP iff R is a clean ring iff R is an exchange ring.

All rings that appear in this paper are commutative.

Inspired by LIP, similar lifting properties were studied in various concrete algebraic structures: bounded distributive lattices [8], [35], residuated lattices [17], [18], abelian l-groups [27], orthomodular lattices [32], MV-algebras and BL-algebras [27], [33], pseudo BL- algebras [5], [6], etc.

On the other hand, two kinds of generalizations of these lifting properties were obtained last decade. Firstly, two lifting properties were introduced for congruences of a congruence modular algebra: Congruence Boolean Lifting Property [20],[22] and Factor Congruence Lifting Property [21]. Secondly, in [10] was defined a lifting property for the elements of a quantale as an abstraction of LIP and of other concrete lifting properties.

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Recently, new interesting results on lifting ring ideals were established in [40], [43]. The aim of this paper is to extend a part of these results to the framework of quantales. We obtain some algebraic and topological properties of the lifting elements in a coherent quantale A, as well as some characterizations of the lifting elements. The main tools for proving the results are the transfer properties of the reticulation $L(A)$ of A (see [16], [10]) and the isomorphism between the Boolean algebra $B(A)$ of the complemented elements in A and the Boolean algebra $Clop(Spec(A))$ of clopen subsets of prime spectrum $Spec(A)$.

Recall from [10] that the reticulation $L(A)$ is a bounded distributive lattice whose prime spectrum $Spec(L(A))$ is isomorphic with the prime spectrum $Spec(A)$ of A. Due to Hochster's theorem [28], for any coherent quantale A one can find a commutative ring R such that the reticulations $L(A)$ and $L(R)$ are isomorphic. Therefore, by using reticulation, one can transfer the properties of lifting ring ideals in R to lifting elements of the quantale A. We will follow a route consisting of two steps: firstly, from commutative rings to bounded distributive lattices, and secondly, from bounded distributive lattices to coherent quantales.

In Section 2 we recall some basic notions and results on the prime spectrum $Spec(A)$ of a quantale A and its Zariski topology, the radical elements, the Boolean center, etc. (see [41], [13], [34]). We prove that the Boolean algebras $B(A)$ and $Clop(Spec(A))$ are isomorphic. Section 3 presents some elementary transfer properties of reticulation.

Section 4 concerns the Boolean Lifting Property (abbreviated LP) in a coherent quantale A, a notion introduced in [10]. We define the lifting elements of A and, by using the reticulation $L(A)$, we prove several properties of them. We describe the clopen subsets of the maximal spectrum $Max(A)$ (endowed with the Zariski topology), then we characterize the situation whenever the Jacobson radical $r(A)$ of A is a lifting element.

The main result of Section 5 is a characterization theorem of lifting elements in a coherent quantale A. If A is the quantale $Id(R)$ of ideals in a commutative ring R we obtain as a particular case the characterization of the lifting ideals in R (see Theorem 3.18 of [43]). Applying our characterization theorem we prove that the join of a regular element and a lifting element is a lifting element. Another consequence is the following result of [10]: a coherent quantale A has LP if and only if A is B-normal.

2 Preliminaries on Quantales

In this section we shall recall some definitions and elementary results in quantale theory. The basic references on quantales are the books [41], [13], [37].

Let us fix a quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$ and denote by $K(A)$ the set of its compact elements. In the usual way, the quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$ is denoted by A. The quantale A is said to be *integral* if the structure $(A, \cdot, 1)$ is a monoid and *commutative*, if the multiplication $\cdot$ is commutative. Recall that a *frame* is a quantale in which the multiplication coincides with the meet (see [30], [39]). The quantale A is said to be *algebraic* if any element $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. An algebraic quantale A is said to be *coherent* if 1 is a compact element and the set $K(A)$ of compact elements is closed under the multiplication. Coherent frames are defined in a similar way (see [30], [39]). The main example of coherent quantale (resp. coherent frame) is the set $Id(R)$ of ideals of a unital commutative ring R (resp. the set $Id(L)$ of ideals of a bounded distributive lattice L).

Throughout this paper, the quantales are assumed to be integral and commutative. We shall write ab instead of $a \cdot b$.

Each quantale A can be endowed with a residuation operation (= implication) $a \to b = \bigvee\{x | ax \leq b\}$ and with a negation operation $a^\perp = a^{\perp_A}$, defined by $a^\perp = a \to 0 = \bigvee\{x \in A | ax = 0\}$ (extending the terminology from ring theory [2], $a^\perp$ is also called the annihilator of a). Recall from [41] that for all $a, b, c \in A$ the following residuation rule holds: $a \leq b \to c$ if and only if $ab \leq c$, so $(A, \vee, \wedge, \cdot, \to, 0, 1)$ becomes a (commutative) residuated lattice. Particularly, it follows that for any $a \in A$, $a \leq b^\perp$ if and only if $ab = 0$. In this paper we shall use without mention some elementary arithmetical properties of residuated lattices [15].

An element $p < 1$ of a quantale A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. The m-prime elements of a quantale extend the notions of prime ideals of a commutative ring and the prime ideals of a bounded distributive lattice. It is well–known that if A is an algebraic quantale, then $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. Let us recall the following usual notations: $Spec(A)$ is the set of m-prime elements of A and $Max(A)$ is the set of maximal elements of A. If 1 is a compact element then for any $a < 1$ there exists $m \in Max(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$. We remark that the set $Spec(R)$ of prime ideals in a commutative ring R is the prime spectrum of the quantale $Id(R)$ and the set of prime ideals in a bounded distributive lattice L is the prime spectrum of the frame $Id(L)$. Keeping the terminology, we say that $Spec(A)$ is the m–prime spectrum of the quantale A (abbreviated, $Spec(A)$ is the prime spectrum of A).

If R is a ring, then its Jacobson radical is the ideal $J(A) = \bigcap Max(A)$ (cf. [2]). This notion can be generalized to a quantale A: $r(A) = \bigwedge Max(A)$ is the Jacobson radical of A (cf. [10]).

The paper [14] emphasizes various abstract theories of m–prime elements and of corresponding spectra developed in the last decades.

Recall from [41] that the radical $\rho(a)$ of an element a of A is defined by $\rho(a) = \bigwedge\{p \in Spec(A) | a \leq p\}$ (it is clear that this notion generalizes the radical of an ideal in a commutative ring). If $a = \rho(a)$ then a is said to be a radical element of A. The set $R(A)$ of the radical elements of A is a frame [41], [42]. In [10] it is proven that $Spec(A) = Spec(R(A))$ and $Max(A) = Max(R(A))$. The quantale A is semiprime if the meet $\rho(0)$ of all m-prime elements in A is 0.

The following useful lemma extends to quantales a well–known result in ring theory [2].

Lemma 2.1. [34] *Let A be a coherent quantale and $a \in A$. Then the following hold:*

(1) $\rho(a) = \bigvee\{c \in K(A) | c^k \leq a$ for some integer $k \geq 1\}$;

(2) For any $c \in K(A), c \leq \rho(a)$ iff $c^k \leq a$ for some integer $k \geq 1$.

(3) A is semiprime if and only if for any integer $k \geq 1$, $c^k = 0$ implies $c = 0$.

Let A be a quantale such that $1 \in K(A)$, so $Spec(A)$ and $Max(A)$ are non–empty sets. For any $a \in A$, denote $D_A(a) = D(a) = \{p \in Spec(A) | a \not\leq p\}$ and $V_A(a) = V(a) = \{p \in Spec(A) | a \leq p\}$. For all $a, b \in A$ we have $D_A(a \vee b) = D_A(a) \cup D_A(b)$ and $D_A(a \wedge b) = D_A(ab) = D_A(a) \cap D_A(b)$; for any family $(a_i)_{i \in I}$ of A, $D_A(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} D_A(a_i)$. Then $Spec(A)$ is endowed with a topology whose closed sets are $(V(a))_{a \in A}$ [41]. If the quantale A is algebraic then the family $(D(c))_{c \in K(A)}$ is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum $Spec(R)$ of a commutative ring R [2]) and the Stone topology (defined on the prime spectrum $Spec(L)$ of a bounded distributive lattice L [3]). Then this topology will be also called the Zariski topology of $Spec(A)$ and the corresponding topological space will be also denoted by $Spec(A)$. According to [41], if A is a coherent quantale, then $Spec(A)$ is a *spectral space* in the sense of [28], [12].

Let L be a bounded distributive lattice. For any $x \in L$, denote $D(x) = \{P \in Spec(L) | x \notin P\}$ and $V(x) = \{P \in Spec(L) | x \in P\}$. The family $(D(x))_{x \in L}$ is a basis of open sets for the Stone topology on $Spec(L)$ (see [3],[7]).

Let R be a commutative ring. For any element $x \in R$, we shall denote $D(x) = \{P \in Spec(R) | x \notin P\}$ and $V(x) = \{P \in Spec(R) | x \in P\}$. The family $(D(x))_{x \in R}$ is a basis of open sets for the Zariski topology on $Spec(R)$ (see [2], [30]).

An element e of the quantale A is a complemented element if there exists $f \in A$ such that $e \vee f = 1$ and $e \wedge f = 0$. The set $B(A)$ of complemented elements of A is a Boolean algebra (cf. [7], [29]). Then $B(A)$ will be called the Boolean center of the quantale A. For any $e \in B(A)$ we denote by $\neg e$ the complement of e in $B(A)$.

Lemma 2.2. [29] *For all $a, b \in A$ and $e \in B(A)$ the following hold*

(1) If $a \in B(A)$ if and only if $a \vee a^{\perp} = 1$;

(2) $a \wedge e = ae$;

(3) $e \to a = e^{\perp} \vee a$;

(4) If $a \vee b = 1$ and $ab = 0$ then $a, b \in B(A)$;

(5) $(a \wedge b) \vee e = (a \vee e) \wedge (b \vee e)$;

(6) $\neg e = e^{\perp}$ and $e \to a = \neg e \vee a$.

Lemma 2.3. [10] *$B(A) \subseteq K(A)$.*

Lemma 2.4. *If $c \in B(A)$ then $D_A(c)$ is a clopen subset of $Spec(A)$.*

Proof. If $c \in B(A)$ then there exists $d \in B(A) \subseteq K(A)$ such that $c \vee d = 1$ and $cd = 0$. Therefore we have $D_A(c) \cup D_A(d) = D_A(c \vee d) = D_A(1) = Spec(A)$ and $D_A(c) \cap D_A(d) = D_A(cd) = D_A(0) = \emptyset$, so $D_A(c)$ is a clopen subset of $Spec(A)$. $\square$

If X is a topological space then it is well-known that the set $Clop(X)$ of clopen subsets of X is a Boolean algebra. By Lemma 2.2 one can take the map $D_A|_{B(A)} : B(A) \to Clop(Spec(A))$ defined by the assignment $c \mapsto D_A(c)$.

Recall from [31], [1] the following standard result in ring theory: if R is a commutative ring, then the Boolean algebra $B(R)$ of idempotents in R and the Boolean algebra $Clop(Spec(R))$ of clopen subsets of $Spec(R)$ are isomorphic. This lemma is intensively used in commutative algebra and algebraic geometry (see e.g. [31], [1], [30], [43]). The following proposition is a quantale version of the mentioned lemma.

Proposition 2.5. *The map $D_A|_{B(A)} : B(A) \to Clop(Spec(A))$ is a Boolean isomorphism.*

Proof. That $D_A|_{B(A)} : B(A) \to Clop(Spec(X))$ is a Boolean morphism is easy to check and the injectivity of $D_A|_{B(A)}$ follows by observing that for any $e \in B(A)$, $D_A(e) = \emptyset$ implies $e = 0$. By applying Lemma 21 of [10] it results in the surjectivity of $D_A|_{B(A)}$. $\square$

We shall use many times the previous proposition to prove some basic results of this paper.

3 Reticulation of a Coherent Quantale

Let A be a coherent quantale and $K(A)$ the set of its compact elements. On the set $K(A)$ we define the following equivalence relation: for all $c, d \in K(A)$, $c \equiv d$ iff $\rho(c) = \rho(d)$. The quotient set $L(A) = K(A)/\equiv$ is a bounded distributive lattice. For any $c \in K(A)$ denote by $c/\equiv$ its equivalence class. Consider the canonical surjection $\lambda_A : K(A) \to L(A)$ defined by $\lambda_A(c) = c/\equiv$, for any $c \in K(A)$. The pair $(L(A), \lambda_A : K(A) \to L(A))$ (or shortly $L(A)$) will be called the reticulation of A. In [10], [16] was given an axiomatic definition of the reticulation. We remark that reticulation $L(R)$ of a commutative ring R (defined in [30], [42]) is isomorphic with the reticulation $L(Id(R))$ of the quantale $Id(R)$.

For any $a \in A$ and $I \in Id(L(A))$ let us denote $a^* = \{\lambda_A(c) | c \in K(A), c \leq a\}$ and $I_* = \bigvee\{c \in K(A) | \lambda_A(c) \in I\}$. The assignments $a \mapsto a^*$ and $I \mapsto I_*$ define two order - preserving maps $(\cdot)^* : A \to Id(L(A))$ and $(\cdot)_* : Id(L(A)) \to A$. The following lemma collects the main properties of the maps $(\cdot)^*$ and $(\cdot)_*$.

Lemma 3.1. [10] *The following assertions hold*

(1) If $a \in A$ then a^* is an ideal of $L(A)$ and $a \leq (a^*)_*$;

(2) If $I \in Id(L(A))$ then $(I_*)^* = I$;

(3) If $p \in Spec(A)$ then $(p^*)_* = p$ and $p^* \in Spec(L(A))$;

(4) If $P \in Spec((L(A))$ then $P_* \in Spec(A)$;

(5) If $p \in K(A)$ then $c^* = (\lambda_A(c)]$;

(6) If $c \in K(A)$ and $I \in Id(L(A))$ then $c \leq I_*$ iff $\lambda_A(c) \in I$;

(7) If $a \in A$ and $I \in Id(L(A))$ then $\rho(a) = (a^*)_*$, $a^* = (\rho(a))^*$ and $\rho(I_*) = I_*$;

(8) If $c \in K(A)$ and $p \in Spec(A)$ then $c \leq p$ iff $\lambda_A(c) \in p^*$.

Lemma 3.2. [25] *The following assertions hold*

(1) If $(a_i)_{i\in I}$ is a family of elements in A then $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$;

(2) If $a, b \in A$ then $(ab)^* = (a \wedge b)^* = a^* \cap b^*$.

By Lemma 3.1 one can consider the functions $\delta_A : Spec(A) \to Spec(L(A))$ and $\epsilon_A : Spec(L(A)) \to Spec(A)$, defined by $\delta_A(p) = p^*$ and $\epsilon_A(I) = I_*$, for all $p \in Spec(A)$ and $I \in Spec(L(A))$.

Lemma 3.3. ([10] and [24]) *The functions δ_A and ϵ_A are homeomorphisms, inverse to one another.*

We also observe that δ_A and ϵ_A are also order-isomorphisms. In particular, for any m-prime element p of A, we have $p \in Max(A)$ if and only if $p^* \in Max(L(A))$. $Max(A)$ is a topological space as a subspace of $Spec(A)$; the family $(Max(A) \cap D_A(x))_{x\in K(A)}$ is a basis for this topology.
The functions δ_A and ϵ_A are order isomorphisms, therefore the functions $\delta_A\mid_{Max(A)}: Max(A) \to Max(L(A))$ and $\epsilon_A\mid_{Spec(L(A))}: Max(L(A)) \to Max(A)$ are order-isomorphisms.

Corollary 3.4. *The functions $\delta_A|_{Max(A)}$ and $\epsilon_A|_{Max(L(A))}$ are homeomorphisms, inverse to one another.*

The maximal spectrum $Spec(L)$ of a bounded distributive lattice L is a compact $T1$-space (cf. [30], p. 66). By applying the previous corollary it follows that the maximal spectrum $Max(A)$ of a coherent quantale A is a compact $T1$–space.

For a bounded distributive lattice L we shall denote by $B(L)$ the Boolean algebra of the complemented elements of L. It is well-known that $B(L)$ is isomorphic to the Boolean center $B(Id(L))$ of the frame $Id(L)$ (see [7], [30]).

Lemma 3.5. [24] *Assume $c \in K(A)$. Then $\lambda_A(c) \in B(L(A))$ if and only if there exists an integer $n \geq 1$ such that $c^n \in B(A)$.*

Corollary 3.6. [10] *The function $\lambda_A|_{B(A)} : B(A) \to B(L(A))$ is a Boolean isomorphism.*

4 Quantales with Boolean Lifting Property

Let A, B be two quantales. A function $u : A \to B$ is a morphism of quantales if it preserves the arbitrary joins and the multiplication (in this case we have $u(0) = 0$); f is an integral morphism if $f(1) = 1$. If $u(K(A)) \subseteq K(B)$ then we say that u preserves the compacts. If u is an integral quantale morphism that preserves the compacts then it is called a coherent quantale morphism. In a similar manner one defines the frame morphisms, integral frame morphisms, coherent frame morphism, etc. (cf. [30], [39]).

Let $f : R_1 \to R_2$ be a morphism of (unital) commutative rings. If I is an ideal of R_1 then I^e will denote the extension of I to R_2, i.e. the ideal $R_2 f(I)$ generated by $f(I)$ in R_2 (cf. [2], p. 9). Then the function $f^\bullet : Id(R_1) \to Id(R_2)$, defined by $f^\bullet(I) = I^e$, for any $I \in Id(R_1)$, is a coherent quantale morphism.

Let $f : L_1 \to L_2$ be a morphism of bounded distributive lattices. If I is an ideal of L_1 then $f^\bullet(I)$ is the lattice ideal $(f(I)]$ generated by $f(I)$ in L_2. Then the function $f^\bullet : Id(L_1) \to Id(L_2)$, defined by $I \mapsto f^\bullet(I)$, for any $I \in Id(L_1)$, is a coherent frame morphism.

The following result is a straightforward generalization of Lemma 3.8(1) of [11] (for the sake of completeness we shall present its proof).

Lemma 4.1. *If $u : A \to B$ is a surjective coherent morphism of quantales then $u(K(A)) = K(B)$.*

Proof. By the definition of a coherent quantale morphism we have $u(K(A)) \subseteq K(B)$. In order to establish the converse inclusion $K(B) \subseteq u(K(A))$ let us consider an arbitrary element d of $K(B)$ so there exists $x \in A$ such that $d = u(x)$. Since A is a coherent quantale we have $x = \bigvee_{i\in I} c_i$, for some family $(c_i)_{i\in I}$ of compact elements in A, therefore $d = u(\bigvee_{i\in I} c_i) = \bigvee_{i\in I} u(c_i)$. According to $d \in K(B)$ it follows that $d = \bigvee_{i\in J} u(c_i)$, for some finite subset J of I. Denoting $c = \bigvee_{i\in J} c_i$ we have $c \in K(A)$ and $d = u(c)$, so $d \in u(K(A))$. Thus the inclusion $K(B) \subseteq u(K(A))$ is proven. □

Proposition 4.2. [10] *Let $u : A \to B$ be a coherent quantale morphism. Then there exists a morphism of bounded distributive lattices $L(u) : L(A) \to L(B)$ such that the following diagram is commutative*

$$\begin{array}{ccc} K(A) & \xrightarrow{\;u|_{K(A)}\;} & K(B) \\ \Big\downarrow{\scriptstyle \lambda_A} & & \Big\downarrow{\scriptstyle \lambda_B} \\ L(A) & \xrightarrow{\;L(u)\;} & L(B) \end{array}$$

Let R be a commutative ring, $B(R)$ the Boolean algebra of its idempotents, $R(Id(R))$ the frame of radical ideals in A and $B(R(Id(R)))$ the Boolean algebra of complemented elements of $R(Id(R))$.

According to [40], [43], an ideal I of R is said to be a lifting ideal if the canonical ring morphism $R \to F/I$ lifts the idempotents: for any $y \in B(R/I)$ there exists $x \in B(R)$ such that $x/I = y$. We say that R satisfies the lifting idempotent property (LIP) if any ideal of R is a lifting ideal (see [36]). The two Boolean algebras $B(R)$ and $B(R(Id(R)))$ are isomorphic and the condition LIP can be expressed in terms of the frame $R(Id(R))$ (see [4]).

Similarly, following [9] we say that an ideal I of a bounded distributive lattice L satisfies the $Id-Boolean$ $Lifting$ $Property$ (abbreviated $Id-BLP$) if the lattice morphism $L \to L/I$ lifts the complemented elements: for any $y \in B(L/I)$ there exists $x \in B(L)$ such that $x/I = y$. If any ideal of L satisfies $Id - BLP$ we say that L satisfies $Id - BLP$.

In what follows we shall generalize the previous lifting properties to the framework of coherent quantales. Firstly, we will develop some preliminary matters.

We fix a coherent quantale A. For any $a \in A$, consider the interval $[a) = \{x \in A | a \leq x\}$ and for all $x, y \in [a)_A$, denote $x \cdot_a y = x \cdot y \vee a$. It is easy to see that $[a)_A$ is closed under the new multiplication $\cdot_a$.

Lemma 4.3. [10] *$([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ is a coherent quantale.*

Let x, y be two elements of the coherent quantale $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$. Denote by $\to^a$ the implication operation in $[a)_A$ and $x^{\perp_a}$ the annihilator of x in $[a)_A$. The negation of an element $x \in B([a)_A)$ will be denoted by $\neg^a(x)$.

Lemma 4.4. *Assume that x, y are two elements of the coherent quantale $[a)_A$. Then the following hold:*

(1) $x \to^a y = x \to y$;

(2) $x^{\perp_a} = x \to a$;

(3) $x \in B(([a)_A)$ *if and only if* $x \vee (x \to a) = 1$;

(4) *If* $e \in B(A)$ *and* $x \in B([e)_A)$ *then* $x \cdot \neg e \in B(A)$ *and* $x = x \cdot \neg e \vee e$;

(5) *If* $f \in B([a)_A)$ *and* $b \in A$ *then* $f \vee b \in B([a \vee b)_A)$.

Proof. The proof of (1) and (2) is easy and (3) follows by Lemma 2.2,(1) and (3). In order to prove (4), assume that $e \in B(A)$ and $x \in B([e)_A)$, so there exists $y \in B([e)_A)$ such that $x \vee y = 1$ and $x \cdot_e y = e$. This last equality implies $xy \leq e$, so $xy \cdot \neg e = 0$. From $x \cdot \neg e \leq x \cdot \neg e$ we get $x \leq \neg e \to x \cdot \neg e$, hence $x \leq x \cdot \neg e \vee e$ (by Lemma 2.2(3)). The converse inequality $x \cdot \neg e \vee e \leq x$ is obvious, so $x = x \cdot \neg e \vee e$. Similarly we have $y = y \cdot \neg e \vee e$.

We observe that $x \cdot \neg e \vee (y \cdot \neg e \vee e) = x \vee y = 1$ and $x \cdot \neg e \cdot (y \cdot \neg e \vee e) = xy \cdot \neg e = 0$, hence $x \cdot \neg e \in B(A)$.

To prove (5), let us assume that $f \in B([a)_A)$, so $f \vee (f \to a) = 1$ (cf. (3)). We remark that $f \to a \leq f \to (a \vee b)$, hence $f \to (a \vee b) = 1$. Thus the following equalities hold:

$(f \vee b) \vee [(f \vee b) \to (a \vee b)] = [f \to (a \vee b)] \wedge [b \to (a \vee b)] = f \to (a \vee b) = 1$.

According to (3), it follows that $f \vee b \in B([a \vee b)_A)$.

□

For an arbitrary $a \in A$, let us consider the function $u_a^A : A \to [a)_A$ defined by $u_a^A(x) = x \vee a$, for any $x \in A$.

Lemma 4.5. *For any* $a \in A$ *the following hold:*

(1) u_a^A *is an integral quantale morphism.*

(2) *If* $c \in K(A)$ *then* $u_a^A(c) \in K([a)_A)$.

(3) $u_a^A(K(A)) = K([a)_A)$.

Proof. The first two assertions are proved in [10] and the third follows by Lemma 4.1.

□

Remark 4.6. *According to Lemma 4.5(2), the quantale morphism* u_a^A *preserves the compacts, so applying Proposition 4.2, the following diagram is commutative:*

$$\begin{array}{ccc} K(A) & \xrightarrow{\;u_a^A\;} & K([a)_A) \\ \lambda \downarrow & & \downarrow \lambda_a \\ L(A) & \xrightarrow{\;L(u_a^A)\;} & L([a)_A) \end{array}$$

where $\lambda = \lambda_A$ and $\lambda_a = \lambda_{[a)_A}$.

Proposition 4.7. [10] *For any* $a \in A$*, the bounded distributive lattices* $L([a)_A)$ *and* $L(A)/a^*$ *are isomorphic.*

By Lemma 4.5, u_a^A is a coherent quantale morphism, so we can consider the Boolean morphism $B(u_a^A) = u_a^A|_{B(A)} : B(A) \to B([a)_A)$. The following diagram is commutative:

$$\begin{array}{ccc} B(A) & \xrightarrow{B(u_a^A)} & B([a)_A) \\ \big\downarrow & & \big\downarrow \\ K(A) & \xrightarrow{u_a^A|_{K(A)}} & K([a)_A) \end{array}$$

where the vertical arrows are the inclusion maps (cf. Lemma 2.3).

Definition 4.8. [10] *An element $a \in A$ has the (Boolean)* lifting property (LP) *if the Boolean morphism $B(u_a^A)$ is surjective. The quantale A has LP if every element $a \in A$ has LP.*

If I is an ideal of a commutative ring R then it is easy to see that I is a lifting ideal if and only if I has LP in the quantale $Id(R)$. Keeping this terminology, if $a \in A$ has LP we shall say that a is a lifting element of the quantale A.

Similarly, an ideal I of a bounded distributive lattice L has $Id - BLP$ if and only if I has LP in the frame $Id(L)$.

Let A, A' be two coherent quantale and $u : A \to A'$ a coherent quantale morphism. We say that u lifts the complemented elements if for each $e' \in B(A')$ there exists $e \in B(A)$ such that $f(e) = e'$. Then an element a of a quantale A is a lifting element if and only if the quantale morphism $u_a^A : A \to [a)_A$ lifts the complemented elements. If I is an ideal in a commutative ring R and $p_I : A \to A/I$ is the associated ring morphism, then I is a lifting ideal in R if and only if the quantale morphism $p_I^{\bullet} : Id(R) \to Id(R/I)$ lifts the complemented elements. A similar result for lifting ideals in bounded distributive lattices is valid.

Let A be a coherent quantale. If a is an element of A such that $B([a)_A) = \{a, 1\}$ then it is clear that a is a lifting element. In particular, any minimal non–zero element of A is a lifting element.

Let p be an m–prime element of A. If $x \in B([p)_A)$ then $x \cdot_a \neg^p(x) = p$, so $x = p$ or $\neg^p(x) = p$. Since $Spec([p)_A) = Spec(A) \cap [p)_A$, it results that $p \in Spec([p)_A)$. Thus $B([a)_A) = \{p, 1\}$, hence the m–prime element p is a lifting element. Particularly, any maximal element of A is a lifting element.

Any complemented element e of A is a lifting element. Indeed, if $x \in B([e)_A)$ then, by using Lemma 4.4(4), it follows that $x \cdot \neg e \in B(A)$ and $x = x \cdot \neg e \vee e = u_e^A(x \cdot \neg e)$, i.e. e is a lifting element.

For any $a \in A$ denote $X_a = Spec([a)_A) = \{p \in Spec(A) | p \leq a\}$. We remark that $X_a = X_{\rho(a)}$ and $Clop(X_a) = Clop(X_{\rho(a)})$. According to Proposition 2.5, there exists a Boolean isomorphism $v_a : B([a)_A) \to Clop(X_a)$, defined by $v_a(e) = D_A(e) \cap [a)_A$, for any $e \in B([a)_A)$.

Theorem 4.9. *For any $a \in A$ the following are equivalent:*

(1) a has LP in the quantale A;

(2) $\rho(a)$ has LP in the quantale A;

(3) $\rho(a)$ has LP in the frame $R(A)$.

Proof. (1) $\Leftrightarrow$ (2) Recall that $v_{\rho(a)}$ and v_a are Boolean isomorphisms and $Clop(X_a) = Clop(X_{\rho(a)})$, so there exists a Boolean isomorphism $w : B([\rho(a)_A) \to B([a)_A)$ such that the following diagram is commutative:

$$\begin{array}{ccc} B[\rho(a))_A & \xrightarrow{v_{\rho(a)}} & Clop(X_{\rho(a)}) \\ w \big\downarrow & & \big\downarrow id \\ B([a)_A) & \xrightarrow{v_a} & Clop(X_a) \end{array}$$

Consider the Boolean morphisms $v_{\rho(a)} \circ u^A_{\rho(a)} : B(A) \to Clop(X_{\rho(a)})$ and $v_a \circ u^A_a : B(A) \to Clop(X_a)$.

An easy computation shows that for any $e \in B(A)$ the following equalities hold:

$(v_{\rho(a)} \circ u^A_{\rho(a)})(e) = \{p \in Spec(A) | e \vee \rho(a) \not\leq p\} \cap [\rho(a))_A$

$(v_a \circ u^A_a)(e) = \{p \in Spec(A) | e \vee a \not\leq p\} \cap [a)_A$.

We remark that for any $p \in Spec(A)$ we have $a \leq p$ iff $\rho(a) \leq p$ and $e \vee a \not\leq p$ iff $e \vee \rho(a) \not\leq p$, therefore $(v_{\rho(a)} \circ u^A_{\rho(a)})(e) = (v_a \circ u^A_a)(e)$. Since $Clop(X_{\rho(a)}) = Clop(X_a)$ it follows that $v_{\rho(a)} \circ u^A_{\rho(a)} = v_a \circ u^A_a$. But $v_{\rho(a)}$ and v_a are Boolean isomorphisms, hence the following diagram is commutative:

$$\begin{array}{ccc} B(A) & \xrightarrow{B(u^A_{\rho(a)})} & B([\rho(a))_A) \\ & {\scriptstyle B(u^A_a)} \searrow \quad \nearrow {\scriptstyle w} & \\ & B([a)_A) & \end{array}$$

Recall that w is a Boolean isomorphism so $B(u^A_{\rho(a)})$ is surjective iff $B(u^A_a)$ is surjective, therefore $\rho(a)$ has LP iff a has LP.

(2) $\Leftrightarrow$ (3) According to Lemma 19 of [10], the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & R(A) \\ {\scriptstyle u^A_{\rho(a)}} \downarrow & & \downarrow {\scriptstyle u^{R(A)}_{\rho(\rho(a))}} \\ [\rho(a))_A & \xrightarrow{\rho_{\rho(a)}} & [\rho(\rho(a)))_{R(A)} \end{array}$$

Since $\rho(\rho(a)) = \rho(a)$ we obtain the following commutative diagram in the category of Boolean algebras:

$$\begin{array}{ccc} B(A) & \longrightarrow & B(R(A)) \\ {\scriptstyle B(u^A_{\rho(a)})} \downarrow & & \downarrow {\scriptstyle B(u^{R(A)}_{\rho(a)})} \\ B([\rho(a))_A) & \longrightarrow & B([\rho(a))_{R(A)}) \end{array}$$

where the horizontal arrows are Boolean isomorphisms (in virtue of Proposition 6 of [10]).

From the previous commutative diagram we get that $B(u^A_{\rho(a)})$ is surjective iff $B(u^{R(A)}_{\rho(a)})$ is surjective, so the equivalence of (2) and (3) follows.

□

Corollary 4.10. [10] *Let a be an element of the coherent quantale A. Then a has LP if and only if the ideal a^* of the lattice $L(A)$ has $Id - BLP$.*

Proof. Recall from Lemma 3.1 (7) that $a^* = (\rho(a))^*$. We know from Corollary 2 of [10] that the frames $R(A)$ and $Id(L(R))$ are isomorphic, hence the following properties are equivalent:

- $\rho(a)$ has LP in the frame $R(A)$;
- $(\rho(a))^*$ has LP in the frame $Id(L(R))$;
- a^* has LP in the frame $Id(L(R))$;
- the ideal a^* of the lattice $L(A)$ has $Id - BLP$.

By applying the previous theorem it follows that a has LP if and only if the ideal a^* of the lattice $L(A)$ has $Id - BLP$.

□

Corollary 4.11. [10] *A quantale A has LP if and only if the reticulation $L(A)$ has $Id-BLP$.*

Corollary 4.12. *Let I be an ideal of the reticulation $L(A)$. Then I has $Id-BLP$ if and only if I_* has LP.*

Proof. We know that $I=(I_*)^*$ (cf. Lemma 3.1(2)), hence, by two applications of Corollary 4.10 we obtain: I has $Id-BLP$ iff $(I_*)^*$ has $Id-BLP$ iff I^* has LP. □

Corollary 4.13. *Let a and b be two elements of A such that $\rho(a)=\rho(b)$. Then a has LP if and only if b has LP.*

Proof. By Theorem 4.9, the following equivalences hold: a has LP iff $\rho(a)$ has LP iff $\rho(b)$ has LP iff b has LP. □

Corollary 4.14. *If a is an element of A such that $a\leq\rho(0)$ then a has LP.*

Proof. If $a\leq\rho(0)$ then $\rho(a)=\rho(0)$. It is obvious that 0 has LP. By applying Corollary 4.13 it follows that a has LP. □

In particular, from Corollary 4.14 it follows that $\rho(0)$ is a lifting element.

Following [10], we say that a quantale A is hyperarchimedean if for any $c\in K(A)$ there exists an integer $n\geq 1$ such that $c^n\in B(A)$.

Corollary 4.15. *If the quantale A is hyperarchimedean then any element $a\in A$ has LP.*

Proof. Let a be an element of the hyperarchimedean quantale A. By Theorem 1 of [10], the reticulation $L(A)$ of A is a Boolean algebra. It is straightforward to see that any ideal of a Boolean algebra has $Id-BLP$ (see [9]). Thus the ideal a^* of $L(A)$ has $Id-BLP$, so, by applying Corollary 4.10, it follows that a has LP. □

Remark 4.16. *The equivalence of assertions (1) and (2) of Theorem 4.9 is a quantale generalization of Corollary 3.2 of [43]. Among the consequences of this theorem we mention an important result of [10]: Corollary 4.11is exactly Theorem 2 of [10]. We remark that Theorem 4.9 can be obtained as a corollary of Theorem 2 of [10]. We shall give here a short proof.*

Let a be an element of the quantale A and $L(A)$ the reticulation of A. We know that $a^*=(\rho(a))^*$ (cf. Lemma 3.1(7)) so by a double application of Theorem 2 of [10] the following equivalences hold: a has LP iff a^* has $Id-BLP$ iff $(\rho(a))^*$ has $Id-BLP$ iff $\rho(a)$ has LP.

Theorem 4.17. *Let a and b two elements of the quantale A such that $a\leq b$ and $Max([a)_A)=Max([b)_A)$. If b has LP then a has LP.*

Proof. Firstly we observe that $Max([a)_A)=Max(A)\cap[a)_A$ and $Max([b)_A)=Max(A)\cap[b)_A$. Let us consider an element $x\in B(([a)_A))$, so $x\vee(x\to a)=1$ (by Lemma 4.4(3)). From $a\leq b$ we get $x\to a\leq x\to b$, hence $1=x\vee(x\to a)\leq x\vee(x\to b)$. Thus $x\vee b\vee((x\vee b)\to b)= x\vee b\vee((x\to b)\wedge(b\to b))=x\vee(x\to b)=1$, so $x\vee b\in B([b)_A)$ (cf. Lemma 4.4(3)).

By hypothesis, b has LP, so there exists a complemented element e of A such that $e\vee b=u_b^A(e)=x\vee b$. We shall prove that $V_A(e\vee a)=V_A(x)$.

Firstly, we shall establish the inclusion $V_A(e\vee a)\subseteq V_A(x)$. Assume that $p\in V_A(e\vee a)$, i.e. p is an m–prime element of A such that $e\vee a\leq p$. Consider a maximal element m of A such that $p\leq m$, so $e\leq m$ and $a\leq m$. According to the hypothesis $Max([a)_A)=Max([b)_A)$, from $a\leq m$ we obtain $b\leq m$, hence $x\vee b=e\vee b\leq m$.

Let us assume that $x\not\leq p$. We observe that $p\in Spec([a)_A)$ (because $p\in Spec(A)$ and $a\leq p$), so $x\to a=\neg^a(x)\leq p\leq m$. From $x\in B([a)_A)$, $x\leq m$ and $x\to a\leq m$ we obtain $1=x\vee(x\to a)\leq m$. This contradicts $m\in Max(A)$, so $x\leq p$, i.e. $p\in V_A(x)$.

In order to prove the converse inclusion $V_A(x) \subseteq V_A(e \vee a)$, assume that $p \in V_A(x)$, so $p \in Spec(A)$ and $x \leq p$. Consider a maximal element m of A such that $p \leq m$, so $a \leq x \leq p \leq m$. By using the hypothesis $Max([a)_A) = Max([b)_A)$, we get $b \leq m$, hence $e \vee b = x \vee b \leq m$.

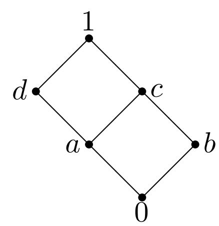
Let us assume that $e \vee a \not\leq p$, so $e \not\leq p$ (because $a \leq p$ and $p \in Spec(A)$). Thus $\neg e \leq p \leq m$, so $1 = e \vee \neg a \leq m$, contradicting $m \in Max(A)$. Then $e \vee a \leq p$, hence $p \in V_A(e \vee a)$.

One remark that $V_{[a)_A}(e \vee a) = V_A(e \vee a) = V_A(x) = V_{[a)_A}(x)$, therefore $D_{[a)_A}(e \vee a) = D_{[a)_A}(x)$. Since $e \vee a, x \in B([a)_A)$ one can apply Proposition 2.5, so from $D_{[a)_A}(e \vee b) = D_{[a)_A}(x)$ we get $u_a^A(e) = e \vee a = x$. Therefore a has LP.

□

One can ask if the element $r(A)$ has or doesn't have LP. The following two examples show that both situations are possible.

Example 4.18. Let us consider the quantale structure defined on the set $A = \{0, a, b, c, d, 1\}$ by the following diagram and table (cf. [8]):

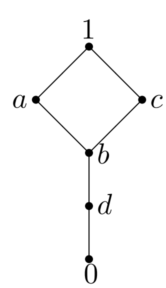


$\odot$	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	0	0	0	a
b	0	b	0	b	b	0
c	0	c	0	b	b	a
d	0	d	a	0	a	d

The maximal spectrum of the quantale A is $Max(A) = \{c, d\}$, so $r(A) = c \wedge d = a$. The Boolean center of A is $B(A) = \{0, b, d, 1\}$. We observe that $[r(A))_A = [a)_A = \{a, c, d, 1\}$ is a Boolean algebra, hence $B([r(A))_A) = B([a)_A) = \{a, c, d, 1\}$.

Now we consider the Boolean morphism $B(u^A_{r(A)}) : B(A) \to B([r(A))_A)$ (in fact, $B(u_a^A) : \{0, b, d, 1\} \to \{a, c, d, 1\}$). An easy computation gives $u_a^A(0) = a, u_a^A(b) = a \vee b = c, u_a^A(d) = a \vee d = d, u_a^A(1) = 1$, so $B(u^A_{r(A)})$ is a surjective map. Then $r(A)$ has LP.

Example 4.19. Let us consider the quantale structure defined on the set $A = \{0, a, b, c, d, 1\}$ by the following diagram and table (cf. [8]):



$\odot$	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	a	b	b	0
b	0	b	b	b	b	0
c	0	c	b	b	c	d
d	0	d	0	0	d	0

In this case, we have $Max(A) = \{a, c\}$, $r(A) = a \wedge c = b$ and $[r(A))_A = [b)_A = \{b, a, c, 1\}$ is a Boolean algebra. We observe that $B(A) = \{0, 1\}$ and $B([b)_A) = [b)_A = \{b, a, , c, 1\}$. It is clear that $B(u_b^A) : B(A) \to B([b)_A)$ is not surjective, so $r(A) = b$ doesn't have LP.

Corollary 4.20. *Assume that a is an element of A such that $a \leq r(A)$. If $r(A)$ has LP then a has LP.*

Proof. If $a \leq r(A)$ then $Max([a)_A) = Max(A) = Max([r(A))_A)$. By applying Theorem 4.17, it follows that a has LP. $\square$

Theorem 4.21. *For any subset U of $Max(A)$ the following are equivalent:*

(1) U is a clopen subset of $Max(A)$;

(2) There exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$.

Proof. (1) $\Rightarrow$ (2) Assume that U is a clopen subset of $Max(A)$. We know that $Max(A)$ is compact (cf. Corollary 3.4, $Max(A)$ and $Max(L(A))$ are homeomorphic spaces and by [30], p.66, the maximal spectrum $Max(L(A))$ of the bounded distributive lattice $L(A)$ is compact) and $U, Max(A) - U$ are closed subsets of $Max(A)$, so they are compact. Therefore there exist two positive integers n, m and the compact elements $c_1, \cdots, c_n$, $d_1, \cdots, d_m$ such that $U = \bigcup_{i=1}^{n}(Max(A) \cap D_A(c_i))$ and $Max(A) - U = \bigcup_{j=1}^{m}(Max(A) \cap D_A(d_j))$.

We remark that for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$ the following equalities hold: $Max(A) \cap D_A(c_i d_j) = Max(A) \cap D_A(c_i) \cap D_A(d_j) = \emptyset$. Then for each $m \in Max(A)$ we have $m \notin D_A(c_i d_j)$, hence $c_i d_j \leq m$, therefore $c_i d_j \leq r(A)$.

Denoting $c = \bigvee_{i=1}^{n} c_i$ and $d = \bigvee_{j=1}^{m} d_j$ it results that $U = Max(A) \cap D_A(c)$ and $Max(A) - U = Max(A) \cap D_A(d)$. Then one obtains the equalities $Max(A) = Max(A) \cap (D_A(c) \cup D_A(d)) = Max_A(A) \cap D_A(c \vee d)$. Since $c_i d_j \leq r(A)$ for all i and j it follows that

$cd = \bigvee\{c_i d_j | i = 1, \cdots, n \ ; \ j = 1, \cdots, m\} \leq r(A)$.

Assume by absurdum that $c \vee d < 1$, so $c \vee d \leq m$ for some maximal element m of A. Then $m \notin D_A(c \vee d)$, contradicting $Max(A) = Max(A) \cap D_A(c \vee d)$. We conclude that $c \vee d = 1$.

(2) $\Rightarrow$ (1) Assume that there exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$. We shall prove the equality $Max(A) \cap D_A(c) = Max(A) \cap V_A(d)$.

Let m be an element of $Max(A) \cap D_A(c)$, hence $c \nleq m$. Since m is m - prime and $cd \leq r(A) \leq m$ we get $d \leq m$, so $m \in Max(A) \cap V_A(d)$. Thus we obtain the inclusion $Max(A) \cap D_A(c) \subseteq Max(A) \cap V_A(d)$.

In order to prove the converse inclusion $Max(A) \cap V_A(d) \subseteq Max(A) \cap D_A(c)$, let us assume that $m \in Max(A) \cap V_A(d)$, so $d \leq m$. If $c \leq m$ then $1 = c \vee d \leq m$, contradicting that m is a maximal element. It follows that $c \nleq m$, hence $m \in Max(A) \cap V_A(d)$. Thus the inclusion $Max(A) \cap V_A(d) \subseteq Max(A) \cap D_A(c)$ is established, therefore $Max(A) \cap D_A(c) = Max(A) \cap V_A(d)$. This equality shows that U is a clopen subset of $Max(A)$. $\square$

Corollary 4.22. *If $x \in B([r(A))_A)$ then $Max(A) \cap D_A(x)$ is a clopen subset of $Max(A)$.*

Proof. If $x \in B([r(A))_A)$ then there exists $y \in B([r(A))_A)$ such that $x \vee y = 1$ and $x \cdot_{r(A)} y = r(A)$. By applying the previous theorem to the elements x and y of the quantale $[r(A))_A$, it follows that $Max(A) \cap D_{[r(A))_A}(x)$ is a clopen subset of $Max([r(A))_A)$. We remark that $Max(A) = Max([r(A))_A)$, hence $Max(A) \cap D_A(x) = Max([r(A))_A) \cap D_{[r(A))_A}(x)$, so $Max(A) \cap D_A(x)$ is a clopen subset of $Max(A)$. $\square$

According to this corollary, the assignment $x \mapsto Max(A) \cap D_A(x)$ defines a map $f : B([r(A))_A) \to Clop(Max(A))$.

Proposition 4.23. *The map $f : B([r(A))_A) \to Clop(Max(A))$ is a Boolean isomorphism.*

Proof. Firstly, we prove that f is a Boolean morphism. Assume that x and y are two elements of the Boolean algebra $B([r(a))_A)$. We remark that the following equalities hold: $f(x \vee y) = Max(A) \cap D_A(x \vee y) = (Max(A) \cap (D_A(x) \cup D_A(y)) = Max(A) \cap D_A(x)) \cup (Max(A) \cap D_A(y)) = f(x) \cup f(y)$. Similarly, we have $f(x \wedge y) = f(x) \cap f(y)$. It is clear that $f(r(A)) = Max(A) \cap D_A(r(A)) = \emptyset$ and $f(1) = Max(A)$, hence f is a Boolean morphism.

If $f(x) = \emptyset$, then $Max(A) \cap D_A(x) = \emptyset$, therefore $m \notin D_A(x)$, for any maximal element m of A. It follows that $x \leq m$, for any $m \in Max(A)$, i.e. $x \leq r(A)$. Since $r(A) \leq x$, we get $x = r(A)$. This shows that f is an injective Boolean morphism.

Assume now that U is a clopen subset of $Max(A)$. In accordance with Theorem 4.21 there exist $c, d \in K(A)$ such that $c \vee d = 1$, $cd \leq r(A)$ and $U = Max(A) \cap D_A(c)$.

By using $c \vee d = 1$ and $cd \leq r(A)$ the following equalities hold:

$(c \vee r(A)) \vee (d \vee r(A)) = 1$

$(c \wedge r(A)) \cdot_{r(A)} (d \wedge r(A)) = cd \vee r(A) = r(A)$,

hence $e = c \vee r(A) \in B([r(A))_A)$ (by Lemma 2.2(4)). We have observed that $Max(A) \cap D_A(r(A)) = \emptyset$, therefore we get

$f(e) = Max(A) \cap D_A(e) = (Max(A) \cap D_A(c)) \cup ((Max(A) \cap D_A(r(A)) = Max(A) \cap D_A(c) = U$.

Thus f is a surjective map, so it is a Boolean isomorphism. □

Lemma 4.24. *The Boolean morphism $B(u^A_{r(A)}) : B(A) \to B([r(A))_A)$ is injective.*

Proof. Assume that $e \in B(A)$ and $B(u^A_{r(A)})(e) = r(A)$. Thus $e \vee r(A) = r(A)$, hence $e \leq r(A)$. If $e \neq 0$ then $\neg e \neq 1$, so $\neg e \leq m$, for some maximal element m of A. On the other hand we have $e \leq r(A) \leq m$, hence $1 = e \vee \neg e \leq m$, contradicting that $m \in Max(A)$. It follows that $e = 0$, so $B(u^A_{r(A)})$ is injective. □

Now consider the map $g : B(A) \to Clop(Max(A))$ defined by $g(e) = Max(A) \cap D_A(e)$, for any $e \in B(A)$. It is straightforward that g is a Boolean morphism.

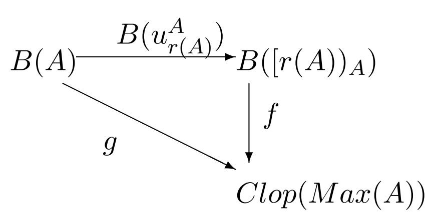
Recall that in general, $r(a)$ does not have LP (see Example 4.19). The following result characterizes the situation whenever $r(A)$ has LP.

Theorem 4.25. *The following properties are equivalent:*

(1) $r(A)$ has LP;

(2) $g : B(A) \to Clop(Max(A))$ is a Boolean isomorphism.

Proof. For any $e \in B(A)$ the following equalities hold: $f(B(u^A_{r(A)})(e)) = f(e \vee r(A)) = Max(A) \cap D_A(e \vee r(A)) = Max(A) \cap (D_A(e) \cup D_A(r(A))) = Max(A) \cap D_A(e) = g(e)$ (because $Max(A) \cap D_A(r(A)) = \emptyset$). It follows that the following diagram is commutative in the category of Boolean algebras:



Recall that f is a Boolean isomorphism (cf. Proposition 4.23) and $B(u^A_{r(A)})$ is injective (cf. Lemma 4.24), therefore, according to the previous commutative diagram it results that g is injective. Then the following equivalences hold: $r(A)$ has LP iff $B(u^A_{r(A)})$ is surjective iff g is surjective iff g is a Boolean isomorphism. □

5 A Characterization Theorem and Its Consequences

We start this section by recalling the following characterization theorem of lifting ideals in a commutative ring R (see Theorems 1.5 and 3.2 of [40] and Theorem 3.18 of [43]).

Theorem 5.1. [43] *If I is an ideal of the commutative ring R then the following are equivalent*

(1) I is a lifting ideal of R;

(2) If J_1, J_2 are two coprime ideals of R such that $J_1J_2 \subseteq I$ then there exists an idempotent e of R such that $e \in I + J_1$ and $\neg e \in I + J_2$;

(3) If J_1, J_2 are two coprime ideals of R such that $J_1J_2 = I$ then there exists an idempotent e of R such that $e \in J_1$ and $\neg e \in J_2$;

(4) If M is a maximal ideal of R and $M^\diamond$ is the ideal of R generated by $\{f \in I | f = f^2\}$ then the quotient ring $R/(I + M^\diamond)$ has no nontrivial idempotents.

We remark that the property (4) of the previous theorem says that the Boolean algebra $B(R/(I + M^\diamond))$ of idempotents of $R/(I + M^\diamond)$ is isomorphic to $L_2 = \{0, 1\}$ (we include the case whenever L_2 is a trivial Boolean algebra).

Theorem 5.1 and some of its consequences were extended in [26] to the lifting congruences in a semidegenerate congruence modular algebra (see Theorem 6.3 of [26]).

In this section we shall prove a new extension of Theorem 5.1. We shall characterize the lifting elements of a coherent quantale. Then we shall present some consequences of Theorem 5.1.

Let A be a coherent quantale. For any element $a \in A$ we shall denote $a^\diamond = \bigvee\{e \in B(A)|e \leq a\}$ (this new element in a quantale was firstly defined in [24]). We remark that in the particular case when A is the quantale $Id(R)$ of the ideals in a commutative ring R we find the ring "diamond construction".

If R is a commutative ring and $I \in Id(R)$ then it is easy to see that $I^\diamond$ is equal to the ideal of R generated by $I \cap B(R)$; if L is a bounded distributive lattice and $I \in Id(L)$ then $I^\diamond$ is equal to the ideal of L generated by $I \cap B(R)$ (i.e. $I^\diamond = [I \cap B(L))$).

Recall that two elements a and b of the quantale A are coprime if $a \vee b = 1$. If a and b are coprime then $ab = a \wedge b$ (see Lemma 2(1) of [10]). For any set Ω we denote by $|\Omega|$ its cardinal number.

The following lemma emphasizes the way in which the reticulation preserves the diamond construction.

Lemma 5.2. *If $a \in A$ then $a^{*\diamond} = a^{\diamond *}$.*

Proof. Firstly, we observe that $a^{*\diamond}$ is the ideal $[a^* \cap B(L(A)))$ of the lattice $L(A)$ generated by $a^* \cap B(L(A))$ and $a^{\diamond *} = \{\lambda_A(c)|c \in K(A), c \leq a^\diamond\}$.

For proving the inclusion $a^{\diamond *} \subseteq a^{*\diamond}$, let x be an element of $a^{\diamond *}$, so $x = \lambda_A(c)$ for some $c \in K(A)$ such that $c \leq a^\diamond$. Since $a^\diamond = \bigvee\{e \in B(A)|e \leq a\}$ and c is compact it follows that there exists $e \in B(A)$ such that $c \leq e \leq a$. Recall that the map λ_A is isotone and preserves the complemented elements. Then $\lambda_A(c) \leq \lambda_A(e)$ and $\lambda_A(e) \in B(L(A)) \cap a^*$, so $x = \lambda_A(c) \in [B(L(A)) \cap a^*) = a^{*\diamond}$. Thus we obtain the inclusion $a^{\diamond *} \subseteq a^{*\diamond}$.

In order to prove the converse inclusion $a^{*\diamond} \subseteq a^{\diamond *}$ it suffices to check that $B(L(A)) \cap a^* \subseteq a^{\diamond *}$. If $x \in B(L(A)) \cap a^*$ then $x = \lambda_A(c)$ for some compact element c of A such that $c \leq a$. By using Lemma 3.5, from $\lambda_A(c) = x \in B(L(A))$ we get $c^n \in B(A)$, for some integer $n \geq 1$. From $c^n \leq a$ and $c^n \in B(A)$ we obtain $c^n \leq a^\diamond$, hence $x = \lambda_A(c) = \lambda_A(c^n) \in a^{\diamond *}$, therefore $a^{*\diamond} \subseteq a^{\diamond *}$.

□

Recall a famous theorem of Hochster [28] (see also [12]): if L is a bounded distributive lattice then there exists a commutative ring R such that the reticulation $L(R)$ of R is isomorphic with L.

Let A be a coherent quantale and $L(A)$ its reticulation. By applying the Hochster theorem one can find a commutative ring R such the reticulations $L(A)$ and $L(R)$ are isomorphic lattices (we shall identify $L(A)$ and $L(R)$). For any element a of A, we know that a^* is an ideal of the bounded distributive lattice $L(A) = L(R)$, so there exists a ring ideal I of R such that $a^* = I^*$ (cf. Lemma 3.1(2)).

This previous construction is a bridge between rings and coherent quantales: by applying the transfer properties of reticulations some results of ring theory can be exported to quantale theory and viceversa. The following propositions are the first illustration of this thesis.

Proposition 5.3. *Keeping the previous notations the following are equivalent*

(1) a is a lifting element of A;

(2) The lattice ideal $a^* = I^*$ has $Id - BLP$;

(3) The ring ideal I of R is a lifting ideal.

Proof. By applying twice Corollary 4.10. □

Proposition 5.4. *Keeping the previous notations the following are equivalent*

(1) The quantale A has LP;

(2) The isomorphic lattices $L(A)$ and $L(R)$ have $Id - BLP$;

(3) The ring R has the Lifting Idempotent Property.

Proof. By applying twice Corollary 4.11. □

Let a be an arbitrary element of the coherent quantale A and m a maximal element of A. Then a^* is an ideal of the lattice $L(A)$ and m^* a maximal ideal of $L(A)$. By using Corollary 3.4 one can find an ideal I of R and a maximal ideal M of the ring R such that $a^* = I^*$ and $m^* = M^*$.

Lemma 5.5. *Keeping the previous notations the following equality holds:*
$|B([a \vee m^\diamond)_A)| = |B(R/(I \vee M^\diamond))|$.

Proof. In accordance with Lemma 3.2(1) we have $(a \vee m^\diamond)^* = a^* \vee m^{\diamond *}$, so by using Proposition 4.7 we get the following isomorphisms in the category of bounded distributive lattices:
$L([a \vee m^\diamond)_A) = L(A)/(a \vee m^\diamond)^* = L(A)/(a^* \vee m^{\diamond *})$

Thus, according to Corollary 3.6, we obtain the following sequence of Boolean isomorphisms:
$B([a \vee m^\diamond)_A) \simeq B(L([a \vee m^\diamond)_A)) \simeq B(L(A)/(a \vee m^\diamond)^*) \simeq B(L(A)/(a^* \vee m^{\diamond *})$.

In a similar way we obtain the Boolean isomorphism: $B(R/(I \vee M^\diamond)) \simeq B(L(R)/(I^* \vee M^{\diamond *}))$.

By Lemma 5.2 and $m^* = M^*$ we have $m^{\diamond *} = m^{*\diamond} = M^{*\diamond} = M^{\diamond *}$, hence the Boolean algebras $B([a \vee m^\diamond)_A)$ and $B(L(R)/(I^* \vee M^{\diamond *}))$ are isomorphic, so their cardinal numbers are equal.
□

Now we are ready to state and prove the following characterization theorem of lifting elements in a coherent quantale.

Theorem 5.6. *Let A be a coherent quantale. For any $a \in A$ the following are equivalent*

(1) a has LP;

(2) If c, d are two coprime elements of A such that $cd \leq a$ then there exists $e \in B(A)$ such that $e \leq a \vee c$ and $\neg e \leq a \vee d$;

(3) If c, d are two coprime elements of A such that $cd = a$ then there exists $e \in B(A)$ such that $e \leq c$ and $\neg e \leq d$;

(4) If m is a maximal element of A then $|B([a \vee m^\diamond)_A)| \leq 2$.

Proof. (1) $\Rightarrow$ (2) Assume that a has LP and c, d are two coprime elements of A such that $cd \leq a$. Then $(a \vee c) \vee (a \vee d) = 1$ and $(a \vee c)(a \vee d) = a \vee cd = a$, so $a \vee c$ and $a \vee d$ are two elements of the Boolean algebra $B([a)_A)$ such that $\neg^a(a \vee c) = a \vee d$.

According to the hypothesis that a has LP there exists $e \in B(A)$ such that $a \vee e = B(u_a^A)(e) = a \vee c$. Since $B(u_a^A)$ is a Boolean morphism it follows that $a \vee d = \neg^a(a \vee c) = \neg^a(B(u_a^A)(e)) = B(u_a^A)(\neg e) = a \vee \neg e$. We conclude that $e \leq a \vee c$ and $\neg e \leq a \vee d$.

(2) $\Rightarrow$ (3) Assume that c, d are two coprime elements of A such that $cd = a$. By the hypothesis (2), there exists $e \in B(A)$ such that $e \leq a \vee c$ and $\neg e \leq a \vee d$. Remark that $c = c(c \vee d) = cd \vee c^2 = a \vee c^2$, hence $a \leq c$. Similarly, we have $a \leq d$, so $e \leq a \vee c = c$ and $\neg e \leq a \vee d = d$, hence the assertion (3) follows.

(3) $\Rightarrow$ (1) Let us consider that $x \in B([a)_A)$ so there exists $y \in B([a)_A)$ such that $y = \neg^a(x)$, hence $x \vee y = 1$ (i.e. x, y are coprime) and $xy \vee a = x \cdot_a y = a$, so $xy \leq a$. Since $a \leq x$, $a \leq y$ and x, y are coprime we have $a \leq x \wedge y = xy$, so $xy = a$. Then one can apply the hypothesis (3), so there exists $e \in B(A)$ such that $e \leq x$ and $\neg e \leq y$.

Recall that u_a^A is a quantale morphism and $u_a^A|_{B(A)} = B(u_a^A)$ is a Boolean morphism. Thus from $e \leq x$ we get $u_a^A(e) \leq u_a^A(x) = x$. On the other hand, $\neg e \leq y$ implies $\neg^a(u_a^A(e)) = u_a^A(\neg e) \leq u_a^A(y) = y$, hence $x = \neg^a(y) \leq u_a^A(e)$. It follows that $x = u_a^A(e) = B(u_a^A)(e)$, so the map $B(u_a^A) : B(A) \to B([a)_A)$ is surjective. Then a has LP.

(1) $\Leftrightarrow$ (4) We shall prove this equivalence by using the reticulation construction in order to obtain a translation of results from lifting ring ideals to lifting elements of the quantale A.

Let R be a commutative ring such that $L(A)$ and $L(R)$ are isomorphic lattices (we will assume that $L(A) = L(R)$). Consider an element a of A; a^* is an ideal of $L(A)$ so one can find an ideal I of R such that $a^* = I^*$. Due to Corollary 3.4, both spaces $Max(A)$ and $Max(R)$ are homeomorphic with $Max(L(A)) = Max(L(R))$. Then for any $m \in Max(A)$ one can find $M \in Max(R)$ such that $m^* = M^*$; conversely, for any $M \in Max(R)$ one can find $m \in Max(A)$ such that $M^* = m^*$.

According to Theorem 5.1, Corollary 4.10 and Lemma 5.5, the following properties are equivalent:

- a has LP (in the quantale A);
- $a^* = I^*$ has $Id - BLP$ (in the lattice $L(A)$);
- I is a lifting ideal of R;
- If M is a maximal ideal of R then $R/(I \vee M^\diamond)$ has no nontrivial idempotents;
- If m is a maximal element of A then $|B([a \vee m^\diamond)_A)| \leq 2$.

Therefore the equivalence (1) $\Leftrightarrow$ (4) follows.

□

Remark 5.7. *The previous proof of the equivalence* (1) $\Leftrightarrow$ (4) *is based on the corresponding equivalence* (1) $\Leftrightarrow$ (4) *in Theorem 5.1. One can pose the problem to do a direct proof of this equivalence, without using Theorem 5.1 and the reticulation. We shall present here a direct proof of implication* (4) $\Rightarrow$ (1).

Let us assume that for any maximal element m of A we have $|B([a \vee m^\diamond)_A)| \leq 2$.

In order to prove that a has LP, consider an element f of $B([a)_A)$. We want to prove that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond = 1$. Assume by absurdum that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond < 1$, so that $(f \to a)^\diamond \vee (\neg^a(f) \to a)^\diamond \leq m$, for some maximal element m of A.

According to the hypothesis (4), $B([a \vee m^\diamond)_A) = \{a \vee m^\diamond, 1\}$. By applying Lemma 4.4(5), $f \in B([a)_A)$ implies $f \vee m^\diamond \in B([a \vee m^\diamond)_A)$. We shall distinguish two cases:

(a) $f \vee m^\diamond = a \vee m^\diamond$. We remark that $f \leq a \vee m^\diamond = \bigvee\{a \vee e | e \in B(A), e \leq m\}$ and f is a compact element of $[a)_A$, so there exists $e \in B(A)$ such that $e \leq m$ and $f \leq a \vee e$. By Lemma 2.2(3) we have $f \leq \neg e \to a$, hence $\neg e \leq f \to a$. Since $\neg e \in B(A)$, it follows that $\neg e \leq (f \to a)^\diamond \leq m$, contradicting $e \leq m$ (because $e \leq m$ and $\neg e \leq m$ imply $1 = e \vee \neg e \leq m$ and m is a maximal element of A).

(b) $\neg^{a \vee m^\diamond}(f \vee m^\diamond) = a \vee m^\diamond$. We remark that $B([a \vee m^\diamond)_A) = B([a \vee m^\diamond)_{[a)_A})$ and

$\neg^{a\vee m^\diamond}(f\vee m^\diamond)=\neg^{a\vee m^\diamond}(u^{[a)_A}_{a\vee m^\diamond}(f))=u^{[a)_A}_{a\vee m^\diamond}(\neg^a(f))=\neg^a(f)\vee a\vee m^\diamond=\neg^a(f)\vee m^\diamond$.

Then $\neg^a(f)\vee m^\diamond=a\vee m^\diamond$, hence $\neg^a(f)\leq a\vee m^\diamond$. By using the compactness of $\neg^a(f)$ in $[a)_A$ and the inequality $\neg^a(f)\leq\bigvee\{a\vee e|e\in B(A),e\leq m\}$, we get $\neg^a(f)\leq a\vee e$, for some $e\in B(A)$ with the property that $e\leq m$. Thus $\neg^a(f)\leq\neg e\to a$, hence $\neg e\leq\neg^a(f)\to a$, resulting $\neg e\leq(\neg^a(f)\to a)^\diamond\leq m$ (because of $\neg e\in B(A)$). It follows that $1=e\vee\neg e\leq m$, contradicting that $m\in Max(A)$.

In both cases (a) and (b) we obtained a contradiction, therefore the equality $(f\to a)^\diamond\vee(\neg^a(f)\to a)^\diamond=1$ holds. According to this equality, there exist $e_1,e_2\in B(A)$ such that $e_1\vee e_2=1$, $e_1\leq f\to a$ and $e_2\leq\neg^a(f)\to a$.

From $e_1\vee e_2=1$ and $e_1\leq f\to a$ we get $\neg e_2\leq e_1\leq f\to a$, therefore $\neg e_2\vee a\leq f\to a$ (because $a\leq f\to a$). Therefore $\neg^a(e_2\vee a)=(e_2\vee a)\to a=e_2\to a=\neg e_2\vee a\leq f\to a=\neg^a(f)$. We observe that f and $e_2\vee a$ are elements of $B([a)_A)$, so from $\neg^a(e_2\vee a)\leq\neg^a(f)$ we get $f\leq e_2\vee a$.

On the other hand, we remark that the inequality $e_2\leq\neg^a(f)\to a$ implies $\neg^a(f)\leq e_2\to a=(e_2\vee a)\to a=\neg^a(e_2\vee a)$, which implies $e_2\vee a\leq f$. We conclude that $f=e_2\vee a=B(u^A_a)(e_2)$, so a has LP.

Remark 5.8. *It is clear that Theorem 5.6 extends Theorem 5.1 to the abstract framework of quantale theory. On the other hand, Theorem 6.3 of [26] generalizes Theorem 5.1 to a framework of universal algebra: it characterizes the lifting congruences of a semidegenerate congruence modular algebra M. We observe that in general Theorem 6.3 of [26] is not a consequence of Theorem 5.6 (because the set $Con(M)$ of congruences of the algebra M does not have a quantale structure). In the very special case when the commutator operation of $Con(M)$ is associative, $Con(M)$ becomes a coherent quantale and Theorem 5.6 can be applied, so Theorem 6.3 of [26] can be deduced from our main result.*

An ideal I of a commutative ring R is regular if it is generated by a set of idempotents of A (see [1]). Then I is regular if and only if $I=I^\diamond$. Similarly, an ideal J of a bounded distributive lattice L is regular if it is equal to the ideal $[J\cap B(L))$ generated by $J\cap B(L)$. It is clear that J is regular if and only if $J=J^\diamond$. According to [24], an element a of a quantale A is regular if it is a join of complemented elements. It is obvious that a is regular if and only if $a=a^\diamond$. Of course, $a^\diamond$ is a regular element of A.

The following lemma shows that the function $(\cdot)^*:A\to Id(L(A))$ maps the regular elements of A to regular ideals of $L(A)$.

Lemma 5.9. *If a is a regular element of A then a^* is a regular ideal of $L(A)$.*

Proof. We want to prove that $a^*=[a^*\cap B(L(A)))$. Assume that x is an element of a^*. By Lemma 3.2(1), the following hold:

$a^*=(\bigvee\{e|e\in B(A),e\leq a\})^*=\bigvee\{e^*|e\in B(A),e\leq a\}$

hence there exist $e_1,\cdots,e_n\in B(A)$ such that $e_i\leq a$, for $i=1,\cdots,n$ and $x\in\bigvee_{i=1}^n e_i^*=(\bigvee_{i=1}^n e_i)^*$ (the last equality follows by using Lemma 3.2(1)).

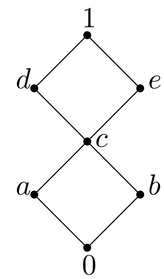
Denoting $e=\bigvee_{i=1}^n e_i$ we have $e\in B(A)$, $e\leq a$ and $x\in e^*$, so there exists $c\in K(A)$ such that $x=\lambda_A(c)$ and $c\leq e$. Therefore $x=\lambda_A(c)\leq\lambda_A(e)$, hence $x\in[a^*\cap B(L(A)))$. We conclude that $a^*\subseteq[a^*\cap B(L(A)))$, and the converse inclusion is obvious.

□

Lemma 5.10. *If I is a regular ideal of $L(A)$ then there exists a regular element b of A such that $I=b^*$.*

Proof. Assume that I is a regular ideal of $L(A)$, so $I^\diamond=I$. Let us define $b=(I_*)^\diamond$. Then b is a regular element of A and, by using Lemmas 5.2 and 3.1(2), we get $b^*=((I_*)^\diamond)^*=((I_*)^*)^\diamond=I^\diamond=I$. □

Example 5.11. Let us consider the frame $A=\{0,a,b,c,d,e,1\}$, defined by the following diagram:



We observe that $B(A) = \{0, 1\}$, $B([a)_A) = B(\{a, c, d, e, 1\} = \{a, 1\}$ and $B([b)_A) = B(\{b, c, d, e, 1\} = \{b, 1\}$, therefore a and b are lifting elements of A. On the other hand, $B([c)_A) = B(\{c, d, e, 1\}) = \{c, d, e, 1\}$, hence the Boolean morphism $B(u_c^A) : B(A) \to B([c)_A)$ is not surjective. We conclude that $c = a \vee b$ is not a lifting element of A.

According to the previous example, the join $a \vee b$ of two lifting elements a and b of a coherent quantale A is not necessarily a lifting element of A. The following proposition and its corollary emphasize an important case whenever it happens.

Proposition 5.12. *Let a and b be two elements of the quantale A. If a has LP and b is a regular element of A then $a \vee b$ is a lifting element of A.*

Proof. According to Theorem 5.6, in order to prove that $a \vee b$ is a lifting element of A it suffices to check that for any maximal element m of A we have
$|B([a \vee b \vee m^\diamond)_A)| \leq 2$.

Assume that $m \in Max(A)$. Let R be a commutative ring such that the reticulations $L(R)$ and $L(A)$ of R and A are identical. One can find an ideal I of the ring R and a maximal ideal M of R such that $I^* = a^*$ and $M^* = m^*$.

By Lemma 5.9, b^* is a regular ideal of the lattice $L(A)$, hence, by using Lemma 5.10, one can find a regular ideal J of R such that $J^* = b^*$. We observe that $(I \vee J)^* = I^* \vee J^* = a^* \vee b^* = (a \vee b)^*$ (cf. Lemma 3.2(1)).

An application of Lemma 5.5 for the pairs $(a \vee b, m)$, $(I \vee J, M)$ gives the equality $|B([a \vee b \vee m^\diamond)_A)| = |B(R/(I \vee J \vee M^\diamond))|$.

Since a has LP it results that the ideal $I^* = a^*$ of $L(A)$ has $Id - BLP$ (cf. Corollary 4.10). A new application of Corollary 4.10 shows that I is a lifting ideal of R. Recall that J is a regular ideal of R, so, by using Corollary 3.19 of [43], it follows that $I \vee J$ is a lifting ideal of R.

According to Theorem 5.6, we have $|B(R/(I \vee J \vee M^\diamond))| \leq 2$, therefore we conclude that $|B([a \vee b \vee m^\diamond)_A)| \leq 2$. Thus $a \vee b$ is a lifting element of A. □

We have seen in Section 4 that any complemented of A is a lifting element. This property will be generalized by the following corollary.

Corollary 5.13. *Any regular element of the coherent quantale A is a lifting element of A.*

Proof. Assume that b is a regular element of A. It is obvious that 0 has LP. If one takes $a = 0$ in Proposition 5.12 it follows that $b = 0 \vee b$ has LP. □

Let us consider the quantale $A = \{a, b, c, d, 1\}$ from Example 4.19. We observe that $B(A) = \{0, 1\}$, $B([a)_A) = \{a, 1\}$ and $B([c)_A) = \{c, 1\}$, hence a and c are lifting elements. On the other hand, we have $B([b)_A) = \{b, a, c, 1\}$, so $b = a \wedge c$ is not a lifting element. This example shows that in general the meet of two lifting elements is not a lifting element. The following proposition achieves a particular case whenever the meet of two lifting elements is a lifting element.

Proposition 5.14. *Let a, b be two non-coprime elements of A. If a has LP and $|B([b)_A)| \leq 2$ then $a \wedge b$ has LP.*

Proof. Assume that a, b are two non-coprime elements of A, a has LP and $|B([b)_A)| \leq 2$.

Let R be a commutative ring such that $L(A)$ and $L(R)$ are identical. We can find two ideals I, J of R such that $I^* = a^*$ and $J^* = b^*$. Then I is a lifting ideal of the ring R (by Proposition 5.3).

By using Proposition 4.7 we get $|B([b)_A)| = |B(L(A)/b^*)| = |B(L(R)/J^*)| = |B(R/J)|$.

According to the hypothesis $|B([b)_A)| \leq 2$, we get the inequality $|B(R/J)| \leq 2$, i.e. the quotient ring R/J has no nontrivial idempotents. Since $a \vee b < 1$, we have $a \vee b \leq m$, for some maximal element m of A. One can find a maximal ideal M of R such that $m^* = M^*$. In accordance with Lemma 3.2(1) we have $(I \vee J)^* = I^* \vee J^* = a^* \vee b^* = (a \vee b)^* \subseteq m^* = M^*$, hence, by applying Lemma 3.1(2) we get $I \vee J = ((I \vee J)^*)_* \subseteq (M^*)_* = M$. Thus I, J are non–coprime ideals of the ring R, therefore we can apply Proposition 1.2 of [40] or Proposition 3.2 of [43]. It follows that $I \cap J$ is a lifting ideal of R. By using Lemma 3.2(2) we have $(a \wedge b)^* = a^* \cap b^* = I^* \cap J^* = (I \cap J)^*$, hence $a \wedge b$ is a lifting element of A (by Proposition 5.3).

□

Recall from [36] that a ring R is said to be a clean ring if any element of R is the sum of a unit and an idempotent. An important theorem of [36] asserts that a commutative ring R is a clean ring if and only if any ideal of R is a lifting ideal (i.e. the quantale $Id(R)$ has LP). By Corollary 4.11, the following lemma holds:

Lemma 5.15. *A commutative ring R is a clean ring if and only if the reticulation $L(R)$ of R has $Id-BLP$.*

Proposition 5.16. *Let a be an element of a coherent quantale A such that $a \leq r(A)$. If the quantale $[a)_A$ has LP then A has LP.*

Proof. Let us consider a commutative ring R such that $L(A) = L(R)$ so there exists an ideal I of R such that $a^* = I^*$. Assume that M is a maximal ideal of the ring R, so M^* is a maximal ideal of the lattice $L(A) = L(R)$. Thus one can find a maximal element m of A such that $m^* = M^*$. By the hypothesis, we have $a \leq m$, so $I^* = a^* \subseteq m^* = M^*$. It follows that $I \subseteq \sqrt{I} = (I^*)_* \subseteq (M^*)_* = M$ (by Lemma 3.1,(2) and (7)), therefore $I \subseteq Rad(R)$ (recall that $Rad(R)$ is the Jacobson radical of the ring R).

By using Proposition 4.7 we get the following lattice isomorphisms: $L([a)_A) \simeq L(A)/a^* \simeq L(R)/I^* \simeq L(R/I)$, hence, by applying Corollary 4.11, the following implications hold:

$[a)_A$ has $LP \Rightarrow L([a)_A)$ has $Id-BLP \Rightarrow L(R/I)$ has $Id-BLP \Rightarrow R/I$ is a clean ring.

According to Proposition 1.5 of [36], R is a clean ring. Then the lattice $L(A) = L(R)$ has $Id-BLP$, so A has LP (by Corollary 4.11).

□

Following [10], a quantale A is said to be B-normal if for all coprime elements a, b of A there exist $e, f \in B(A)$ such that $a \vee e = b \vee f = 1$ and $ef = 0$. By using Theorem 5.6 we present here a short proof of the following result of [10]:

Proposition 5.17. *A coherent quantale A has LP if and only if A is B-normal.*

Proof. Assume that A has LP and a, b are two coprime elements of A. Let us denote $x = ab$. By applying the condition (3) of Theorem 5.6 for x, one can find $e \in B(A)$ such that $e \leq a$ and $\neg e \leq b$. Denoting $f = \neg e$ it follows that $a \vee f = b \vee e = 1$ and $ef = 0$, hence A is B–normal.

Conversely, let us suppose that the quantale A is B–normal. Let a, x, y be three elements of A such that x, y are coprime and $a = xy$. Since A is B–normal, there exists $e, f \in B(A)$ such that $x \vee e = b \vee f = 1$ and $ef = 0$. Then $\neg e \rightarrow x = \neg f \rightarrow y = 1$ (cf. Lemma 2.2(3)) and $e \leq \neg f$, therefore $e \leq \neg f \leq y$ and $\neg e \leq x$. Thus the property (3) of Theorem 5.6 is verified, so A has LP.

□

Recall from [37] that a quantale A is normal if for all coprime elements a, b of A there exist $x, y \in A$ such that $a \vee x = b \vee y = 1$ and $xy = 0$.

Proposition 5.18. [10] *If A is a normal coherent quantale then the Jacobson radical $r(A)$ of A is a lifting element.*

Proof. It suffices to verify the condition (3) of Theorem 5.6. Let $c, d \in K(A)$ such that $c \vee d = 1$ and $cd = r(A)$. Since A is normal there exist $x, y \in A$ such that $c \vee x = d \vee y = 1$ and $xy = 0$, so $x \vee y \vee c = x \vee y \vee d = 1$. According to Lemma 2(2) of [10] we have $x \vee y \vee (cd) = 1$, hence $x \vee y \vee r(A) = 1$. By applying Lemma 22 of [10], it follows that $x \vee y = 1$. From $x \vee y = 1$ and $xy = 0$ we get $x, y \in B(A)$ and $y = \neg x$ (cf. Lemma 2.2(4)).

We observe that $x \vee c = 1$ implies $y = y(x \vee c) = xy \vee cy = cy$, so $y \leq c$. Similarly, $y \vee d = 1$ implies $y \leq d$. We have proven that $y \in B(A)$, $y \leq c$ and $\neg y \leq d$, therefore the condition (3) of Theorem 5.6 is verified.

□

6 Concluding Remarks

The study of the coherent quantales with Boolean Lifting Property (LP) and the elements with LP began in [10]. The results of [10] are focused on quantales with LP rather than on elements with LP. The present paper deals with the lifting elements of a coherent quantale (= the elements that satisfy LP).

We proved many algebraic and topological properties of lifting elements in a coherent quantale. Among them we mention a characterization theorem and a lot of its consequences (new or old results). The inspiration points of the paper are the results obtained in [40] and [43] for the lifting ideals in commutative rings.

Two methods are applied to prove the results of this paper. The first one consists in applying a proposition that provides an isomorphism between the Boolean center $B(A)$ of a coherent quantale A and the Boolean algebra of clopen subsets of the prime spectrum $Spec(A)$ of A. The second method uses the transfer properties of the reticulation $L(A)$ of A. According to a Hochster theorem [28], for any coherent quantale A one can find a commutative ring R such that the results on the lifting ideals of R can be transferred to the lifting elements of A.

Our general results can be applied to concrete algebraic structures for which a Boolean Lifting Property can be defined: rings, bounded distributive lattices, commutative l-groups, orthomodular lattices, MV-algebras, BL- algebras, pseudo- BL algebras, residuated lattices, etc.

An important open problem is to extend the construction of reticulation and the study of lifting properties from quantales to more larger classes of multiplicative lattices.

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Fuzzy Mathematics and Nonstandard Analysis Application to the Theory of Relativity

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Abstract. In this paper, we extend some results of nonstandard analysis to include concepts from fuzzy mathematics. We then apply our results to issues from special and general relativity and the theory of light-clocks. The extension includes concepts of fuzzy numbers, continuity, and differentiability. Our goal is to provide a new research area in fuzzy mathematics.

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1 Introduction

The purpose of this paper is to open the door for a new research area in fuzzy mathematics. This new area is based on nonstandard analysis. Mathematicians and physicists solved problems by considering infinitesimally small pieces of a shape, or movement along a path by infinitesimal amounts. Infinitesmals were ultimately rejected as mathematically unsound. However, in 1960 Abraham Robinson developed nonstandard analysis by rigorously extending the reals $\mathbb{R}$ to a field $\mathbb{R}^*$ which includes infintesimal numbers and infinite numbers. The goal was to create a system of analysis that was more intuitively appealing than standard analysis, but without losing any rigor of standard analysis, [1]. The standard notation for the field of hyperreals is ${}^*\mathbb{R}$, but for our purposes the notation $\mathbb{R}^*$ is easier to work with.

Our approach to introducing these concepts to fuzzy mathematics essentially involves replacing the interval $[0, 1]$ with an extension of it to $\mathbb{R}^*$. There are two possible extensions. One is replacing $[0, 1]$ with its natural extension $[0, 1]^*$ or with $]^-0, 1^+[$. A few scholars have replaced $[0, 1]$ with $]^-0, 1^+[$ in the definition of certain fuzzy sets, but have never used it in their research. These extensions have been discussed in [8]. In [3], Herrmann applies nonstandard analysis to explain issues from special and general relativity and the theory of light-clocks. In this paper, we extend some of the results in [3] to nonstandard fuzzy analysis. We do this in terms of nonstandard fuzzy functions and nonstandard fuzzy numbers. Related works can be seen in [4, 6, 7].

We let $\mathbb{N}$ denote the positive integers, and $\mathbb{R}$ the set of real numbers. We let $\wedge$ denote minimum or infimum and $\vee$ denote maximum or supremum. If X is a set, $\mathcal{P}(X)$ denotes the power set of X. If X and Y are sets, $X \backslash Y$ denotes set difference. If Y is a subset of X, we sometimes write Y^c for $X \backslash Y$.

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In Section 2, we present some basic concepts, definitions, and results from nonstandard analysis. The material presented here is a summary of known basic results needed in our presentation.

In Section 3, we extend these concepts to fuzzy mathematics. We first review the basic definitions and results of fuzzy numbers. We then extend the notion of a fuzzy number to that of a nonstandard fuzzy number.

In Section 4, we review some results concerning continuity and differentiability of functions that are pertinent to nonstandard analysis and which can be used in our application to the theory of relativity established in [3].

In Section 5, we show how the concepts of fuzzy numbers, continuity and differentiability can be applied to nonstandard analysis and an application to the theory of relativity. Much of the discussion is from [3].

2 Nonstandard Analysis

Much of the following is from [1].

Definition 2.1. *(Free Ultrafilter) A **filter** $\mathcal{U}$ on a set J is a subset of $\mathcal{P}(J)$ satisfying properties* $(1)-(3)$. *A filter $\mathcal{U}$ is called an **ultrafilter** if it satisfies* (4) *and an ultrafilter is called **free** if it satisfies* (5).

(1) *Proper:* $\emptyset \notin \mathcal{U}$,

(2) *Finite intersection property: If* $A, B \in \mathcal{U}$, *then* $A \cap B \in \mathcal{U}$.

(3) *Superset property: If* $A \in \mathcal{U}$ *and* $A \subseteq B \subseteq J$, *then* $B \in \mathcal{U}$,

(4) *Maximality: For all* $A \subseteq J$, *either* $A \in \mathcal{U}$ *or* $J \backslash A \in \mathcal{U}$,

(5) *Freeness:* $\mathcal{U}$ *contains no finite subsets.*

It is important to note that by $(1), (2)$ and (4), if $A \subseteq J$, then either $A \in \mathcal{U}$ or $J \backslash A \in \mathcal{U}$, but not both.

Lemma 2.2. *Let U be an ultrafilter on $\mathbb{N}$ and let $\{A_1, ..., A_n\}$ be a finite collection of disjoint subsets of $\mathbb{N}$ such that $\cup_{j=1}^{n} A_i = \mathbb{N}$. Then $A_i \in \mathcal{U}$ for exactly one $i \in \{1, ..., n\}$.*

Proof. Suppose that $\mathcal{U}$ contains no A_i. Then by (4), $\mathcal{U}$ contains A_i^c for each i. Thus by (2), contains $\cap_{i=1}^{n} A_i^c = (\cup_{i=1}^{n} A_i)^c = \mathbb{N}^c = \emptyset$, contrary to (1). Thus $\mathcal{U}$ contains some A_i. Suppose that $\mathcal{U}$ contains A_i and $A_j, i \neq j$. Then by (2), $\mathcal{U}$ contains $A_i \cap A_j = \emptyset$, contrary to (1). □

The proof of the next result uses Zorn' Lemma.

Lemma 2.3. *(Ultrafilter) Let A be a set and $F_0 \subseteq \mathcal{P}(A)$ be a filter on A. Then F_0 can be extended to an ultrafilter $\mathcal{F}$ on A.*

Proposition 2.4. *Free Ultrafilters exist.*

Definition 2.5. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real-valued sequences. Define the relation $=_{\mathcal{U}}$ on $\mathbb{R}^{\mathbb{N}}$ by $\forall (a_n), (b_n) \in \mathbb{R}^{\mathbb{N}}$, $(a_n) =_{\mathcal{U}} (b_n)$ if and only if $\{n \in \mathbb{N} | \ a_n = b_n\} \in \mathcal{U}$.*

Proposition 2.6. *$=_{\mathcal{U}}$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let $[(a_n)]_{\mathcal{U}}$ denote the equivalence class of $=_{\mathcal{U}}$ determined by (a_n).

Let $\mathbb{R}^* = \{[(a_n)]_U | \ a_n \in \mathbb{R}, \ n = 1, 2, ...\}$. Define addition $+$ and multiplication $\bullet$ on $\mathbb{R}^*$ as follows: $\forall [(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in R^*$.

$$\begin{aligned} [(a_n)]_{\mathcal{U}} +_U [(b_n)]_{\mathcal{U}} &= [(a_n + b_n)]_{\mathcal{U}}, \\ [(a_n)]_{\mathcal{U}} \bullet_U [(b_n)]_{\mathcal{U}} &= [(a_n \bullet b_n)]_{\mathcal{U}}. \end{aligned}$$

It follows that $\mathbb{R}^*$ is a field under these operations. The result actually holds from the Transfer Principle. □

Theorem 2.7. *$(\mathbb{R}^*, +_{\mathcal{U}}, \bullet_{\mathcal{U}})$ is a field.*

Definition 2.8. *Define $\leq_{\mathcal{U}}$ on $\mathbb{R}^*$ as follows: $\forall[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in \mathbb{R}^*$, $[(a_n)]_{\mathcal{U}} \leq_U [(b_n)]_{\mathcal{U}}$ if and only if $\{n \in \mathbb{N}|\ a_n \leq b_n\} \in \mathcal{U}$.*

Let $[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}}, [(b_n]_{\mathcal{U}} \in \mathbb{R}^*$. Suppose that $[(a_n)]_{\mathcal{U}} \leq [(b_n)]_{\mathcal{U}}$ and $[(a_n)]_{\mathcal{U}} \leq [(b_n)]_{\mathcal{U}}$. Then $\{j \in \mathbb{N}|\ a_j \leq b_j\} \in \mathcal{U}$ and $\{j \in \mathbb{N}|\ a_j \leq b_j\} \in \mathcal{U}$ By the finite intersection property, it follows that $\{j \in \mathbb{N}|\ a_j \leq c_j\} \in \mathcal{U}$ and so $[(a_n)]_{\mathcal{U}} \leq [(c_n)]_{\mathcal{U}}$.

Let $[(a_n)]_{\mathcal{U}}, [(b_n)]_{\mathcal{U}} \in \mathbb{R}^*$. Let $X = \{j \in \mathbb{N}|\ a_j \leq b_j\}$. Then either $X \in \mathcal{U}$ or $\mathbb{N}\backslash X \in \mathcal{U}$. If $X \in \mathcal{U}$, then $[(a_n)]_{\mathcal{U}} \leq_{\mathcal{U}} [(b_n)]_{\mathcal{U}}$. If $X \notin U$, then $\mathbb{N}\backslash X \in \mathcal{U}$, but $\mathbb{N}\backslash X = \{j \in \mathbb{N}|\ a_n > b_n\}n$ and so $[(a_n)]_{\mathcal{U}} >_U [(b_n)]_{\mathcal{U}}$. Thus $\leq_U$ is a total ordering on $\mathbb{R}^*$.

Definition 2.9. *A hyperreal number $[(a_n)]_{\mathcal{U}}$ in $\mathbb{R}^*$ is said to be **infinitesimal** if $[(a_n)]_{\mathcal{U}} \leq_{\mathcal{U}} [(j)]_{\mathcal{U}}$ for every $j \in \mathbb{N}$ and **infinite** if $[(j)]_{\mathcal{U}} \leq_{\mathcal{U}} [(a_n)]_{\mathcal{U}}$ for every $j \in \mathbb{N}$.*

Consider $[(1, 2, 3, ...)]_{\mathcal{U}}$. Let $j \in \mathbb{N}$. Since $\mathcal{U}$ is free, it contains all cofinite subsets. Thus $\mathcal{U}$ contains $\{m \in \mathbb{N}|\ m \geq j\}$. Hence $[(1, 2, 3, ...)]_{\mathcal{U}} \geq_{\mathcal{U}} [(j, j, j, ...)]_{\mathcal{U}}$ for all $j \in \mathbb{N}$. Thus $\mathbb{R}^*$ contains infinite elements. Similarly, it can be shown that $[(1, \frac{1}{2}, \frac{1}{3}, ...)]_{\mathcal{U}} \leq_{\mathcal{U}} [(\frac{1}{j}, \frac{1}{j}, \frac{1}{j}, ...)]_{\mathcal{U}}$ for fixed $j \in \mathbb{N}$. Hence $\mathbb{R}^*$ contains infinitesimal elements.

Define the function $f : \mathbb{R} \to \mathbb{R}^*$ by for all $a \in \mathbb{R}, f(a) = [(a, a, a, ...)]_{\mathcal{U}}$. It is easily shown that f is a one-to-one function of $\mathbb{R}$ into $\mathbb{R}^*$ that preserves addition and multiplication. It also follows easily for all $a, b \in \mathbb{R}$ that $a \leq b$ if and only if $f(a) \leq_{\mathcal{U}} f(b)$.

We review some postulates given in [2] that a nonstandard universe should possess. Actually, this universe has been rigorously constructed.

Let $\mathbb{R}^*$ denote a nonstandard universe with the following properties:

$(NS1)$ $(\mathbb{R}, +, \cdot, 0, 1, <)$ is an ordered subfield of $(\mathbb{R}^*, +, \cdot, 0, 1, <)$.

$(NS2)$ $\mathbb{R}^*$ has a positive infinitesimal element, that is $\varepsilon \in \mathbb{R}^*$ such that $\varepsilon > 0$, but $\varepsilon < r$ for all positive real numbers r.

$(NS3)$ For all $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a natural extension $f : (\mathbb{R}^*)^n \to \mathbb{R}^*$. The natural extensions of the field operations $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ coincide with the field operations in $\mathbb{R}^*$. Similarly, for every $A \subseteq R^n$, there is a subset $A^* \subseteq (\mathbb{R}^*)^n$ such that $A^* \cap \mathbb{R}^n = A$.

$(NS4)$ $\mathbb{R}^*$, equipped with the above assignments of extensions of functions and subsets, behaves logically like $\mathbb{R}$.

Definition 2.10. *$\mathbb{R}^*$ is called the ordered field of hyperreals.*

Now ε has an additive inverse $-\varepsilon$. It is easily seen that $-\varepsilon$ is a negative infinitesimal. Since $\varepsilon \neq 0$, it has a multiplicative inverse ε^{-1}. For any positive real number $r, \epsilon^{-1} > r$ since $\varepsilon < r$. Thus ε^{-1} is a positive infinite element and $-\varepsilon^{-1}$ is a negative infinite element.

Definition 2.11. (1) *Let $\mathbb{R}_{fin} = \{x \in \mathbb{R}^*|\ |x| \leq n$ for some $n \in \mathbb{N}\}$. $\mathbb{R}_{fin}$ is called the set of **finite hyperreals**.*

(2) *Let $\mathbb{R}_{\inf} = \mathbb{R}^*\backslash\mathbb{R}_{fin}$. $\mathbb{R}_{\inf}$ is called the set of **infinite hyperreals**.*

(3) *Let $\mu(0) = \{x \in \mathbb{R}^*|\ |x| \leq \frac{1}{n}$, for all $n \in \mathbb{N}\}$. $\mu(0)$ is called the set of **infinitesimal hyperreals**.*

We see that $\mu(0) \subseteq \mathbb{R}_{fin}, \mathbb{R} \subseteq \mathbb{R}_{fin}$, and $\mu(0) \cap \mathbb{R} = \{0\}$. If $\delta \in \mu(0)\backslash\{0\}$, then $\delta^{-1} \notin \mathbb{R}_{fin}$.

Proposition 2.12. (1) *$\mathbb{R}_{fin}$ is a subring of $\mathbb{R}^*$.*

(2) *$\mu(0)$ is an ideal of $\mathbb{R}_{fin}$.*

Definition 2.13. *Define the relation $\approx$ on $\mathbb{R}^*$ by for all $x, y \in \mathbb{R}^*, x \approx y$ if and only if $x - y \in \mu(0)$. If $x \approx y$, we say that x and y are **infinitely close**.*

It follows immediately that $\approx$ is an equivalence relation on $\mathbb{R}^*$. It also follows that $\approx$ is a congruence relation on $\mathbb{R}_{fin}$. This follows since $\mu(0)$ is an ideal of $\mathbb{R}_{fin}$.

Theorem 2.14. *(Existence of Standard Parts) Let $r \in \mathbb{R}_{fin}$. Then there exists a unique $s \in \mathbb{R}$ such that $r \approx s$. We call s the **standard part** of r and write $st(r) = s$.*

Corollary 2.15. $\mathbb{R}_{fin} = \mathbb{R} + \mu(0)$ *and* $\mathbb{R} \cap \mu(0) = \{0\}$.

Corollary 2.16. *Define $st : \mathbb{R}_{fin} \to \mathbb{R}$ by for all $r \in \mathbb{R}, st(r) = s$, where s is the standard part of r. Then st is a homomorphism of $\mathbb{R}_{fin}$ onto $\mathbb{R}$ such that $Ker(st) = \mu(0)$.*

Corollary 2.17. *The quotient ring $\mathbb{R}_{fin}/\mu(0)$ is isomorphic to $\mathbb{R}$, $\mu(0)$ is a maximal ideal of $\mathbb{R}_{fin}$, and is in fact the unique maximal ideal of $\mathbb{R}_{fin}$.*

Proof. Let $a \in \mathbb{R}_{fin} \backslash \mu(0)$. Then $a^{-1} \in \mathbb{R}^*$. However, $a^{-1} \notin \mathbb{R}^* \backslash \mathbb{R}_{fin}$ since $a \notin \mu(0)$. Thus $a^{-1} \in \mathbb{R}_{fin}$. That is, every element in $\mathbb{R}_{fin}$, but not in $\mu(0)$ has an inverse.

Let F_0 be the filter consisting of all cofinite subsets of $\mathbb{N}$. Let U be a free ultrafilter. Let $A \in F_0$. Then A or A^c is in U. However, A^c is not in U since A^c is finite. Thus $A \in U$. Hence $F_0 \subseteq U$.

Let (x_i) and (y_i) be sequences of real numbers. Define the relation $\simeq$ by $(x_i) \simeq (y_i)$ if and only if $\{i \in \mathbb{N} |\ x_i = y_i\} \in U$. Then $\simeq$ is an equivalence relation. Let $[(x_i)]_U$ denote the equivalence class of (x_i) with respect to $\simeq$.

Hence $[(x_i)]_U = [(y_i)]_U$ if and only if $\{i \in \mathbb{N} |\ x_i = y_i\} \in U$. □

If we replace the notation $\leq_U$ by $\leq$, we have the following.

Definition 2.18. *Let $[(x_1, x_2, ...)]_U \leq [(y_1, y_2, ...)]_U$ if and only if $\{i \in \mathbb{N} | x_i \leq y_i\} \in U$.*

Define $\geq, <, >$ on $\mathbb{R}^*$ similarly.

Definition 2.11(3) becomes $\mu(0) = \{[(x_i)]_U | [(x_i)]_U < [(r, r, , ...)]_U$ for all $r \in \mathbb{R}, r > 0\}$.

Definition 2.13 is equivalent to $[(x_i)]_U \approx [(y_i)]_U$ if and only if $[(x_i)]_U - [(y_i)]_U \in \mu(0)$, i.e., $[(x_i - y_i)]_U < [(r, r, ...)]_U$ for all $r \in \mathbb{R}, r > 0$.

Definition 2.19. [[10], p.10] *Let $A \subseteq \mathbb{R}$. The **natural extension** of A to $\mathbb{R}^*$ is the set A^* defined to be the set of all $[(r_n)]_U$ such that $\{n \in \mathbb{N} |\ r_n \in A\} \in U$.*

Definition 2.20. [[10], p.10] *Let $f : X \to \mathbb{R}$, where X is a subset of $\mathbb{R}$. The natural extension of f to $\mathbb{R}^*$ is the function $f^* : X^* \to \mathbb{R}^*$ defined as follows:*

$$f^*([(r_n)]_U) = [(f(r_n))]_U.$$

Consequently, the natural extension of $[0, 1]$ to $\mathbb{R}^*$ is $[0, 1]^* = \{x \in \mathbb{R}^* |\ 0 \leq x \leq 1\}$.

Proposition 2.21. *Let $[a, b]$ be a closed interval in $\mathbb{R}$. Then $[a, b]^* = \{x \in \mathbb{R}^* |\ a \leq x \leq b\}$.*

Proof. We have that

$$\begin{aligned} [(r_n)]_U \in [a, b]^* &\Leftrightarrow \{n \in \mathbb{N} |\ r_n \in [a, b]\} \in U \\ &\Leftrightarrow \{n \in \mathbb{N} |\ a \leq r_n \leq b\} \in U \\ &\Leftrightarrow a = [(a, a, ...)]_U \leq [(r_n)]_U \leq [(b, b, ...)]_U = b. \end{aligned}$$

Consequently, the natural extension of $[0, 1]$ to $\mathbb{R}^*$ is $[0, 1]^* = \{x \in \mathbb{R}^* | 0 \leq x \leq 1\}$. □

It is shown in [8] that $[0, 1]^* \neq]^-0, 1^+[= [0, 1] + \mu(0)$.

Let $a = [(a, a, a, ...,)]_U$ and $m = [(1, 1/2, 1/3, ...)]_U$. Then $a + m > a$. Define $A(a) = a$ for all $a \in \mathbb{R}$. Let A^* denote the natural extension of A to $\mathbb{R}^*$. Then $A^*([(x_1, x_2, ...)]_U) = [(A(x_1), A(x_2), ...]_U = [(x_1, x_2, ...)]_U$. Thus $A^*(a + m) > A(a)$.

3 Fuzzy Numbers

We review some basics of fuzzy numbers.

Definition 3.1. [[6], p.97] *Let A be a fuzzy subset of $\mathbb{R}$. Then A is a fuzzy number if the following conditions hold.*
(1) *There exists $x \in \mathbb{R}$ such that $A(x) = 1$.*
(3) *A^{α} is a closed bounded interval for all $\alpha \in (0, 1]$.*
(3) *The support of A is bounded.*

Theorem 3.2. [[6]. p.98] *Let A be a fuzzy subset of $\mathbb{R}$. Then A is a fuzzy number if and only if there is a closed interval $[c, d]$ and functions $l : (-\infty, c) \to [0, 1], r : (d, \infty) \to [0, 1]$, and $a, b \in \mathbb{R}, a \leq c \leq d \leq b$ such that*

$$A(x) = \begin{cases} 1 & ; \text{ if } x \in [c, d] \\ l(x) & ; \text{ if } x \in (-\infty, c) \\ r(x) & ; \text{ if } x \in (d, \infty), \end{cases}$$

where l is monotonic increasing, continuous from the right and such that $l(x) = 0$ for $x \in (-\infty, a); r$ is monotonic decreasing, continuous from the left and such that $r(x) = 0$ for $x \in (b, \infty)$.

Theorem 3.3. [[6], p.41] *Let A be a fuzzy subset of $\mathbb{R}$. Then $A = \cup_{\alpha \in [0,1]\alpha} A$, where ${}_{\alpha}A(x) = \alpha A^{a}(x)$ and $(\cup_{\alpha \in [0,1]\alpha} A)(x) = \vee\{_{\alpha}(A)(x) | \ x \in [0, 1]\}$ for all $x \in \mathbb{R}$.*

We next proceed to the second method for developing fuzzy arithmetic, which is the extension principle. Employing this principle, standard arithmetic operations on real numbers are extended to fuzzy numbers.

Let $*$ denote any of the four basic arithmetic operations and let A, B denote fuzzy numbers. Then define $A * B$ by for all $z \in \mathbb{R}$.

$$(A * B)(z) = \vee\{A(x) \wedge B(y) | \ z = x * y, x, y \in \mathbb{R}\}.$$

Theorem 3.4. [6] *Let $* \in \{+, -, \bullet, /\}$ and let A, B denote continuous fuzzy numbers. Then the fuzzy subset $A * B$ is a continuous fuzzy number.*

Let A be a fuzzy subset of $\mathbb{R}$. Let A^* be the natural extension of A to $\mathbb{R}^*$. Let $B = 1 - A$, i.e., for all $x \in \mathbb{R}, B(x) = 1 - A(x)$. Let B^* be the natural extension of B to $\mathbb{R}^*$. Let $[(x_1, x_2, ..., x_n, ...)]_U$. Then

$$\begin{aligned} B^*([(x_1, x_2, ..., x_n, ...)]_U) &= [(B(x_1), B(x_2), ..., B(x_n), ...]_U \\ &= [(1 - A(x_1), 1 - A(x_2), ..., 1 - A(x_n), ...]_U \\ &= [(1, 1, ..., 1, ...)]_U - [A(x_1), A(x_2), ..., A(x_n), ...]_U \\ &= 1 - A^*([(x_1, x_2, ..., x_n, ...)]_U. \end{aligned}$$

In the following, let A and B be continuous fuzzy numbers. Let A^*, B^*, and $(A + B)^*$ denote the natural extensions of A, B, and $A + B$ to $\mathbb{R}^*$.

Definition 3.5. Define $A^* + B^*$ as follows:

$$\begin{aligned} (A^* + B^*)(a + m) &= (A + B)(a) + m, \text{ if } a \in \mathbb{R}, m \in \mu(0), \\ (A^* + B^*)(x) &= 0 \text{ if } x \in \mathbb{R} \backslash \mathbb{R}_{fin}. \end{aligned}$$

Let $a \in \mathbb{R}$. Then $(A^* + B^*)(a) = (A + B)(a) = (A + B)^*(a)$. Now $(A + B)^*(a + m) \approx (A + B)(a) \approx (A^* + B^*)(a + m)$ since $(A^* + B^*)(a + m) = (A^* + B^*)(a) + m$.

Definition 3.6. A^* is a nonstandard fuzzy number if the following properties hold:

(1) There exist $x \in \mathbb{R}^*$ such that $A^*(x) \approx 1$.

(2) $\forall \alpha \in [0,1]^*$, there exists $c_\alpha, d_\alpha \in [0,1]^*$ such that $c_\alpha \leq d_\alpha$ and $A^{*\alpha} = \{x \in \mathbb{R}^* | \ c_\alpha \lessapprox x \lessapprox d_\alpha\}$.

(3) There exists $c, d \in \mathbb{R}_{fin}$ such that $c \leq d$ and $\text{NSupp}(A^*) \subseteq \{x \in R^* | \ c \lessapprox x \lessapprox d\}$.

Theorem 3.7. *Suppose A is continuous. Then A is a fuzzy number if and only if A^* is a nonstandard fuzzy number.*

Proof. Suppose A is a fuzzy number.

(1) Then there exists $x \in \mathbb{R}$ such that $A(x) = 1$. Hence $A^*(x) = 1$.

(2) Let $\alpha \in [0,1]^*$ and $y \in \mathbb{R}^*$. Suppose $A^*(y) \gtrapprox \alpha$. Since A^* is microcontinuous, $A^*(y) \approx A(st(y))$ and so $A(st(y)) \gtrapprox \alpha$. Thus $A(st(y)) \geq st(\alpha)$. Hence there exist $c_{st(\alpha)}, d_{st(\alpha)} \in [0,1]$ such that $c_{st(\alpha)} \leq d_{st(\alpha)}$ and $c_{st(\alpha)} \leq st(y) \leq d_{st(\alpha)}$. Thus $c_{st(\alpha)} \lessapprox y \lessapprox d_{st(\alpha)}$.

(3) Suppose $A^*(y) \notin \mu(0)$, where $y \in \mathbb{R}^*$. Now $A^*(y) \approx A(st(y))$. Thus $A(st(y)) > 0$. Hence there exists $c, d \in R$ such that $c \leq st(y) \leq d$. Thus $c \lessapprox y \lessapprox d$.

Conversely, suppose A^* is a nonstandard fuzzy number.

(1) Then there exists $y \in \mathbb{R}^*$ such that $A^*(y) \approx 1$. Hence $A(st(y)) = A^*(st(y)) = 1$.

(2) Let $\alpha \in [0,1]$ and $x \in \mathbb{R}$. Suppose $A(x) \geq \alpha$. Then $A^*(x) = A(x) \geq \alpha$. Thus $A^*(x) \gtrapprox \alpha$. Hence there exists $c_\alpha, d_\alpha \in [0,1]$ with $c \leq d$ such that $c_\alpha \lessapprox x \lessapprox d_\alpha$. Since $x \in R, c_\alpha \leq x \leq d_\alpha$.

(3) Suppose $A(x) > 0$, where $x \in \mathbb{R}$. Then $A^*(x) > 0$ and so $x \notin \mu(0)$. Thus there exists $c, d \in [0,1]^*$ such that $c \lessapprox x \lessapprox d$. Since $x \in \mathbb{R}, st(c) \leq x \leq st(d)$. Thus $\text{Supp}(A) \subseteq [st(c), st(d)]$. □

Proposition 3.8. *Let C and D be nonstandard fuzzy subsets of $\mathbb{R}^*$. If C is a nonstandard fuzzy number and $C(y) \approx D(y)$ for all $y \in \mathbb{R}^*$, then D is a nonstandard fuzzy number.*

Proof. There exists $y \in \mathbb{R}$ such that $A(y) \approx 1$. Thus $B(y) \approx 1$.

Let $y \in R^*$. Let $\alpha \in [0,1]^*$. Then $B(y) \gtrapprox \alpha$ if and only if $A(y) \gtrapprox \alpha$. Thus B^α is bounded.

Now $A(y) \notin \mu(0)$ if and only if $B(y) \notin \mu(0)$ since $A(y) \approx B(y)$. Hence $\text{NSupp}(B) = \text{NSupp}(A)$. Thus $\text{NSupp}(B)$ is bounded. □

Corollary 3.9. *Let A and B be fuzzy subsets of $\mathbb{R}$. Then $A + B$ is a fuzzy number if and only if $A^* + B^*$ is a nonstandard fuzzy number.*

Proof. $A + B$ is a fuzzy number if and only if $(A + B)^*$ is a fuzzy number. Now $(A + B)^*(y) \approx (A^* + B^*)(y)$ for all $y \in \mathbb{R}^*$. □

Proposition 3.10. *Let $a \in \mathbb{R}$ and $m \in \mu(0)$. Let $m = m' + m''$, where $m', m'' \in \mu(0)$. Then $(A^* + B^*)(a + m) = \vee\{(A^* \circ st)(b + m') \wedge (B^* \circ st(c + m'')|a = b + c\}$.*

Proof. For all such m', m'' (held fixed),

$$\begin{aligned} \vee\{(A^* \circ st)(b + m') \wedge (B^* \circ st(c + m'')|a &= b + c\} \\ &= \vee\{A^*(b) \wedge B^*(c)|a = b + c\} \\ &= \vee\{A(b) \wedge B(c)|a = b + c\} \\ &= (A + B)(a). \end{aligned}$$

Let A be a fuzzy subset of $\mathbb{R}$. Assume there exist real numbers a, b with $a \leq b$ such that $A(y) = 0$ for all $y \notin [a, b]$. □

Proposition 3.11. *If A^* is the natural extension of A to $\mathbb{R}^*$, then $A^*(y) = 0$ for all $y \in \mathbb{R}^* \backslash [a, b]^*$.*

Proof. Suppose $[(y_n)]_U \in \mathbb{R}^* \backslash [a, b]^*$. Then $\{n \in \mathbb{N} | \ a \leq y_n \leq b\} \notin U$ else $[(y_n)]_U \in [a, b]^*$. Hence $\{n \in \mathbb{N} | y_n \notin [a, b]\} \in U$ since either $\{n \in \mathbb{N} | \ a \leq y_n \leq b\} \in U$ or $\{n \in \mathbb{N} | \ a \leq y_n \leq b\}^c \in U$, but not both. Thus $\{n \in \mathbb{N} | \ A(y_n) = 0\} \in U$. Hence $A^*([(y_n)]_U) = [A(y_n)]_U = [(0, 0, ...)]_U$. It follows that A^* maps every element of $\mathbb{R}^* \backslash \mathbb{R}_{fin}$ to 0. □

4 Continuity and Differentiation

Definition 4.1. (See [10]) *(Nonstandard Definition of Continuity) Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous at a if and only if $\forall \delta \approx 0, f^*(a+\delta) - f(a) \approx 0$, where f^* is the natural extension of f to $\mathbb{R}^*$.*

Let $A : \mathbb{R} \to \mathbb{R}$ (or $[0,1]$) and let A^* be the natural extension of A to $\mathbb{R}^* \to \mathbb{R}^*$(or $[0,1]^*$). Let $a \in \mathbb{R}$. Then in $\mathbb{R}^*, a = [(a, a, ..., a, ..)]_U$ and $A^*(a) = [(A(a), A(a), ..., A(a), ...)]_U = A(a)$. That is, $A^*|_{\mathbf{R}} = A$.

Definition 4.2. [[10], p.11] *Let $X \subseteq \mathbb{R}^*$. Then a function $f : X \to \mathbb{R}^*$ is said to be **microcontinuous** at $x_0 \in X$ if $x \approx x_0$ implies $f(x) \approx f(x_0)$ for all $x \in X$.*

Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. If f is continuous at a, then f^* is microcontinuous at $a + m$ for all $m \in \mu(0)$. Note that if f is continuous at a, then $f^*(a+m) \approx f(a)$ for all $m \in \mu(0)$ and so $f(a+m) \approx f(a+m')$ for all $m, m' \in \mu(0)$.

Theorem 4.3. [[10], p.11] *A function $f : X \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if and only if f^* is microcontinuous at c.*

Theorem 4.4. [[12], p.21] *The nonstandard definition of continuity given above is equivalent to the classic definition of continuity: f is continuous at a if and only if $lim_{x\to a} f(x) = f(a)$.*

Let A be a continuous fuzzy number. Then $\forall a \in \mathbb{R}$ and $\forall m \in \mu(0), A^*(a+m) \approx A(a)$. Since $A^*(a) = A(a)$ and $A^* : \mathbb{R}\backslash\mathbb{R}_{fin} \to \{0\}$,we have a "picture" of A^*.

Let A^* be the natural extension of A to $\mathbb{R}^*$. We could define A^* to be a nonstandard fuzzy number if A is a fuzzy number. Note $A^*|_{\mathbb{R}} = A$.

We need to review some results dealing with the differentiation and integration of fuzzy functions in order to make their connection to the work of [H] concerning relativity.

We denote the space of all fuzzy-valued functions on $[a, b]$ by $\mathcal{F}[a, b]$, or simply $\mathcal{F}$.

Definition 4.5. *Define the fuzzy subset $\widetilde{F}$ of $\mathbb{R}$ by $\forall y \in \mathbb{R}$, $\widetilde{F}(y) = \int_a^b \widetilde{f}(x)(y)dx = \vee\{\wedge\{\widetilde{f}(x)(g(x))|\ a \leq x \leq b\}|g \in \mathcal{I}(a, b), y = \int_a^b g(t)dt\}$, where $\mathcal{I}(a, b)$denotes the set of all integrable functions whose domain is $[a, b]$.*

Definition 4.6. *Let $\widetilde{f}$ be a fuzzy-valued function with level sets $[f^-(a, x), f^+(\alpha, x)]$ such that $f^-(a, \_)$ and $f^+(\alpha, \_)$ are integrable functions on the interval $[a, b]$. Let $\widetilde{I} \in \widetilde{P}(\mathbb{R})$ be defined by $\forall y \in \mathbb{R}$,*

$$\widetilde{I}(y) = \begin{cases} \vee\{\alpha \in (0,1)|\int_a^b f^-(\alpha, x)dx \leq y \leq \int_a^b \widetilde{f}(\alpha, x)dx\}, & \\ 0 & ;\ otherwise \end{cases}$$

Theorem 4.7. [[9], p.40] *Let $\widetilde{I}$ be defined as in Definition 4.6 Then $\widetilde{I} = \widetilde{F}$.*

Definition 4.8. [[10], p.12] *Let $f : A \to \mathbb{R}$. We say that f is differentiable at $x_0 \in A$ if there exists $L \in \mathbb{R}^*$ such that for every nonzero infinitesimal ε, we have*

$$\frac{f^*(x_0+\varepsilon) - f^*(x_0)}{\varepsilon} \approx L.$$

If so, we define the derivative of f at x_0 to be the standard part of L, $f'(x_0) = st(L)$.

Theorem 4.9. [9] *Let $\widetilde{f}$ be a function of $\mathbb{D}$ into $\widetilde{P}(\mathbb{R})$ such that $\widetilde{f}(x)$ is a fuzzy number for all x in $\mathbb{D}$. Suppose that $\forall x \in \mathbb{D}, \forall \alpha \in [0,1], \widetilde{f}(x)_\alpha$ is a closed bounded interval. Then there exist unique functions f^-, f^+ of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that*

(1) $\forall x \in \mathbb{D},\ f^-(\_, x)\ (f^+(\_, x))$ *is a nondecreasing (nonincreasing) function of* α.

(2) $\forall(\alpha, x) \in [0,1] \times \mathbb{D},\ f^-(\alpha, x) \leq f^+(\alpha, x)$.

(3) $\forall(\alpha, x) \in [0,1] \times \mathbb{D},\ f(x)_\alpha = [f^-(\alpha, x), f^+(\alpha, x)]$.

(4) $\forall x \in \mathbb{D},\ f^-(1, x) = f^+(1, x)$.

Theorem 4.10. [9] *Let g and h be functions of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that $\forall x \in \mathbb{D},\ g(\_, x)\ (h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall(\alpha, x) \in [0,1] \times \mathbb{D}, g(\alpha, x) \leq h(\alpha, x)$. Let $\widetilde{f}$ be the function $\mathbb{D} \times \mathbb{R}$ into $[0,1]$ defined as follows: $\forall(x, y) \in \mathbb{D} \times \mathbb{R}$,*

$$\widetilde{f}(x, y) = \begin{cases} \vee\{\beta \in [0,1] | y \in g(\beta, x), h(\beta, x)]\} & ; \text{ if } y \in [g(0, x), h(0, x)]\} \\ 0 & ; \text{ otherwise.} \end{cases}$$

If $\forall x \in \mathbb{D},\ g(\_, x)$ and $h(\_, x)$ are continuous from the left, then $\widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)] \forall \alpha \in [0,1]$.

Theorem 4.11. [9] *Let g and h be functions of $[0,1] \times \mathbb{D}$ into $\mathbb{R}$ such that $\forall x \in \mathbb{D}, g(\_, x)\ (h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall(\alpha, x) \in [0,1] \times \mathbb{D}, g(\alpha, x) \leq h(\alpha, x)$. Suppose there exists a function $\widetilde{f}$ of $[0,1] \times \mathbb{D}$ into $[0,1]$ such that $\forall \alpha \in [0,1], \widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)]$. Then $\forall x \in \mathbb{D}, g(\_, x)$ and $h(\_, x)$ are continuous functions of α from the left. Furthermore,*

$$\widetilde{f}(x)(y) = \begin{cases} \vee\{\beta \in [0,1] | y \in g(\beta, x), h(\beta, x)]\} & ; \text{ if } y \in [g(0, x), h(0, x)]\} \\ 0 & ; \text{ otherwise.} \end{cases}$$

Let $x \in \mathbb{D}$ and hold x fixed. Let $f(x)^*$ be the natural extension of $f(x)$ to $\mathbb{R}^*$. ($f(x)$ is a fuzzy number so $f(x) : \mathbb{R} \to [0,1]$.We assume $f(x)$ is a continuous fuzzy number throughout.) Thus $f(x)^* : \mathbb{R}^* \to [0,1]^*$ since $f((x)^*([(r_1, r_2, ...)\}]_U = [(f(x)(r_1), f(x)(r_2), ...)]_U$ and $f(x)(r_i) \in [0,1], i = 1, 2, ...$ Note that since $f(x)(r_i) \geq 0, [(f(x)(r_1), f(x)(r_2), ...)]_U \geq [(0, 0, ...)]_U = 0$.

Write $f^-(x)$ for $f^-(\_, x)$ and $f^+(x)$ for $f^+(\_, x)$. Then $f^-(x)$ and $f^+(x)$ map $[0,1]$ into $\mathbb{R}$. Let $f^-(x)^*$ and $f^+(x)^*$ be the natural extensions of $f^-(x)$ and $f^+(x)$ to mappings of $[0,1]^*$ to $\mathbb{R}^*$, respectively. Let $\alpha \in [0,1]$. Then $f(x)_\alpha = [f^-(\alpha, x), f^+(\alpha, x)]$. Since $f(x)$ is a fuzzy number, there exists $a, b \in R$ such that $a \leq b$ and $f(x)(y) = 0$ if $y \notin (a, b)$. (a and b are dependent on x.) Now $a \leq f^-(x)(\alpha) \leq f^+(x)(\alpha) \leq b$ for $\alpha > 0$.

Proposition 4.12. *Let $a, b, c \in \mathbb{R}^*$. Then $a \lessapprox c \lessapprox b$ if and only if $st(a) \leq st(c) \leq st(b)$.*

Proof. We have that $a \approx c \Leftrightarrow st(a) = st(c)$. Also, $a < c \Leftrightarrow st(a) < st(c)$ or ($st(a) = st(c)$ and $nst(a) < nst(c)$). Similar arguments hold for c and b. □

Proposition 4.13. *Let $x \in \mathbb{D}$. Let $f(x)$ be a fuzzy number and $f(x)^*$ its natural extension to $\mathbb{R}^*$. Then for all $\alpha \in [0,1], f^-(x)^*(\alpha^*) \lessapprox y \lessapprox f^+(x)^*(\alpha^*)$ if and only if $f^-(\alpha, x) \leq st(y) \leq f^+(\alpha, x)$.*

Proof. Since $f^-(x)$ and $f^+(x)$ are continuous on $[0,1]$, $f^-(x)^*$ and $f^+(x)^*$ are micro-continuous on $[0,1]^*$. Now $f^-(x)^*(\alpha^*) \approx f^-(x)(\alpha)$ and $f^+(x)^*(\alpha^*) \approx f^+(x)(\alpha)$. Also, $st(f^-(x)^*(\alpha^*)) = f^-(x)(\alpha)$ and $st(f^+(x)^*(\alpha^*)) = f^+(x)(\alpha)$.

For $x \in \mathbb{D}$ fixed, $f(x)$ gives the shape of the fuzzy number at x. □

5 Relativity

We next make a connection to the theory of relativity. Let $q : \mathbb{D} \times \mathbb{R} \to [0,1]$. For example, let $\mathbb{D}$ be a time interval $[a, b]$ and let r be distance, [3]. Then $q(t, r)$ is the intensity with which an object travels r units in time t. We next consider $q^* : \mathbb{D}^* \times \mathbb{R}^* \to [0,1]^*$. The following Proposition shows that q^* can be considered to be the natural extension of q to $(\mathbb{D} \times \mathbb{R})^*$.

Proposition 5.1. *Define f from $\mathbb{D}^* \times \mathbb{R}^*$ into $(\mathbb{D} \times \mathbb{R})^*$ by for all* $([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U) \in \mathbb{D}^* \times \mathbb{R}^*$,

$$f(([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U)) = [((t_1, r_1), (t_2, r_2), ...)]_U.$$

Then f is a one-to-one function of $\mathbb{D}^* \times \mathbb{R}^*$ onto $(\mathbb{D} \times \mathbb{R})^*$.

Proof. We have

$$\begin{aligned} ([(t_1, t_2, ...)]_U, [(r_1, r_2, ...)]_U) &= ([(t'_1, t'_2, ...)]_U, [(r'_1, r'_2, ...)]_U) \Leftrightarrow \\ [(t_1, t_2, ...)]_U &= [(t'_1, t'_2, ...)]_U \text{ and } [(r_1, r_2, ...)]_U = [(r'_1, r'_2, ...)]_U \Leftrightarrow \\ (t_1, t_2, ...) &\approx (t'_1, t'_2, ...) \text{ and } (r_1, r_2, ...) \approx (r'_1, r'_2, ...) \Leftrightarrow \\ \{i | t_i &= t'_i\} \in U \text{ and } \{j | r_j = r'_j\} \in U. \end{aligned}$$

Now $\{k|\ t_k = t'_k \text{ and } r_k = r'_k\} = \{i|\ t_i = t'_i\} \cap \{j|\ r_j = r'_j\} \in U$.
Also,

$$\begin{aligned} [((t_1, r_1), (t_2, r_2), ...)]_U &= [((t'_1, r'_1), (t'_2, r'_2), ...)]_U \Leftrightarrow \\ ((t_1, r_1), (t_2, r_2), ...) &\approx ((t'_1, r'_1), (t'_2, r'_2), ...) \Leftrightarrow \\ \{k | (t_k, r_k) &= (t'_k, r'_k)\} \in U \Leftrightarrow \\ \{k | t_k &= t'_k \text{ and } r_k = r'_k\} \in U. \end{aligned}$$

Thus f is a one-to-one function of $\mathbb{D}^* \times \mathbb{R}^*$ onto $(\mathbb{D} \times \mathbb{R})^*$.

Let $l : \mathbb{D} \times \mathbb{R} \to [0,1]$ and $l^* : \mathbb{D}^* \times \mathbb{R}^* \to [0,1]^*$, where $l(t, r)$ is the intensity of the velocity $r \in \mathbb{R}$ of a particle at time $t \in \mathbb{D}$. □

The following is from [3].

In [[3], p.10], it is stated that Newton's approach created a schism in the philosophy of mathematical modeling. One group of scientists believed that there exists actual real world entities that can be characterized in terms of infinitesimal measures of time, mass, volume, and charge. Another group assumed that such terms are auxiliary in character and do not correspond to objective reality. The mathematical model called *the nonstandard physical world* {i.e. NSP-world) uses the corrected theory of the infinitesimally small and infinitely large, with other techniques, along with a new physical language theory of correspondence.

In [[3], p.27], it is stated that experiments show that for small time intervals $[a, b]$ the Galilean theory of average velocities suffices to give accurate information relative to the compositions of such velocities. Let there be an internal function $q : [a, b]^* \to \mathbb{R}^*$ where q represents the NSP-world distance function. Also, let nonnegative and internal $l : [a, b]^* \to \mathbb{R}^*$ be a function that yields the NSP-world velocity of the electromagnetic propagation at a time $t \in [a, b]^*$. As usual $\mu(t)$ denotes the monad of standard time $\mu(t)$, where "t" is an absolute NSP-world "time" parameter.

The general and correct methods of infinitesimal modeling state that, within the internal portion of the NSP-worlds, two measures m_1 and m_2 are *indistinguishable* for dt (i.e., infinitely close of order one) (notation $m_1 \sim m_2$) if and only if $0 \neq dt \in \mu(0)$,

$$\frac{m_1}{dt} - \frac{m_2}{dt} \in \mu(0).$$

Intuitively, indistinguishable in this sense means that, although within the NSP-world the two measures are only equivalent and not necessarily equal, the *first level* (or first-order) effects these measures represent over dt are indistinguishable within the N-world (i.e., they appear to be equal.)

In [3], some continuity conditions are placed on q and l. It is argued that for each $t \in [a, b]$ and $t' \in \mu(t) \cap [a, b]^*$,

$$\frac{q(t')}{t'} - \frac{q(t)}{t} \in \mu(0) \text{ and } l(t') - l(t) \in \mu(0).$$

The above expressions give relations between nonstandard time $t' \in \mu(t)$ and the standard time t. Recall that if $x, y \in R^*$, then $x \approx y$ if and only if $x - y \in \mu(0)$. It thus follows that for each $dt \in \mu(0)$ such that $t + dt \in \mu(t) \cap [a, b]^*$,

$$\begin{aligned} \frac{q(t+dt)}{t+dt} &\approx \frac{q(t)}{t}, \\ l(t+dt) + \frac{q(t+dt)}{t+dt} &\approx l(t) + \frac{q(t)}{t}. \quad (4.1) \end{aligned}$$

Hence

$$(l(t+dt) + \frac{q(t+dt)}{t+dt})dt \sim (l(t) + \frac{q(t)}{t})dt$$

Thus, it follows [3] that

$$q(t+dt) - q(t) \sim (l(t+dt) + \frac{q(t+dt)}{t+dt})dt$$

and

$$q(t+dt) - q(t) \sim (l(t) + \frac{q(t)}{t})dt. \quad (4.2)$$

It is stated in [3] that Expression (4.2) is the basic result that will lead to conclusions relative to the Special Theory of relativity. In order to find out exactly what standard functions will satisfy (4.2), let arbitrary $t_1 \in [a, b]$ be the standard time at which electromagnetic propagation from position F_1. Next, the definition of $\sim$, yields

$$\frac{s^*(t+dt) - s(t)}{dt} \approx l(t) + \frac{s(t)}{t}. \quad (4.3)$$

Note that l is microcontinuous on $[a, b]^*$. For each $t \in [a, b]$, the value of $l(t)$ is limited. Hence let $st(l(t)) = v(t) \in \mathbb{R}$. From Theorem 1.1 in [3] or 7.6 in [11], v is continuous on $[a, b]$. Now (4.3) may be rewritten as

$$(\frac{d(s(t)/t)}{dt})^* = \frac{v^*(t)}{t}, \quad (4.4)$$

where all functions in (4.4) are $*$-continuous on $[a, b]^*$. Consequently, we may apply the *-integral to both sides of (4.4). Now (4.4) implies that for $t \in [a, b]$

$$\frac{s(t)}{t} =^* \int_{t_1}^{t} \frac{v^*(x)}{x} dx$$

for $t_1 \in [a, b]$, $s(t_1)$ has been initialized to be zero.

We next provide a possible extension of these results to nonstandard fuzzy numbers. We define a nonstandard fuzzy number to be the natural extension of a fuzzy number to $\mathbb{R}^*$ into $[0, 1]^*$. Consider the function l above. Let l be a function of $[a, b]^*$ into the set of nonstandard fuzzy numbers. Then for all $t' \in [a, b]^*$ and $r' \in \mathbb{R}_{fin}, l(t', r') \in [0, 1]^*$ is the intensity with the velocity is r' at time t'. Define (it is argued that) $l(t', r') \wedge l(t, r)$ to be the intensity with which $r' - r \in \mu(0)$. Note that if $l(t', r') = 1$ and $l(t, r) = 1$, then the intensity with which $r' - r \in \mu(0)$ equals 1. Similar, interpretations can be given to the other equations given above. For example, let $v(t) = q(t)/t$ and consider v be a function of $[a,b]^*$ into the set of nonstandard fuzzy numbers.

We next consider the sum in (4.1). Let $v : [a,b] \times \mathbb{R} \to [0,1]$ be such that for all $t \in [a,b], v(t)$ is a fuzzy number. Let v^* be the natural extension of v. Then $v^* : [a,b]^* \times \mathbb{R}^* \to [0,1]^*$. Consider two such v_1 and v_2. Define $v_1^* + v_2^* : [a,b]^* \times \mathbb{R}_{fin} \to [0,1]^*$ as follows: For all $t' \in [a,b]^*$ and $r' \in \mathbb{R}_{fin}$,

$$\begin{aligned}(v_1^* + v_2^*)(t_1')(r') &= \vee\{st(v_1(t', r_1') \wedge v_2(t', r_2'))|r' = r_1' + r_2'; r_1', r_2' \in \mathbb{R}_{fin}\} \\ &\quad + nst(r_1') \vee nst(r_2'),\end{aligned}$$

where $nst(r_i')$ denotes the nonstandard part of r_i', $i = 1, 2$.

Consider the definition of $m_1 \sim m_2$. Let $m_1 = [(1, \frac{1}{4}, \frac{1}{9}, ...)]_U$ and $m_2 = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$. Then $m_1 \approx m_2$. Let $t = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$. Then $t \in \mu(0)$ and $\frac{m_1}{t} = [(1, \frac{1}{2}, \frac{1}{3}, ...)]_U$ and $\frac{m_2}{t} = [(1, 1, 1, ...)]_U$. Thus it's not the case that $\frac{m_1}{t} \approx \frac{m_2}{t}$. Hence it is not the case that $m_1 \sim m_2$.

Recall that $s(t)$ is the distance traveled at time t so $s : [a,b] \times \mathbb{R} \to [0,1]$ gives the intensity that the distance traveled at time t is $r, s(t,r) \in [0,1]$.

Also $v(t)$ is the velocity of a particle at time t so $v : [a,b] \times \mathbb{R} \to [0,1]$ gives the intensity that a particle is traveling r at time $t., v(t,r) \in [0,1]$.

Thus $(s(t)/t)(r)$ is the intensity that the velocity at time t is r. Also $l(t)(r)$, is the intensity that the velocity is r at time t. Hence $l(t) + s(t)/t$ is the sum of two fuzzy numbers which we define as follows: Given $t \in [a,b]$,

$$(l(t) + \frac{s(t)}{t})(r) = \vee\{l(t)(r_1) \wedge \frac{s(t)}{t}(r_2)|r = r_1 + r_2, r_1, r_2 \in \mathbb{R}\}$$

for all $r \in \mathbb{R}$.

Let f be integrable on the interval $[a,b]$. For all $t \in [a,b]$, define the function F of $[a,b] \to \mathbb{R}$ by for all $t \in [a,b], F(t) = \int_a^t f(x)dx$. Let f^* and F^* be the natural extensions of f and F to $\mathbb{R}$, respectively. Define $\int_a^t f^*(x)dx$ to be $F^*(t)$.

6 Conclusion

In this paper, we laid a foundation for a new research area in fuzzy mathematics, namely the use of nonstandard analysis. This can be accomplished by extending the field of real numbers to the field of hyperreals $\mathbb{R}^*$. Then the closed interval $[0,1]$ can be replaced by its natural extension to $[0,1]^*$. We point out that many theoretical results in $\mathbb{R}^*$ will automatically hold by the transfer principle. The use of $]^-0, 1^+[$ instead of $[0,1]^*$ would be more general, but would be a little more difficult since $]^-0, 1^+[$ is not the natural extension of $[0,1]$. Along these lines, scholars should be aware of the work of Klement an Mesiar, [5], where it is shown that many results of certain variations of fuzzy sets automatically hold from results of ordinary fuzzy sets.

Conflict of Interest: The authors declare no conflict of interest.

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A Modified Novel Method for Solving the Uncertainty Linear Programming Problems Based on Triangular Neutrosophic Number

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Abstract. Generally, linear programming (LP) problem is the most extensively utilized technique for solving and optimizing real-world problems due to its simplicity and efficiency. However, to deal with the inaccurate data, the neutrosophic set theory comes into play, which creates a simulation of the human decision-making process by considering all parts of the choice (i.e., agree, not sure, and disagree). Keeping the benefits in mind, we proposed the neutrosophic LP models based on triangular neutrosophic numbers (TNN) and provided a method for solving them. Fuzzy LP problem can be converted into crips LP problem based on the defined ranking function. The provided technique has been demonstrated with numerical examples given by Abdelfattah. Finally, we found that, when compared to previous approaches, the suggested method is simpler, more efficient, and capable of solving all types of fuzzy LP models.

AMS Subject Classification 2020: 90C70; 90C05
Keywords and Phrases: Linear Programming Problem, Triangular Neutrosophic Number, Ranking Function, Nutrosophic Linear Programming Problem.

1 Introduction

One of the most extensively used optimizations approaches in real-world applications is linear programming and it is a type of mathematical programming that has a linear objective function and a set of linear equality and inequality constraints. However, in real world issues, data precision is largely misleading, which has an impact on the best solution to LP problems. With erroneous and ambiguous data, probability distributions failed to transact. Zadeh [30] in 1965 proposed fuzzy sets to deal with ambiguous and imprecise data. Zimmermann [31] in 1978 offered the first definition and solution of the fuzzy LP problem. Zimmermann [32] in 1987, divided the fuzzy LP problems into two groups: symmetric and non-symmetric problems. In symmetric fuzzy LP problems, the weights of objectives and constraints are equal, whereas in non-symmetric fuzzy LP problems, the weights of objectives and constraints are not equal. Leung [20] in 2013 divided fuzzy LP problems into different categories, these are

1. problems with crisp objective and fuzzy constraint.
2. problems with crisp constraint and fuzzy objective.

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3. problems with fuzzy objectives and fuzzy constraints.

4. challenges with robust programming.

Kumar et al. [19] provide a fuzzy LP problem with equality and inequality constraints. Several authors proposed several methods for solving fuzzy LP with inequality constraints, as well as first converting fuzzy LP problems to their equivalent crisp model and then getting the best fuzzy solution to the original scenario. A large number of authors have explored the various aspects of fuzzy LP problems and provided various solutions. Lotfi et al. [21] introduced the entire fuzzy LP difficulties. Some researchers have proposed a ranking function for converting fuzzy LP problems into crisp LP analogues, which can then be solved using standard approaches. Ebrahimnejad and Tavana [12] proposed a novel technique for tackling fuzzy LP problems based on symmetric trapezoidal fuzzy numbers.

However, because it solely examines the truthiness function, the fuzzy set does not effectively represent unclear and imprecise information. Then, by considering both the truth and falsity functions, Atanassov [6] in 1986 created the notion of the intuitionistic fuzzy set to handle unclear and imprecise information. Bharati and Singh [8] proposed completely intuitionistic fuzzy LP problems that are based on the sign distance between triangular intuitionistic fuzzy integers. Gani and Ponnalagu [15] proposed a method of solving a fuzzy LP problems based on the intuitionistic triangular fuzzy numbers. Sidhu and Kumar [25] employed a ranking algorithm to solve intuitionistic fuzzy LP problems. To defuzzify triangular intuitionistic fuzzy numbers, Nagoorgani and Ponnalagu [23] devised an accuracy function.

However, the intuitionistic fuzzy set does not accurately represent the human decision-making process. Because making the best decision is basically a matter of organising and explaining facts, Smarandache [27] in 1999 proposed the notion of neutrosophic set theory to deal with ambiguous, imprecise, and inconsistent data. Neutrosophic set theory replicates human decision-making by taking into account all parts of the process. The phrase "neutrosophic set" refers to popularisation of fuzzy and intuitionistic fuzzy sets, in which each element has a membership function for truth, indeterminacy, and falsehood. As a result, the neutrosophic set may swiftly and effectively ingest incorrect, unclear, and maladjusted information [11]. In uncertainty modelling, neutrosophic sets play a significant role. The advancement of uncertainty theory is essential in the formulation of real-life scientific mathematical models and its extensions have been applied in a wide variety of fields [28] including computer science [14], engineering [17], mathematics [9, 4], health care [5, 22] etc. In addition, they have been applied to much multi-criteria decision making problems [16, 24, 2]. A neutrosophic set's main advantage is that it enhances decision-making by accounting for degrees of truth, falsehood, and indeterminacy. The degree of indeterminacy is frequently seen as a free component with a significant commitment in decision-making. Because real-world situations are unpredictable, triangular neutrosophic linear programming is preferred to classical linear programming. The neutrosophic LP problems are more beneficial than crisp LP problems since the decision maker is not needed to establish a rigorous formulation in his or her formulation of the problem. It is recommended that neutrosophic LP concerns be employed to minimise unrealistic modelling. Abdel-Basset et al. [1] proposed a novel method for solving the fully neutrosophic linear programming problems based on Tripezoidal neutrosophic numbers which was modified by Singh et al. [26] to solve fully neutrosophic linear programming problems. Edalatpanah [13] proposed a direct model to solve triangular neutrosophic linear programming. Wang et al. [29] used a triangular neutrosophic numbers to solve multi objective linear programming problems. Khatter [18] proposed a model to convert each triangular neutrosophic number in a linear programming problem to a weighted value using a possibilistic mean to determine the crisp linear programming problem. Das and Chakraborty [10] employed a pentagonal neutrosophic number and developed a method for translating it to the corresponding crisp LP problem using a ranking function. Bera and Mahapatra [7] used a single valued trapezoidal neutrosophic number to linear programming problems in the simplex method. Abdelfattah [3] proposed a parametric approach to solve neutrosophic linear programming models. We may now define a neutrosophic LP problem as one in which at least one coefficient is represented by a neutrosophic number as a result of ambiguous, inconsistent, and uncertain data. We proposed a study to solve NLP challenges based on past research. Ranking functions have been introduced to transform neutrosophic LP difficulties into crisp problems, one for each problem type. The proposed model was used to address both maximisation and minimization problems as well as mixed constraint problems.

The remainder of this study is organised as follows:

In Section 2, introduces some basic arithmetic operations of the neutrosophic set. Section 3 presents the formularization of neutrosophic LP models, whereas Section 4 presents the recommended strategy for addressing neutrosophic LP problems. In Section 5, the suggested technique is used to solve numerical examples given by Abdelfattah [3]. Finally, in Section 6, the benefits of current methods are emphasised, and future directions are discussed.

2 Preliminary

Definition 2.1. [11] *Triangular neutrosophic number (TNN) is denoted by $\widehat{X} = \langle x^L, x^M, x^U; \phi_x, \varphi_x, \psi_x\rangle$, where the three membership functions for the truth, indeterminacy, and falsity of x can be defined as follows:*

$$\tau(x) = \begin{cases} \dfrac{x-x^L}{x^M-x^L}\phi_x, & x^L \le x \le x^M \\ \phi_x, & x = x^M \\ \dfrac{x^U-x}{x^U-x^M}\phi_x, & x^M \le x \le x^U \\ 0, & otherwise \end{cases} \tag{1}$$

$$\iota(x) = \begin{cases} \dfrac{x-x^L}{x^M-x^L}\varphi_x, & x^L \le x \le x^M \\ \varphi_x, & x = a^M \\ \dfrac{x^U-x}{x^U-x^M}\varphi_x & x^M \le x \le x^U \\ 1, & otherwise \end{cases} \tag{2}$$

$$\nu(x) = \begin{cases} \dfrac{x-x^L}{x^M-x^L}\psi_x, & x^L \le x \le x^M \\ \psi_x, & x = x^M \\ \dfrac{x^U-x}{x^U-x^M}\psi_x, & x^M \le x \le x^U \\ 1, & otherwise \end{cases} \tag{3}$$

where $0 \le \tau(x) + \iota(x) + \nu(x) \le 3,\ x \in \widehat{X}$.

Definition 2.2. [11] *Suppose $\widehat{X}_1 = \langle x_1^L, x_1^M, x_1^U; \phi_{x_1}, \varphi_{x_1}, \psi_{x_1}\rangle$ and $\widehat{X}_2 = \langle x_2^L, x_2^M, x_2^U; \phi_{x_2}, \varphi_{x_2}, \psi_{x_2}\rangle$ two TNNs. Then The arithmetic relationships are stated as follows:*

1. $\widehat{X}_1 \oplus \widehat{X}_2 = \langle x_1^L + x_2^L, x_1^M + x_2^M, x_1^U + x_2^U; \phi_{x_1} \wedge \phi_{x_2}, \varphi_{x_1} \vee \varphi_{x_2}, \psi_{x_1} \vee \psi_{x_2}\rangle$.
2. $\widehat{X}_1 - \widehat{X}_1 = \langle x_1^L - x_2^U, x_1^M - x_2^M, x_1^U - x_2^L; \phi_{x_1} \wedge \phi_{x_2}, \varphi_{x_1} \vee \varphi_{x_2}, \psi_{x_1} \vee \psi_{x_2}\rangle$.
3. $\widehat{X}_1 \otimes \widehat{X}_1 = \langle x_1^L x_2^L, x_1^M x_2^M, x_1^U x_2^U; \phi_{x_1} \wedge \phi_{x_2}, \varphi_{x_1} \vee \varphi_{x_2}, \psi_{x_1} \vee \psi_{x_2}\rangle$.

4. $\lambda \widehat{X}_1 = \begin{cases} \langle \lambda x_1^L, \lambda x_1^M, \lambda x_1^U; \phi_{x_1}, \varphi_{x_1}, \psi_{x_1} \rangle, & \lambda > 0 \\ \langle \lambda x_1^U, \lambda x_1^M, \lambda x_1^L; \phi_{x_1}, \varphi_{x_1}, \psi_{x_1} \rangle, & \lambda < 0 \end{cases}$
where $a \wedge b = \min(a, b)$ *and* $a \vee b = \max(a, b)$.

Definition 2.3. *Based on the definition (2.2), the ranking function can be defined as*

$$R(\widehat{X}) = \begin{cases} \dfrac{2(x^L + x^U) - x^M}{3} + \phi_x - \varphi_x - \psi_x, & \text{if } \widehat{X} \text{ be a TNN} \\ \widehat{X}, \quad \text{if } \widehat{X} \text{ is real number} \end{cases} \tag{4}$$

$$R(\lambda\widehat{X}) = \begin{cases} \lambda\Big(R(\widehat{X}) - (\phi_x - \varphi_x - \psi_x)\Big) + (\phi_x - \varphi_x - \psi_x), & \text{if } \lambda > 0 \\ \lambda\Big(R(\widehat{X}) - (\phi_x - \varphi_x - \psi_x)\Big) - (\phi_x - \varphi_x - \psi_x), & \text{if } \lambda < 0 \end{cases} \tag{5}$$

Definition 2.4. *Suppose* $\widehat{X_1}$ *and* $\widehat{X_2}$ *be two TNNs, then two triangular number can be compared by*

1. $\widehat{X_1} \leq \widehat{X_2}$ iff $R(\widehat{X_1}) \leq R(\widehat{X_2})$.
2. $\widehat{X_1} = \widehat{X_2}$ iff $R(\widehat{X_1}) = R(\widehat{X_2})$.

where $R(.)$ *is a ranking function.*

Example 2.5. Let us consider $\widehat{X_1} = \langle 10, 14, 17; 0.6, 0.2, 0.3 \rangle$ and $\widehat{X_2} = \langle 10, 16, 18; 0.4, 0.5, 0.7 \rangle$ are the TNN. Then

(a) $R(\widehat{X_1}) = \dfrac{2(10 + 17) - 14}{3} + (0.6 - 0.2 - 0.3) = 13.433.$

(b) $5\widehat{A} = \langle 50, 70, 85; 0.6, 0.2, 0.3 \rangle$ then $R(5\widehat{X_1}) = \dfrac{2(50 + 85) - 70}{3} + (0.6 - 0.2 - 0.3) = 66.76$, by using equation (5) we have $R(5\widehat{X_1}) = 5\Big(13.433 - (0.6 - 0.2 - 0.3)\Big) + (0.6 - 0.2 - 0.3) = 66.76$

(c) $-5\widehat{X_1} = \langle -85, -70, -50; 0.6, 0.2, 0.3 \rangle$ then $R(-5\widehat{X_1}) = \dfrac{2(-50 - 85) + 70}{3} - (0.6 - 0.2 - 0.3) = -66.76$, by using equation (5) we have $R(-5\widehat{X_1}) = -5\Big(13.433 - (0.6 - 0.2 - 0.3)\Big) - (0.6 - 0.2 - 0.3) = -66.76$

(d) Since $R(\widehat{X_1}) = 13.433$ and $R(\widehat{X_2}) = 12.533$, then $\widehat{X_1} > \widehat{X_2}$.

Theorem 2.6. *Let us consider* $\widehat{X}_1 = \langle x_1^L, x_1^M, x_1^U; \phi_{x_1}, \varphi_{x_1}, \psi_{x_1} \rangle$ *and* $\widehat{X}_2 = \langle x_2^L, x_2^M, x_2^U; \phi_{x_2}, \varphi_{x_2}, \psi_{x_2} \rangle$ *are the TNN. Then*

$$\begin{aligned} R(\widehat{X_1} - \widehat{X_2}) = R(\widehat{X_1}) - R(\widehat{X_2}) - \big[(\phi_{x_1} - \phi_{x_2}) - (\varphi_{x_1} - \varphi_{x_2}) - (\psi_{x_1} - \psi_{x_2})\big] + \phi_{x_1} \wedge \phi_{x_2} \\ - \varphi_{x_1} \vee \varphi_{x_2} - \psi_{x_1} \vee \psi_{x_2}. \end{aligned} \tag{6}$$

Proof. Since, $\widehat{X}_1 - \widehat{X}_1 = \langle x_1^L - x_2^L, x_1^M - x_2^M, x_1^U - x_2^U; \phi_{x_1} \wedge \phi_{x_2}, \varphi_{x_1} \vee \varphi_{x_2}, \psi_{x_1} \vee \psi_{x_2} \rangle$, then

$$\begin{aligned} R(\widehat{X_1} - \widehat{X_2}) &= \frac{2(x_1^L - x_2^L + x_1^U - x_2^U) - (x_1^M - x_2^M)}{3} + \phi_{x_1} \wedge \phi_{x_2} - \varphi_{x_1} \vee \varphi_{x_2} - \psi_{x_1} \vee \psi_{x_2} \\ &= \frac{2(x_1^L + x_1^U) - x_1^M}{3} - \frac{2(x_2^L + x_2^U) - x_2^M}{3} + \phi_{x_1} \wedge \phi_{x_2} - \varphi_{x_1} \vee \varphi_{x_2} - \psi_{x_1} \vee \psi_{x_2} \\ &= \big[R(\widehat{X_1}) - (\phi_{x_1} - \varphi_{x_1} - \psi_{x_1})\big] - \big[R(\widehat{X_2}) - (\phi_{x_2} - \varphi_{x_2} - \psi_{x_2})\big] \\ &\quad + \phi_{x_1} \wedge \phi_{x_2} - \varphi_{x_1} \vee \varphi_{x_2} - \psi_{x_1} \vee \psi_{x_2} \\ &= R(\widehat{X_1}) - R(\widehat{X_2}) - \big[\phi_{x_1} - \phi_{x_2} - (\varphi_{x_1} - \varphi_{x_2}) - (\psi_{x_1} - \psi_{x_2})\big] \\ &\quad + \phi_{x_1} \wedge \phi_{x_2} - \varphi_{x_1} \vee \varphi_{x_2} - \psi_{x_1} \vee \psi_{x_2} \end{aligned}$$

□

Theorem 2.7. *Let $\widehat{X_i} = \langle x_i^L, x_i^M, x_i^U; \phi_{x_i}, \varphi_{x_i}, \psi_{x_i}\rangle$ be n TNNs. Then*

$$R\Big(\sum_{i=1}^{n}\widehat{X_i}\Big) = \sum_{i=1}^{n} R(\widehat{X_i}) - \sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \tag{7}$$

Proof. Let $\widehat{X_i} = \langle x_i^L, x_i^M, x_i^U; \phi_{x_i}, \varphi_{x_i}, \psi_{x_i}\rangle$, then

$$\sum_{i=1}^{n}\widehat{X_i} = \Big\langle \sum_{i=1}^{n} x_i^L, \sum_{i=1}^{n} x_i^M, \sum_{i=1}^{n} x_i^U; \bigwedge_{i=1}^{n}\phi_{x_i}, \bigvee_{i=1}^{n}\varphi_{x_i}, \bigvee_{i=1}^{n}\psi_{X_i}\Big\rangle$$

$$\begin{aligned}
R\Big(\sum_{i=1}^{n}\widehat{A_i}\Big) &= \frac{2\big(\sum_{i=1}^{n} x_i^L + \sum_{i=1}^{n} x_i^U\big) - \sum_{i=1}^{n} x_i^M}{3} + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\Big(\frac{2(x_i^L + x_i^U) - x_i^M}{3}\Big) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\Big(\frac{2(x_i^L + x_i^U) - x_i^M}{3} + (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - \sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) \\
&\quad + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\Big(R(\widehat{X_i})\Big) - \sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i}.
\end{aligned}$$

□

Theorem 2.8. *Let $\widehat{X_i} = \langle x_i^L, x_i^M, x_i^U; \phi_{x_i}, \varphi_{x_i}, \psi_{x_i}\rangle$ be n TNNs and if $\lambda_i > 0$. Then*

$$R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}\Big) = \sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{A_i})\Big) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \tag{8}$$

Proof.

For $\lambda_i > 0$, $\sum_{i=1}^{n}\lambda_i\widehat{X_i} = \Big\langle \sum_{i=1}^{n}\lambda_i x_i^L, \sum_{i=1}^{n}\lambda_i x_i^M, \sum_{i=1}^{n}\lambda_i x_i^U; \bigwedge_{i=1}^{n}\phi_{x_i}, \bigvee_{i=1}^{n}\varphi_{x_i}, \bigvee_{i=1}^{n}\psi_{x_i}\Big\rangle$, then

From definition 2.2, we have

$$\begin{aligned}
R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}\Big) &= \frac{2\big(\sum_{i=1}^{n}\lambda_i x_i^L + \sum_{i=1}^{n}\lambda_i x_i^U\big) - \sum_{i=1}^{n}\lambda_i x_i^M}{3} + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\lambda_i\Big(\frac{2(x_i^L + x_i^U) - x_i^M}{3}\Big) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\lambda_i\Big(\frac{2(x_i^L + x_i^U) - x_i^M}{3} + (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - \sum_{i=1}^{n}\lambda_i(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) \\
&\quad + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\
&= \sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i}
\end{aligned}$$

□

Theorem 2.9. *Let* $\widehat{A_i} = \langle x_i^L, x_i^M, x_i^U; \phi_{x_i}, \varphi_{x_i}, \psi_{x_i}\rangle$ *be* n *TNNs and if* $\lambda_i < 0$. *Then*

$$R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}\Big) = \sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - 2\sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \tag{9}$$

Proof. For $\lambda_i < 0$, using theorem 2.7 and defination 2.2, we have

$$\begin{aligned} R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}\Big) &= \sum_{i=1}^{n}\Big(R(\lambda_i\widehat{X_i})\Big) - \sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\ &= \sum_{i=1}^{n}\Big(\lambda_i\big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\big) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - \sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) \\ &\quad + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \\ &= \sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - 2\sum_{i=1}^{n}(\phi_{x_i} - \varphi_{x_i} - \psi_{x_i}) + \bigwedge_{i=1}^{n}\phi_{x_i} - \bigvee_{i=1}^{n}\varphi_{x_i} - \bigvee_{i=1}^{n}\psi_{x_i} \end{aligned}$$

□

Theorem 2.10. *Let* $\widehat{X}_i = \langle x_i^L, x_i^M, x_i^U; \phi_{x_i}, \varphi_{x_i}, \psi_{x_i}\rangle$ *and* $\widehat{Y}_j = \langle y_j^L, y_j^M, y_j^U; \phi_{y_j}, \varphi_{y_j}, \psi_{y_j}\rangle$ *are the TNNs, and* $\lambda_i, \delta_j > 0$ *for* $i = 1, 2, 3, \cdots, n \quad j = 1, 2, 3, \cdots, m,$. *Then*

$$\begin{aligned} R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i} - \sum_{j=1}^{m}\delta_j\widehat{Y}_j\Big) &= \sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i}) - (\phi_{x_i} - \varphi_{x_i} - \psi_{x_i})\Big) - \sum_{j=1}^{m}\delta_j\Big(R(\widehat{Y}_j) - (\phi_{y_j} - \varphi_{y_j} - \psi_{y_j})\Big) \\ &\quad + \Big(\bigwedge_{i=1}^{n}\phi_{x_i} \wedge \bigwedge_{j=1}^{m}\phi_{y_j}\Big) - \Big(\bigvee_{i=1}^{n}\varphi_{x_i} \vee \bigvee_{j=1}^{m}\varphi_{y_j}\Big) - \Big(\bigvee_{i=1}^{n}\psi_{x_i} \vee \bigvee_{j=1}^{m}\psi_{y_j}\Big) \end{aligned} \tag{10}$$

Proof.

$$R\Big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}-\sum_{j=1}^{m}\delta_j\widehat{Y_j}\Big)=R\big(\sum_{i=1}^{n}\lambda_i\widehat{X_i}\big)-R\big(\sum_{j=1}^{m}\delta_j\widehat{Y_j}\big)-\Big[\bigwedge_{i=1}^{n}\phi_{x_i}-\bigwedge_{j=1}^{m}\phi_{y_j}-\Big(\bigvee_{i=1}^{n}\varphi_{x_i}-\bigvee_{j=1}^{m}\varphi_{y_j}\Big)$$
$$-\Big(\bigvee_{i=1}^{n}\psi_{x_i}-\bigvee_{j=1}^{m}\psi_{y_j}\Big)\Big]+\Big(\bigwedge_{i=1}^{n}\phi_{x_i}\wedge\bigwedge_{j=1}^{m}\phi_{y_j}\Big)-\Big(\bigvee_{i=1}^{n}\varphi_{x_i}\vee\bigvee_{j=1}^{m}\varphi_{y_j}\Big)-\Big(\bigvee_{i=1}^{n}\psi_{x_i}\vee\bigvee_{j=1}^{m}\psi_{y_j}\Big)$$
$$=\sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i})-(\phi_{x_i}-\varphi_{x_i}-\psi_{x_i})\Big)+\bigwedge_{i=1}^{n}\phi_{x_i}-\bigvee_{i=1}^{n}\varphi_{x_i}-\bigvee_{i=1}^{n}\psi_{x_i}-\Big[\sum_{j=1}^{m}\delta_j\Big(R(\widehat{Y_j})-\bigvee_{j=1}^{m}\varphi_{y_j}\Big)$$
$$-(\phi_{y_j}-\varphi_{y_j}-\psi_{y_j})\Big)+\bigwedge_{j=1}^{m}\phi_{y_j}-\bigvee_{i=1}^{m}\varphi_{y_j}-\bigvee_{i=1}^{m}\psi_{y_j}\Big]-\Big[\bigwedge_{i=1}^{n}\phi_{x_i}-\bigwedge_{j=1}^{m}\phi_{y_j}-\Big(\bigvee_{i=1}^{n}\varphi_{x_i}$$
$$-\Big(\bigvee_{i=1}^{n}\psi_{x_i}-\bigvee_{j=1}^{m}\psi_{y_j}\Big)\Big]+\Big(\bigwedge_{i=1}^{n}\phi_{x_i}\wedge\bigwedge_{j=1}^{m}\phi_{y_j}\Big)-\Big(\bigvee_{i=1}^{n}\varphi_{x_i}\vee\bigvee_{j=1}^{m}\varphi_{y_j}\Big)-\Big(\bigvee_{i=1}^{n}\psi_{x_i}\vee\bigvee_{j=1}^{m}\psi_{y_j}\Big)$$
$$=\sum_{i=1}^{n}\lambda_i\Big(R(\widehat{X_i})-(\phi_{x_i}-\varphi_{x_i}-\psi_{x_i})\Big)-\sum_{j=1}^{m}\delta_j\Big(R(\widehat{Y_j})-(\phi_{y_j}-\varphi_{y_j}-\psi_{y_j})\Big)+\Big(\bigwedge_{i=1}^{n}\phi_{x_i}\wedge\bigwedge_{j=1}^{m}\phi_{y_j}\Big)$$
$$-\Big(\bigvee_{i=1}^{n}\varphi_{x_i}\vee\bigvee_{j=1}^{m}\varphi_{y_j}\Big)-\Big(\bigvee_{i=1}^{n}\psi_{x_i}\vee\bigvee_{j=1}^{m}\psi_{y_j}\Big)$$

□

3 Triangular Neutrosophic Linear Programming Problem

Consider the standard form of linear programming problem with m constraints and n variables.

$$\begin{aligned}\text{Min / Max }&\sum_{j=1}^{n}c_jx_j\\ \text{s.t. }&\sum_{j=1}^{n}\alpha_{ij}x_j\ (\le,=,\ge)\ b_j,\quad\forall\ i=1,2,3,\cdots,m.\end{aligned}\tag{11}$$

The corresponding neutrosophic linear programming problem having all coefficients and resources are represented triangular neutrosophic numbers as follows:

$$\begin{aligned}\text{Min / Max }&\sum_{j=1}^{n}\widehat{c_j}x_j\\ \text{s.t. }&\sum_{j=1}^{n}\widehat{\alpha_{ij}}x_j\ (\le,=,\ge)\ \widehat{b_i}\quad\forall i=1,2,3,\cdots,m.\end{aligned}\tag{12}$$

where $\widehat{c_j}=\langle c_j^L,c_j^M,c_j^U;\phi_{c_j},\varphi_{c_j},\psi_{c_j}\rangle$, $\widehat{\alpha_{ij}}=\langle\alpha_{ij}^L,\alpha_{ij}^M,\alpha_{ij}^U;\phi_{\alpha_{ij}},\varphi_{\alpha_{ij}},\psi_{\alpha_{ij}}\rangle$ and $\widehat{b_i}=\langle b_i^L,b_i^M,b_i^U;\phi_{b_i},\varphi_{b_i},\psi_{b_i}\rangle$, that is

$$\begin{aligned} \text{Min / Max} \ & \sum_{j=1}^{n} \langle c_j^L, c_j^M, c_j^U; \phi_{c_j}, \varphi_{c_j}, \psi_{c_j} \rangle x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \langle \alpha_{ij}^L, \alpha_{ij}^M, \alpha_{ij}^U; \phi_{\alpha_{ij}}, \varphi_{\alpha_{ij}}, \psi_{\alpha_{ij}} \rangle x_j \ (\leq, =, \geq) \ \langle b_i^L, b_i^M, b_i^U; \phi_{b_i}, \varphi_{b_i}, \psi_{b_i} \rangle \\ & \quad \forall \ i = 1, 2, 3, \cdots, m. \end{aligned}$$

which is the general form of fully Triangular neutrosophic linear programming problem.

4 Method for Solving Triangular Neutrosophic Linear Programming Problem

Consider two scenarios for a completely triangular neutrosophic LP problem with n variables and m constraints in a standard form.

Step 1: Check, if the triangular neutrosophic linear programming problem is one of the scenarios provided.

Scenario 1: Suppose the triangular neutrosophic LP problem does not contain any negative term in the objective function and constraint.

$$\begin{aligned} \text{Min / Max} \ & \sum_{j=1}^{n} \widehat{c_j} x_j \\ \text{s.t.} \quad & \sum_{j=1}^{k} \widehat{\alpha_{ij}} x_j \ (\leq, =, \geq) \ \widehat{b_i}, \quad \forall \ i = 1, 2, 3, \cdots, m. \end{aligned} \tag{13}$$

Scenario 2: Suppose the TNLP problem contain any negative term in the objective function and constraint.

$$\begin{aligned} \text{Min / Max} \ & \sum_{j=1}^{s} \widehat{c_j} x_j - \sum_{j=s+1}^{n} \widehat{c_j} x_j \\ \text{s.t.} \quad & \sum_{j=1}^{k} \widehat{\alpha_{ij}} x_j - \sum_{j=k+1}^{n} \widehat{\alpha_{ij}} x_j \ (\leq, =, \geq) \ \widehat{b_i}, \quad \forall \ i = 1, 2, 3, \cdots, m. \end{aligned}$$

where

$$\begin{aligned} \widehat{c_j} &= \langle c_j^L, c_j^M, c_j^U; \phi_{c_j}, \varphi_{c_j}, \psi_{c_j} \rangle, \\ \widehat{\alpha_{ij}} &= \langle \alpha_{ij}^L, \alpha_{ij}^M, \alpha_{ij}^U; \phi_{\alpha_{ij}}, \varphi_{\alpha_{ij}}, \psi_{\alpha_{ij}} \rangle, \\ \widehat{b_i} &= \langle b_i^L, b_i^M, b_i^U; \phi_{b_i}, \varphi_{b_i}, \psi_{b_i} \rangle. \end{aligned}$$

Step 2: Applying the ranking function in the TNLP problem based on definition (2.2) and (2.3), and theorem (2.6)-(2.10) and convert it into crips LP problem.

Scenario 1:

$$\begin{aligned} \text{Min / Max} \ & R\Big(\sum_{j=1}^{n} \widehat{c_j} x_j\Big) \\ \text{s.t.} \quad & R\Big(\sum_{j=1}^{n} \widehat{\alpha_{ij}} x_j\Big) \ (\leq, =, \geq) \ R\Big(\widehat{b_i}\Big), \quad \forall \ i = 1, 2, 3, \cdots, m. \end{aligned}$$

that is

$$\begin{aligned}
\text{Min / Max } & \sum_{j=1}^{n}\Big(R(\widehat{c_j})-(\phi_{c_j}-\varphi_{c_j}-\psi_{c_j})\Big)x_j+\bigwedge_{j=1}^{n}\phi_{c_j}-\bigvee_{j=1}^{n}\varphi_{c_j}-\bigvee_{j=1}^{n}\psi_{c_j}\\
\text{s.t. } & \sum_{j=1}^{k}\Big(R(\widehat{\alpha_{ij}})-(\phi_{\alpha_{ij}}-\varphi_{\alpha_{ij}}-\psi_{\alpha_{ij}})\Big)x_j+\bigwedge_{j=1}^{n}\phi_{\alpha_{ij}}-\bigvee_{j=1}^{n}\varphi_{\alpha_{ij}}-\bigvee_{j=1}^{n}\psi_{\alpha_{ij}}\ (\leq,=,\geq)\ R\Big(\widehat{b_i}\Big)\\
& \forall\ i=1,2,3,\cdots,m.
\end{aligned}$$

Scenario 2:

$$\begin{aligned}
\text{Min / Max } & R\Big(\sum_{j=1}^{s}\widehat{c_j}x_j-\sum_{j=s+1}^{n}\widehat{c_j}x_j\Big)\\
\text{s.t } & R\Big(\sum_{j=1}^{k}\widehat{\alpha_{ij}}x_j-\sum_{j=k+1}^{n}\widehat{\alpha_{ij}}x_j\Big)(\leq,=,\geq)\ R\Big(\widehat{b_i}\Big),\quad \forall\ i=1,2,3,\cdots,m.
\end{aligned}$$

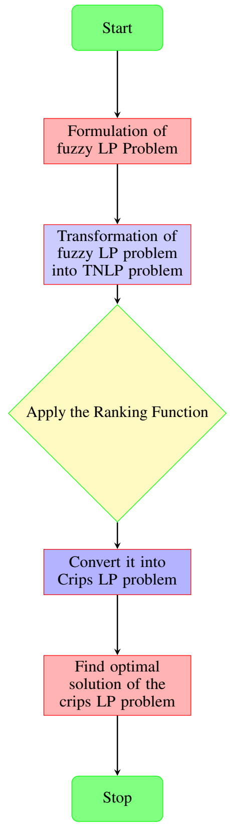
that is

$$\begin{aligned}
\text{Min / Max } & \sum_{j=1}^{s}\Big(R(\widehat{c_j})-(\phi_{c_j}-\varphi_{c_j}-\psi_{c_j})\Big)x_j-\sum_{j=s+1}^{n}\Big(R(\widehat{c_j})-(\phi_{c_j}-\varphi_{c_j}-\psi_{c_j})\Big)x_j\\
& +\bigwedge_{j=1}^{n}\phi_{c_j}-\bigvee_{j=1}^{n}\varphi_{c_j}-\bigvee_{j=1}^{n}\psi_{c_j}\\
\text{s.t. } & \sum_{j=1}^{k}\Big(R(\widehat{\alpha_{ij}})-(\phi_{\alpha_{ij}}-\varphi_{\alpha_{ij}}-\psi_{\alpha_{ij}})\Big)x_j-\sum_{j=k+1}^{n}\Big(R(\widehat{\alpha_{ij})}x_j-(\phi_{\alpha_{ij}}-\varphi_{\alpha_{ij}}-\psi_{\alpha_{ij}})\Big)\\
& +\bigwedge_{j=1}^{n}\phi_{\alpha_{ij}}-\bigvee_{j=1}^{n}\varphi_{\alpha_{ij}}-\bigvee_{j=1}^{n}\psi_{\alpha_{ij}}(\leq,=,\geq)\ R\Big(\widehat{b_i}\Big),\quad \forall\ i=1,2,3,\cdots,m.
\end{aligned}$$

which are the crips LP problem.

Step 3: Solve this crips LP problem using any method and find the optimal solution.

The step by step solution procedure of triangular neutrosophic LP problems is shown in the given flow chart



5 Numerical Example

In this section, to prove the applicability and advantages of our proposed model of neutrosophic LP problems, we solved the same problem which introduced by Abdelfattah [3].

Example 5.1. (Minimization Problem)
Let us consider a minimization problem

$$\begin{aligned} \text{Min} \quad & \langle 2,6,8;1,0,0\rangle x_1 + \langle 1,3,6;1,0,0\rangle x_2 \\ \text{s.t.} \quad & \langle 0.5,2,3;0.7,0.4,0.1\rangle x_1 + \langle 0,4,6;0.6,0.3,0.1\rangle x_2 \geq \langle 12,16,19;0.5,0.3,0.5\rangle, \\ & \langle 1,4,12;0.5,0.4,0.2\rangle x_1 + \langle 1,3,10;0.7,0.4,0.3\rangle x_2 \geq \langle 20,24,28;0.8,0.3,0.3\rangle \\ \text{and } & x_1, x_2 \geq 0. \end{aligned}$$

We have used the ranking function in the above neutrosophic linear programming problem, it follows that

$$\begin{aligned} \text{Min} \quad & R\Big(\langle 2,6,8;1,0,0\rangle x_1 + \langle 1,3,6;1,0,0\rangle x_2\Big) \\ \text{s.t.} \quad & R\Big(\langle 0.5,2,3;0.7,0.4,0.1\rangle x_1 + \langle 0,4,6;0.6,0.3,0.1\rangle x_2\Big) \geq R\Big(\langle 12,16,19;0.5,0.3,0.5\rangle\Big) \\ & R\Big(\langle 1,4,12;0.5,0.4,0.2\rangle x_1 + \langle 1,3,10;0.7,0.4,0.3\rangle x_2\Big) \geq R\Big(\langle 20,24,28;0.8,0.3,0.3\rangle\Big) \\ \text{and } & x_1, x_2 \geq 0. \end{aligned}$$

By using definition (2.2) and theorem (2.8), we have

$$\begin{aligned}
\text{Min} \quad & 4.67x_1 + 3.67x_2 + 1 \\
\text{s.t.} \quad & 1.67x_1 + 2.67x_2 + 0.2 \geq 15.03 \\
& 7.33x_1 + 6.33x_2 - 0.2 \geq 24.2 \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

The optimal solution of the problem as $x_1 = 0.0000,\ x_2 = 5.5543$ and $Z^* = 21.3843$.

Example 5.2. (Maximization Problem)
Let us consider the maximization problem

$$\begin{aligned}
\text{Max } & \langle 30, 40, 50; 0.7, 0.4, 0.3\rangle x_1 + \langle 40, 50, 60; 0.6, 0.5, 0.2\rangle x_2 \\
\text{s.t. } & \langle 0.5, 1, 3; 0.6, 0.4, 0.1\rangle x_1 + \langle 0, 2, 6; 0.6, 0.4, 0.1\rangle x_2 \leq \langle 20, 40, 60; 0.4, 0.3, 0.5\rangle \\
& \langle 1, 4, 12; 0.4, 0.3, 0.2\rangle x_1 + \langle 1, 3, 10; 0.7, 0.4, 0.3\rangle x_2 \leq \langle 100, 120, 140; 0.7, 0.4, 0.3\rangle \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

We have used the ranking function in the above neutrosophic linear programming problem, it follows that

$$\begin{aligned}
\text{Max } & R\Big(\langle 30, 40, 50; 0.7, 0.4, 0.3\rangle x_1 + \langle 40, 50, 60; 0.6, 0.5, 0.2\rangle x_2\Big) \\
\text{s.t. } & R\Big(\langle 0.5, 1, 3; 0.6, 0.4, 0.1\rangle x_1 + \langle 0, 2, 6; 0.6, 0.4, 0.1\rangle x_2\Big) \leq R\Big(\langle 20, 40, 60; 0.4, 0.3, 0.5\rangle\Big) \\
& R\Big(\langle 1, 4, 12; 0.4, 0.3, 0.2\rangle x_1 + \langle 1, 3, 10; 0.7, 0.4, 0.3\rangle x_2\Big) \leq R\Big(\langle 100, 120, 140; 0.7, 0.4, 0.3\rangle\Big) \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

By using definition (2.2) and theorem (2.8), we have

$$\begin{aligned}
\text{Max} \quad & 40x_1 + 50x_2 - 0.2 \\
\text{s.t.} \quad & 2x_1 + 3.33x_2 + 0.1 \leq 39.6 \\
& 7.3x_1 + 6.33x_2 - 0.3 \leq 126.3 \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

The optimal solution of the mixed constrained problem as $x_1 = 14.6008,\ x_2 = 3.0926$ and $Z^* = 738.1623$.

Example 5.3. (Mixed Constraint Problem)

$$\begin{aligned}
\text{Max} \quad & \langle 380, 400, 430; 0.7, 0.4, 0.3\rangle x_1 + \langle 170, 200, 210; 0.6, 0.5, 0.2\rangle x_2 \\
\text{s.t.} \quad & \langle 0.5, 1, 3; 0.6, 0.5, 0.1\rangle x_1 + \langle 1, 2, 4; 0.6, 0.4, 0.2\rangle x_2 = \langle 50, 70, 100; 1, 0, 0\rangle \\
& \langle 1, 2, 5; 0.5, 0.3, 0.2\rangle x_1 + \langle 5, 8, 12; 0.7, 0.6, 0.5\rangle x_2 \geq \langle 72, 80, 89; 1, 0, 0\rangle \\
& \langle 0, 1, 4; 0.7, 0.5, 0.2\rangle x_1 + \langle 0, 0, 3; 0.8, 0.3, 0.2\rangle x_2 \leq \langle 30, 40, 55; 1, 0, 0\rangle \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

We have used the ranking function in the above neutrosophic linear programming problem, it follows that

$$\begin{aligned}
\text{Max} \quad & R\Big(\langle 380, 400, 430; 0.7, 0.4, 0.3\rangle x_1 + \langle 170, 200, 210; 0.6, 0.5, 0.2\rangle x_2\Big) \\
\text{s.t.} \quad & R\Big(\langle 0.5, 1, 3; 0.6, 0.5, 0.1\rangle x_1 + \langle 1, 2, 4; 0.6, 0.4, 0.2\rangle x_2\Big) = R\Big(\langle 50, 70, 100; 1, 0, 0\rangle\Big) \\
& R\Big(\langle 1, 2, 5; 0.5, 0.3, 0.2\rangle x_1 + \langle 5, 8, 12; 0.7, 0.6, 0.5\rangle x_2\Big) \geq R\Big(\langle 72, 80, 89; 1, 0, 0\rangle\Big) \\
& R\Big(\langle 0, 1, 4; 0.7, 0.5, 0.2\rangle x_1 + \langle 0, 0, 3; 0.8, 0.3, 0.2\rangle x_2\Big) \leq R\Big(\langle 30, 40, 55; 1, 0, 0\rangle\Big) \\
\text{and } & x_1, x_2 \geq 0.
\end{aligned}$$

By using definition (2.2) and theorem (2.8), we have

$$\begin{aligned}
\text{Max} \quad & 406.67x_1 + 186.67x_2 - 0.2 \\
\text{s.t.} \quad & 2x_1 + 2.67x_2 - 0.1 = 77.67 \\
& 3.33x_1 + 8.6x_2 - 0.6 \geq 81.67 \\
& 2.33x_1 + 2x_2 \leq 44.33
\end{aligned}$$

The optimal solution of the mixed constrained problem as $x_1^* = 0.0000, x_2^* = 39.4257$ and $Z^* = 7359.4$.

6 Conclusion

Neutrosophic sets are a relatively new academic topic that is rapidly growing in popularity and being used for a wide range of decision-making concerns, particularly mathematical programming problems. The focus of this study is on linear programming models with neutrosophic coefficients. We solved the triangular neutrosophic linear programming problem in this article. We provided a unique ranking function for converting TNNs to their crisp counterparts, and we thoroughly investigated the arithmetic operations of triangular neutrosophic numbers. After utilising this ranking technique to convert the problem to its crisp values and solve it in any traditional way. Real-world modelling of triangular neutrosophic LP optimization may be simplified using the proposed approach, and it may be straightforward to use from a computational viewpoint. We used triangular neutrosophic linear programming problems to explain three basic problems offered by Abdelfattah [3]. We found that our proposed model is simpler, more efficient, and yields better outcomes than others.

Furthermore, researchers can successfully apply the concept of triangular neutrosophic number based linear programming strategy in a broad range of research domains. The real benefit of the proposed technique is that it can handle both symmetric and non-symmetric TNNs. Comparing results allows decision-makers to choose their own acceptance, imprecise, and falsehood criteria.

Conflict of Interest: The authors declare no conflict of interest.

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Min and Max are the Only Continuous &- and $\vee$-Operations for Finite Logics

Vladik Kreinovich

Abstract. Experts usually express their degrees of belief in their statements by the words of a natural language (like "maybe", "perhaps", etc.). If an expert system contains the degrees of beliefs $t(A)$ and $t(B)$ that correspond to the statements A and B, and a user asks this expert system whether "$A \,\&\, B$" is true, then it is necessary to come up with a reasonable estimate for the degree of belief of $A \,\&\, B$. The operation that processes $t(A)$ and $t(B)$ into such an estimate $t(A \,\&\, B)$ is called an &-operation. Many different &-operations have been proposed. Which of them to choose? This can be (in principle) done by interviewing experts and eliciting a &-operation from them, but such a process is very time-consuming and therefore, not always possible. So, usually, to choose a &-operation, we extend the finite set of actually possible degrees of belief to an infinite set (e.g., to an interval $[0, 1]$), define an operation there, and then restrict this operation to the finite set. In this paper, we consider only this original finite set. We show that a reasonable assumption that an &-operation is continuous (i.e., that gradual change in $t(A)$ and $t(B)$ must lead to a gradual change in $t(A \,\&\, B)$), uniquely determines min as an &-operation. Likewise, max is the only continuous $\vee$-operation. These results are in good accordance with the experimental analysis of "and" and "or" in human beliefs.

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1 Introduction

We must represent uncertainty. When we design an expert system, and place the experts' knowledge inside the computer, we must somehow describe the fact that experts may have different degrees of belief in their statements. Some of these statements are believed to be absolutely true, some are true to some extent, and some are only probably true, but an expert is not sure about that. Usually, experts describe their degrees of belief by the words from a natural language (like "for sure", "maybe", "probably", etc.) Since there are only finitely many words in a language, we have only finitely many different degrees of belief.

We must represent these degrees in a computer.

If an expert is absolutely sure about the truth of any statement that he (or anyone else) pronounces, then we have only two degrees of belief: "absolutely sure" and "absolutely sure that it is wrong"; these two degrees of belief are just truth values: "true" and "false". Therefore, in a general case, when different degrees of belief are allowed, these degrees of belief can be viewed as *truth values* that characterize different statements.

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We must deal with uncertainties. Representing the truth values inside a computer is not all: we must be able to process these values. For example, suppose that we know the truth values $t(A)$ and $t(B)$ of two statements A and B, and the user asks a query "$A \,\&\, B$?". Since we are not sure whether A and B are true, we are also not sure whether $A \,\&\, B$ is true or not. Therefore, the only possible answer that we can give to this query is to describe a (reasonable) degree of belief $t(A \,\&\, B)$ in $A \,\&\, B$. If the only information that we have about A and B consists of their truth values, then we must somehow produce this reasonable estimate $t(A \,\&\, B)$ based on the known values $t(A)$ and $t(B)$. In other words, we must have a function (moreover, an algorithm) that would transform $t(A)$ and $t(B)$ into $t(A \,\&\, B)$. If we denote this function by $f_{\&}$, then we can describe the resulting "reasonable" estimate for $t(A \,\&\, B)$ as $f_{\&}(t(A), t(B))$.

In case both $t(A)$ and $t(B)$ coincide with "true" or "false", this function must coincide with the usual & −operation that is defined on a classical set of truth values $\{0, 1\}$. Therefore, this function $f_{\&}$ is called an *&-operation*.

Likewise, there must exist a function $f_{\vee}$ that corresponds to ∨ and is therefore called an ∨-operation, and a function $f_{\neg}$ (an ¬−operation) that generalizes "not" to the bigger set of truth values.

Conclusion: an ideal representation of degrees of uncertainty is by finite logic. A set with logical operations on it ("and", "or", and "not") is usually called a *logic.* A logic that is a finite set is called a *finite logic.* Our finite set of truth values has all these operations, and is therefore a finite logic.

Therefore, an ideal representation of degrees of uncertainty must form a finite logic.

How to choose &- and ∨-operations for finite logics: ideal solution. Since our main objective is to represent experts' beliefs in the most adequate manner, it is reasonable to choose &- and ∨-operations so as to provide the best description of human reasoning with uncertainty. To do this, we must first ask the experts to estimate their degrees of belief in different statements and their logical combinations. Then, we choose a function $f_{\&}$ as follows: For every pair of degrees of belief a and b, we find all the statements in our record for which the degree of belief was a ($t(A) = a$), and all the statement B for which $t(B) = b$. For different A and B, we look for the truth values that the experts assigned to the statements $A \,\&\, B$. For different A and B, these truth values may be different; we find the "average" one (e.g., the one that is most frequent) and use it as $f_{\&}(a, b)$.

In a similar way, we can experimentally determine $f_{\vee}(a, b)$.

This is (in essence) the method that was originally used to choose &- and ∨-operations in one of the first successful expert systems MYCIN; see, e.g., [3]. A similar method was efficiently used to produce &- and ∨-operations on finite logic in MILORD system [1, 12].

How are &- and ∨-operations chosen now, if we cannot afford to elicit them from the experts? If we can afford to perform the above-described procedure, fine, this procedure is the ideal solution to the choice problem. However, already the authors of MYCIN have noticed that it is very expensive and time-consuming procedure [3]. So, what to do if we cannot afford it, but still have to choose &- and ∨-operations?

In this case, we need to develop theoretical methods to choose these operations. The authors of MILORD formulated reasonable conditions that &- and ∨-operations must satisfy [1, 12]. However, there are several different operations that satisfy all these conditions. Hence, the problem of choice remains.

At present, this choice problem is solved in the following manner. In the majority of actual expert systems the set of possible truth values is infinite; see, e.g., [3, 13]; MILORD is one of the few exceptions). Usually, the numbers from the interval $[0, 1]$ are used to represent degrees of belief. The reason for choosing this interval is very simple: inside the computer, "true" is usually represented as 1, and "false" as 0. So, it is reasonable to represent all intermediate degrees of belief by real numbers that are intermediate between 0 and 1.

If we assume that all numbers from $[0, 1]$ are possible, then we need to define &- and ∨-operations as functions from $[0, 1] \times [0, 1]$ to $[0, 1]$. There exist several reasonable approaches that enable us to make a choice of such a function; see, e.g., [7].

Formulation of a problem. These approaches provide us with reasonable &- and $\vee$-operations, but they essentially depend on the assumption that *all* numbers from the interval $[0,1]$ can be truth values. Strictly speaking, this assumption is not true. Therefore, it is reasonable to formulate the following problem: if we are unable to elicit these operations from the experts, can we still choose them using only the actual truth values?

How we are going to solve this problem. In order to solve this problem, we will assume that both &- and $\vee$-operations $f_\&(a,b)$ and $f_\vee(a,b)$ are "continuous" in the following sense. If we gradually (=without skipping any intermediate values) increase our degrees of belief $a = t(A)$ and $b = t(B)$, then the resulting degrees of belief $t(A \,\&\, B) = f_\&(a,b)$ and $t(A \vee B) = f_\vee(a,b)$ must also change gradually.

It turns out that this reasonable demand is satisfied by only one pair of operations: min and max, that were originally proposed by L. Zadeh [14]; see also [2, 6, 8, 9, 10].

This result is in good accordance with the known experiments [5, 11, 15], according to which in many situations, min and max describe human reasoning better than other possible &- and $\vee$-operations.

2 Definitions and the Main Results

Definition 2.1. *By a finite logic, we understand a (partially) ordered finite set L that contains two elements T and F such that $F \le a \le T$ for every $a \in L$. The elements of L will be called truth values, or degrees of belief.*

Motivation. We consider finitely many truth values, that represent different degrees of belief. Sometimes, we are certain that a belief expressed by a degree a is stronger than the belief that is expressed by a degree b. For example, $a =$"for certain" is stronger than $b =$"maybe'. We will denote this by $a > b$. So, on our set of truth values, there is an ordering relation.

In particular, if we denote the degree of belief that expresses our absolute certainty in A, by T (T from "true"), and the degree of belief that expresses the absolute belief in $\neg A$ by F (from "false"), then $F \le a \le T$ for an arbitrary degree of belief a.

It is possible that for some words that describe uncertainty, there is no clear understanding which of them corresponds to a greater belief (e.g., it is difficult to compare "probable" and "possible"). Therefore, we do not require that this ordering is a total (linear) order, it can be only partial.

Definition 2.2. *Let L be a finite logic. By an &-operation on L we mean a function $f_\& : L \times L \to L$ with the following properties:*

- $f_\&(a,b) \le a$;
- $f_\&(a,b) = f_\&(b,a)$;
- $f_\&(a,F) = F$;
- *if* $a \le a'$ *and* $b \le b'$, *then* $f_\&(a,b) \le f_\&(a',b')$.

Motivations.

- The first property is motivated by the following: if we believe in A *and* B, then we must believe in both statements A and B; therefore, our belief in $A \,\&\, B$ is either of the same strength or less strong than our belief in A.
- The second property is motivated by the fact that "$A \,\&\, B$" and "$B \,\&\, A$" are equivalent statements, so it is reasonable to demand that our estimated degree of belief in $A \,\&\, B$ $(= f_\&(t(A), t(B)))$ is the same as the estimated degree of belief in $B \,\&\, A$ $(= f_\&(t(B), t(A)))$.

- The third property expresses the following: if B is false, then "A and B" is false for all A.
- The fourth means that if the degree of belief in A and B increases (i.e., if we found additional reasons to believe in A or B), then the resulting degree of belief in $A \,\&\, B$ must either increase, or stay the same.

Comment. This definition is similar to the usual definition of a t-norm (see, e.g., [2, 6, 8, 9, 10]) and to the definition of an &-operation on a finite logic from [1, 12]. The reader may notice, however, that we do not require some additional properties that are usually required for a t-norm, like associativity ($f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$)). The reason is that in our case, as we will see later, it automatically follows from the other properties.

Definition 2.3. *Let L be a finite logic. By an $\vee$-operation on L we mean a function $f_\vee : L \times L \to L$ with the following properties:*

- $f_\vee(a, b) \geq a$;
- $f_\vee(a, b) = f_\vee(b, a)$;
- $f_\vee(a, T) = T$;
- *if* $a \leq a'$ *and* $b \leq b'$, *then* $f_\vee(a, b) \leq f_\vee(a', b')$.

Motivations for these demands are similar to the ones given for an &-operation.

Definition 2.4. *Let L be a finite logic.*

- *We say that an element $a' \in L$ immediately follows a (and denote it by $a \ll b$, or $b \gg a$) if $a < a'$, and there exists no c such that $a < c < a'$.*
- *We say that a function $f : L \to L$ is discontinuous if there exist elements a, a', c such that $a \ll a'$, and either $f(a) < c < f(a')$, or $f(a') < c < f(a)$.*

Motivation. If such values a, a', c exist, this means that when we gradually increase our degree of belief from a to a' (gradually in the sense that we do not skip any intermediate values), then the resulting value of f "jumps" from $f(a)$ to $f(a')$, skipping an intermediate value c. So, in this sense, the function f is discontinuous.

We can use the same definition for a function of two variables.

Definition 2.5. *A function $f : L \times L \to L$ is called discontinuous is there exist the values a, a', b, b', c for which the following three conditions are true:*

- $a \ll a'$, $a' \ll a$, *or* $a = a'$;
- $b \ll b'$, $b' \ll b$, *or* $b = b'$;
- $f(a, b) < c < f(a', b')$, *or* $f(a', b') < c < f(a, b)$.

Comment.

- The first condition means that a gradually changes into a' (i.e., either a' immediately follows a, or a immediately follows a', or a' equals a).
- The second condition means that b gradually changes into b'.
- The third condition means that there is a "gap" between $f(a, b)$ and $f(a', b')$.

Definition 2.6. *A function is called continuous if it is not discontinuous.*

Comment 1. If a function f is continuous in the intuitive sense of this word, then it cannot have discontinuities in the sense of Definitions 2.4 and 2.5, and therefore it will be continuous in the sense of Definition 2.6. We do not claim, however, that an arbitrary function that satisfies Definition 2.6 is intuitively continuous, because there may be other types of discontinuity. We will prove that this weak continuity is sufficient to select &- and $\vee$-operations.

Comment 2. It is worth mentioning that usually in mathematics, continuity is understood as continuity with respect to some topology. For finite sets, however, this notion is not applicable: on a finite set, we either have a discrete topology (in which case all functions are continuous), or a topology that is reduced to an ordering relation, in which case monotonic functions and only they are continuous; see, e.g., [4]. This monotonicity is not enough for us: we have already included monotonicity in our definitions of &- and $\vee$-operations, and we want to formalize the evident fact that some monotonic operations are "continuous" (in an intuitive sense), and some are not. Hence, we had to use new definitions of continuity.

Now, we are ready to formulate the main results.

Theorem 2.7. *If f is a continuous &-operation on a finite logic L, then L is linearly ordered, and $f(a, b) = \min(a, b)$.*

Comments.

- For a linearly ordered set, $\min(a, b)$ is defined as the smallest of a and b.
- For readers' convenience all the proofs are given in Section 4.

Theorem 2.8. *If f is a continuous $\vee$-operation on a finite logic L, then L is linearly ordered, and $f(a, b) = \max(a, b)$.*

Example 2.9. Let us give an example of an &-operation that is different from min, and show that it is really discontinuous. As a finite logic, let us take the set of 11 numbers $\{0, 0.1, 0.2, \ldots, 0.9, 1.0\}$ with natural order. We thus defined L as a subset of the interval $[0, 1]$. In the same original paper by L. Zadeh [14], another operation on the interval $[0, 1]$ has been proposed for $\& :\ f(a, b) = a \cdot b$. This operation, unlike min, cannot be directly applied to the chosen values, because, e.g., $0.6 \cdot 0.6 = 0.36$, and the number 0.36 does not belong to the set of 11 chosen values. This difficulty is, however, easy to overcome: we can take as $f(a, b)$ the number from L that is the closest to $a * b$ (and if there are two closest numbers, like 0.2 and 0.3 for $0.25 = 0.5 \cdot 0.5$, choose the biggest of these two). For this operation, we will have $f(0.6, 0.6) = 0.4$, $f(0.3, 0.5) = 0.2$, etc.

Let us now consider the case when we have two statements A and B, and our degree of belief in each of them is equal to 0.9. Then, our degree of belief in $A \,\&\, B$ is equal to $f(0.9, 0.9) = 0.8$. In the chosen set L, 1.0 immediately follows 0.9, which means that an increase in the degree of belief from 0.9 to 1.0 can be called gradual. So, we can consider the possibility that our degrees of belief in both A and B gradually increase from 0.9 to 1.0. After this increase, the degree of belief in $A \,\&\, B$ becomes equal to $f(1.0, 1.0) = 1.0$. So, we gradually increased our degrees of belief in A and B, but the resulting degree of belief in $A \,\&\, B$ "jumped" from 0.8 to 1.0, skipping the value 0.9. Hence, this function f is discontinuous.

In Definition 2.5, we can thus take $a = b = 0.9$, $a' = b' = 1.0$, and $c = 0.9$.

3 Operations that Correspond to Negation and Implication

In Section 2, we described continuous "and" and "or" operations, and concluded that L must be linearly ordered. Let us now describe continuous operations with degrees of belief that correspond to other logical connectives.

Definition 3.1. *By a ¬-operation on L we mean a function* $f : L \to L$ *such that* $f(T) = F$ *and* $f(F) = T$.

Motivation. This condition simply means that if A is absolutely true, then $\neg A$ is absolutely false, and vice versa.

Theorem 3.2. *If* $L = \{F = a_0 < a_1 < a_2 < \ldots < a_n = T\}$ *is a linearly ordered finite logic, and* f *is a continuous ¬-operation on* L*, then* $f(a_i) = a_{n-i}$.

Comment. We can represent this result in a manner that is closer to the traditional representation of uncertainty, if we describe each degree of belief a_i by a real number i/n. Then, for each truth value a, $f_\neg(a) = 1 - a$. This is exactly the operation originally proposed by Zadeh. In other words, not only the &- and ∨-operations initially proposed by Zadeh are the only continuous &- and ∨-operations, but his negation operation is the only continuous "not"-operation on a finite logic.

Let us now describe the implication operations.

Definition 3.3. *Let* L *be a finite logic. By a →-operation on* L *we mean a function* $f_\to : L \times L \to L$ *with the following properties:*

- $f_\to(F, a) = T$;
- $f_\to(T, a) = a$;
- $f_\to(a, T) = T$;
- $f_\to(a, a) = 1$;
- *if* $a \le a'$*, then* $f_\to(a, b) \ge f_\to(a', b)$.

Motivations. The intended meaning of the function $f_\to(a, b)$ is as follows: if we know the degrees of belief $a = t(A)$ and $b = t(B)$ in some statements A and B, then $f_\to(a, b)$ is a reasonable degree of belief in the statement $A \to B$ ("A implies B"). With this interpretation in mind:

- The first of the above properties states that anything follows from a false statement.
- The second one states that to believe that A follows from an absolutely true statement is the same as to believe that A is true, and therefore, the corresponding degrees of belief must coincide.
- The third condition means that a true statement follows from everything.
- The fourth that for any statement A, A follows from A (and therefore, the degree of belief in $A \to A$ must be equal to T).
- The last condition is related to the third one: Namely, the third one says that if A is false, then $A \to B$ is always true. Therefore, if for some reason our degree of belief in statement A decreases (from a' to a), then our belief that A can be false will correspondingly increase. Therefore, our degree of belief that $A \to B$ is true, will also increase. Hence, it is reasonable to demand that $f_\to(a', b) \le f_\to(a, b)$.

Theorem 3.4. *If* $L = \{F = a_0 < a_1 < \ldots < a_n = T\}$ *is a linearly ordered finite logic, and* f *is a continuous →-operation on* L*, then* $f(a_i, a_j) = a_{\min(n, n+j-i)}$

Comment. If we describe a_i by a real number i/n, then this →-operation turns into $f(a, b) = \min(1, 1 + b - a)$.

4 Proofs

Proof of Theorem 2.7.

1°. Let us first prove that every element $a \in L$ can be connected to T by a finite chain $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$ $(k \geq 0)$.

Indeed, if $a = T$, then we already have a chain, with $k = 0$.

If $a \neq T$, then according to our definition of a finite logic, we have $a < T$. If $a \ll T$, then we have a chain $a_0 = T$, $a_1 = a$. If $a \not\ll T$, then, according to the definition of $\ll$, it means that there exists a c such that $T > c > a$. If $T \gg c$, and $c \gg a$, then we have the desired chain. Else, we can insert additional elements in between them, etc.

On each step of this procedure, we either have a chain, or we can insert more elements into a sequence $T = a_0 > a_1 > \ldots > a_n = a$. Since there are only finitely many elements in the set L, and all a_i are different, this insertion cannot go on forever. Therefore, sooner or later, it will stop, and we will get the desired chain.

2°. Let us now prove that $f(a, a) = a$ for every $a \in L$.

Indeed, suppose that $a \in L$ is given. According to 1°, there exists a chain $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$ that connects T and a.

If $k = 0$, then $a = T$, and $f(T, T) = T$ follows from the properties of an &-operation.

So, we can assume that $k > 0$. We will prove that $f(a, a) = a$ by reduction to a contradiction. Indeed, suppose that $f(a, a) \neq a$. Hence, $f(a_0, a_0) = a_0$, and $f(a_k, a_k) \neq a_k$. Let us denote by p the smallest integer for which $f(a_p, a_p) \neq a_p$. From this definition of p it follows, in particular, that $f(a_{p-1}, a_{p-1}) = a_{p-1}$.

Since f is an &-operation, we can conclude that $f(a_p, a_p) \leq a_p$. Since $f(a_p, a_p) \neq a_p$ (by choice of p), we conclude that $f(a_p, a_p) < a_p$.

Therefore, we have $a_p \ll a_{p-1}$, and $f(a_p, a_p) < a_p < a_{p-1} = f(a_{p-1}, a_{p-1})$, i.e., f is discontinuous (here, $a = b = a_p, a' = b' = a_{p-1}$, and $c = a_p$). However, we assumed that f is continuous.

This contradiction proves that $f(a, a)$ cannot be different from a, so $f(a, a) = a$ for all a.

3°. Let us prove that L is linearly ordered, i.e., for every two elements $a, b \in L$, either $a = b$, or $a < b$, or $b < a$.

Indeed, let us take $a, b \in L$. Following 1°, we will form chains $T = a_0 \gg a_1 \gg \ldots \gg a_k = a$, and $T = b_0 \gg b_1 \gg \ldots \gg b_l = b$. Let us denote by p the biggest integer for which a_p and b_p are both defined and equal to each other ($a_p = b_p$).

3.1°. If $p = k = l$, then $a = a_k = a_p = b_p = b_l = b$, i.e., $a = b$.

3.2°. If $p = k \neq l$, then $a = a_k = b_p \gg b_{p+1} \gg \ldots \gg b_l = b$, therefore $a > b_{p+1} > \ldots > b_l = b$, and $a > b$.

3.3°. Likewise, if $p = l \neq k$, then $b > a$.

3.4°. Let us prove that the remaining case when $p < k$ and $p < l$, is impossible.

Indeed, in this case, both a_{p+1} and b_{p+1} are defined and different from each other. Since f is an &-operation, we can conclude that $f(a_{p+1}, b_{p+1}) \leq a_{p+1}$ and $f(a_{p+1}, b_{p+1}) = f(b_{p+1}, a_{p+1}) \leq b_{p+1}$.

The first inequality means that we have two possibilities: $f(a_{p+1}, b_{p+1}) = a_{p+1}$, and $f(a_{p+1}, b_{p+1}) < a_{p+1}$. We will show that in both cases, we have a contradiction.

3.4.1°. Suppose first that $f(a_{p+1}, b_{p+1}) = a_{p+1}$. We already know that $f(a_{p+1}, b_{p+1}) \leq b_{p+1}$, so $a_{p+1} \leq b_{p+1}$. We chose p in such a way that $a_{p+1} \neq b_{p+1}$ (and $a_p = b_p$), therefore $a_{p+1} < b_{p+1}$. So, $a_{p+1} < b_{p+1} < b_p = a_p$. The existence of the intermediate value b_{p+1} contradicts the assumption that $a_{p+1} \ll a_p$. So, in this case, we have a contradiction.

3.4.2°. Let us now consider the case when $f(a_{p+1}, b_{p+1}) < a_{p+1}$. Since $a_p = b_p$ (because of our choice of p), and $f(a, a) = a$ for all a (this we have proved), we have $f(a_p, b_p) < a_{p+1} < a_p = f(a_p, a_p) = f(a_p, b_p)$. Therefore, in this case, $a_{p+1} \ll a_p$, $b_{p+1} \ll a_p$, and $f(a_{p+1}, b_{p+1}) < a_{p+1} < f(a_p, b_p)$. Hence, we have a proof that f is discontinuous (with $a = a_{p+1}, b = b_{p+1}, a' = a_p, b' = b_p$, and $a_{p+1} = c$). This contradicts to our assumption that f is continuous.

3.4.3°. Summarizing: in both cases the assumption that $p < k$ and $p < l$ led us to a contradiction. So, either $p = k$, or $p = l$, in which cases, as we have already proved, either $a = b$, or $a < b$, or $b < a$.

We have thus proved that L is linearly ordered.

4°. It now remains to prove that $f(a, b) = \min(a, b)$ for all a, b.

Since L is finite and linearly ordered, we can order all its elements into a sequence

$$F = a_0 < a_1 < \ldots < a_{n-1} < a_n = T.$$

So, each element of L has the form a_i, and $a_i < a_j$ if and only if $i < j$.

In these terms, it is necessary to prove that $f(a_i, a_j) = a_{\min(i,j)}$.

4.1°. If $i = j$, this follows from 2°.

4.2°. Let us now consider the case, when $i < j$, and prove that in this case, $f(a_i, a_j) = a_i$.

Let us fix j. For every i, the value of $f(a_i, a_j) \in L$ is equal to a_k for some k. Let us denote this k by $\phi(i)$. So, in these denotations, $f(a_i, a_j) = a_{\phi(i)}$. The desired equality can be then expressed as $\phi(i) = i$ for all $i \leq j$.

We already know the value of this function $\phi(i)$ for $i = 0$ and $i = j$: Indeed, since f is an &-operation, we have $f(T, a_j) = T$, i.e., in our notations, $f(a_0, a_j) = a_0$, hence $\phi(0) = 0$. From 2°, it follows that $f(a_j, a_j) = a_j$, so $\phi(j) = j$.

Since f is an &-operation, it is monotonically non-decreasing, hence ϕ is also non-decreasing:

$$0 = \phi(0) \leq \phi(1) \leq \phi(2) \leq \ldots \leq \phi(j) = j.$$

Since $a_i \ll a_{i+1}$, and f is continuous, there cannot be a gap between $F(a_i)$ and $F(a_{i+1})$. Therefore, for each i, we must either have $\phi(i+1) = \phi(i)$, or $\phi(i+1) = \phi(i) + 1$. Since

$$j = j - 0 = \phi(j) - \phi(0) =$$

$$(\phi(j) - \phi(j-1)) + \ldots + (\phi(2) - \phi(1)) + (\phi(1) - \phi(0)),$$

the number j is the sum of j differences, each of which is ≤ 1. If one of these differences was equal to 1, then the entire sum would be smaller than j. Since this sum is equal to j, none of these differences can be smaller than 1. Therefore, $\phi(i+1) - \phi(i) = 1$ for all i. This equality is equivalent to $\phi(i+1) = \phi(i) + 1$.

So, we have $\phi(0) = 0$, and $\phi(i+1) = \phi(i)+1$ for all $i < j$. From this, we can conclude (using mathematical induction), that $\phi(i) = i$ for all $i < j$. By definition of ϕ this means that $f(a_i, a_j) = a_{\phi(i)} = a_i$, i.e., that $f(a, b) = \min(a, b)$.

If $i > j$, then the desired equality follows from the fact that f is commutative ($f(a_i, a_j) = f(a_j, a_i)$), and so this case is reduced to the previous one. Q.E.D.

Comment. The ideas of this proof are similar to the proofs from [1, 12].

Proof of Theorem 2.8 is similar, with the only difference that we must use F instead of T, $>$ instead of $<$, and $\ll$ instead of $\gg$.

Proof of Theorem 3.2. For every $a_i \in L$, $f(a_i) = a_k$ for some k. Let us denote this k by $\psi(i)$. In these terms, $f(a_i) = a_{\psi(i)}$. The definition of a negation operation means that $\psi(0) = n$, and $\psi(n) = 0$. Continuity means that for each i, since $a_i \ll a_{i+1}$, there cannot be anything in between $a_{\psi(i)} = f(a_i)$ and $a_{\psi(i+1)} = f(a_{i+1})$. In other words, there cannot be anything in between $\psi(i)$ and $\psi(i+1)$. So, $\psi(i)$ and $\psi(i+1)$ must either coincide, or be neighbors: $|\psi(i+1) - \psi(i)| \leq 1$. In particular, $\psi(i+1) - \psi(i) \geq -1$.

Now, the difference $\psi(n) - \psi(0) = 0 - n = -n$ can be represented as

$$-n = \psi(n) - \psi(0) = (\psi(n) - \psi(n-1)) + \ldots + (\psi(2) - \psi(1)) + (\psi(1) - \psi(0)).$$

So, $-n$ is represented as the sum of n terms each of which is ≥ -1. If one of them was greater than -1, then the entire sum would have been greater than $-n$. Since this sum is equal to $-n$, we can conclude that all the terms in this sum are exactly equal to -1: $\psi(i+1)-\psi(i)=-1$. Therefore, $\psi(0)=n$, and $\psi(i+1)=\psi(i)-1$ for all i. From these two conditions, one can easily conclude that $\psi(i)=n-i$. Hence, $f(a_i)=a_{\psi(i)}=a_{n-i}$. Q.E.D.

Proof of Theorem 3.4. For every i and j, the value $f(a_i,a_j)$ belongs to L and is, therefore, equal to a_k for some k. Let us denote this k by $h(i,j)$, so that $f(a_i,a_j)=a_{h(i,j)}$.

We will consider two cases: $i\le j$, and $i>j$.

1°. Let us first assume that $i\le j$.

According to the definition of an $\to$-operation, $f(a_j,a_j)=T=a_n$, and $f(F,a_j)=f(a_0,a_j)=T=a_n$. In terms of h, it means that $h(j,j)=n$, and $h(0,j)=n$. From the fifth property of an $\to$-operation, we can conclude that

$$h(0,j)\ge h(1,j)\ge\ldots\ge h(j-1,j)\ge h(j,j).$$

Since $h(0,j)=h(j,j)=n$, we can conclude that all the terms in this inequality are equal to n, i.e., $h(i,j)=n$ if $i\le j$.

2°. Let us now consider the case when $i>j$.

According to the definition of a $\to$-operation, for every j, we have $f(T,a_j)=a_j$, and $f(a_j,a_j)=1$. In terms of h, this turns into $h(n,j)=j$ and $h(j,j)=n$. Since f is continuous, we can conclude (just like we did in the proofs of Theorems 1 and 3) that $|h(i+1,j)-h(i,j)|\le 1$. So, the difference between $h(n,j)$ and $h(j,j)$ that is equal to $j-n=-(n-j)$, can be represented as the sum of $n-j$ differences $h(i+1,j)-h(i,j)$ $(j\le i<n)$, each of which is ≥ -1. If one of these differences was >-1, then the entire sum would be $>-(n-j)$. Therefore, all these difference are equal to -1. So, $h(j,j)=n$, and for $i\ge j$, $h(i+1,j)=h(i,j)-1$. Therefore, for $i\ge j$, we have $h(i,j)=n-(i-j)=n+j-i$.

3°. Combining the cases $i\le j$ and $i>j$, we get the desired formula. Q.E.D.

5 Conclusions

Experts use words from natural languages to describe their degree of belief in their statements (e.g., "probably", "for sure", etc). If we want to use these degrees of belief in a computer-based expert system, we must be able to estimate the degree of belief in $A\,\&\,B$ based on the known degrees of belief in A and B. The function that performs this estimate is called an &-operation. The best way to choose an &-operation is to elicit and analyze the experts' degrees of belief in statements $A\,\&\,B$ for different A and B. However, this ideal procedure is very expensive and time-consuming, and is, therefore, in some cases not affordable. For such cases, when we cannot make an empirically justified choice of an &-operation, we need a theoretically justified choice.

In this paper, we formalize the natural demand that gradual changes in $t(A)$ and $t(B)$ must lead to gradual changes in our estimate for $t(A\,\&\,B)$ (we call it continuity). We show that the only continuous &-operation is $\min(a,b)$. Likewise, the only continuous $\vee$-operation is $\max(a,b)$, the only continuous "not"-operation corresponds to $f(a)=1-a$, etc.

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