A Mathematical Model for Determining the Optimal Production Lot Size of Multiple Products on a Machine

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Abstract
In this research, the production of multi-products using one machine is investigated in continuous time. The machine has limited capacity and can produce only one product at any time. To change the product, the machine should be set up. Due to the difference in demand for products, there is no need to equate the number of machine start-ups for these products, by removing this constraint, a nonlinear mathematical model is presented that gives the optimal production lot size for each product. To solve the single-constraint nonlinear model, the Lagrange method is used. For a numerical example, the obtained solution is compared with the method of rotating a constant cycle. Due to the total cost, the solution was better than the solution of the rotation cycle method. Also, contrary to the rotation cycle method, the total holding cost is equal to the total setup costs, which is similar to the Wilson inventory basic formula.

Keywords
Production Lot Size, Continuous Time, One Machine, Multi-product

1. Introduction

In manufacturing companies, productivity should be enhanced and costs should be reduced to survive in today's competitive market. Besides, timely delivery should be also taken into consideration as a vital goal for survival. Therefore, organizations strive to exploit machines at maximum capacity with their fixed capital along with one time delivery of products to customers.
The determination of production quantities and their scheduling is called the Economic Lot Scheduling Problem (ELSP) [1]. ELSP is a complicated problem and classified as NP-hard [2].
The objective of the ELSP is to determine a production schedule that minimizes the sum of inventory holding costs and setup costs.
A machine, with finite capacity, can produce multiple products. To change the product, the machine should be setup. Hence, the lot size of each product is intended to be determined at a minimum cost. The earliest paper discussed economic production quantity is Taft [3], and finite production rate [4]. In a study conducted by Goyal [5], the lot size of various products by a machine was investigated using the random demands for products, and the distribution of the lot size of each product was also determined. In another study done by Kono [6], the problem of determining the lot size of multi-product production by a machine was investigated through considering the product-related transport. Additionally, in a study performed by Hodgson et al. [7], the policies of dynamic lot size in single-machine production were studied and a semi-Markov model was presented. Moreover, a common

cyclic method with insufficient capacity for production was considered in a study done by Khoury et al. [8]. Besides, an integer hybrid model was presented by Cooke et al. [9] to find timely schedules. In some other studies such as the one performed by Yao [10], the problem was studied without capacity limitations, and Leven and Segerstedt [11] provided schedule policies to change the impossible answers to a feasible solution. A new structure was proposed by Nilson and Segerstedt [12] for the holding costs to make the costs more realistic. In another study done by Brander and Segerstedt [13], in addition to the set-up and holding costs, the costs of using machines dependent on time were also considered. Further, a nonlinear model was presented by Sun et al. [14] based on the basic cycle and 2 power policy. In a study performed by Chan and Chung [15], the genetic algorithm was applied to solve the model. In another study done by Gabay et al. [16], the problem was studied when the set-up costs depended on the product orders. Ilic and Radovic [17] analyzed EPQ in eight situations, in cases gradual or instantaneous replenishment, with or without delivery during the production period, and with or without shortages. They considered the same cycle for different products.

Al-Salamah [18] considered a production-inventory system consisting of a single imperfect unreliable machine. The items manufactured may require a rework to be restored to perfect quality. A single-variable expected average cost function is derived to find the optimal lot size. Because of the complexity in the model, the ABC heuristic is proposed and implemented to find a near-optimal value for the lot size. Zhang et al. [19] investigated the problem of single machine lot scheduling where each lot contains one or more jobs and is of identical processing time. They proved that the WSPT (Weighted Shortest Processing Time first) rule is optimal for both models of minimizing total weighted completion time and minimizing total weighted discounted completion time. Zheng and Jin [20] studied the problem of single machine lot scheduling to minimize the total completion time of jobs. Each processing lot has a uniform capacity and is of identical processing time. They proposed an improved Best Fit Random (BFN) algorithm named IBFN, which makes a refined adjustment of job assignment, based on the BFN schedule, to reduce the spare space of each lot as much as possible. Rios-Solis et al. [21] studied a lot-sizing and scheduling problem to maximize the profit of assembled products over several periods. The setting involves a plastic injection production environment where pieces are produced using auxiliary equipment (molds) to form finished products. Each piece may be processed in a set of molds with different production rates on various machines. They developed a two-stage iterative heuristic based on mathematical programming. First, the lot-size of the products is determined together with the mold-machine assignments. And the second stage determines if there is a feasible schedule of the molds with no overlapping.

In this paper, the assumption of the constant rotation cycle is violated for products and a certain cycle is obtained for each of the products. Each product may be produced one time or more. It should be noted that none of the products confront shortage, and given the different demands for products, the different production cycles and the difference in the number of machine set-ups for each product in the cycle are real and logical.

The rest of the paper is organized as follows. In section 2 and 3 assumptions and notation of the paper is introduced. The mathematical model is developed in section 4. The exact mathematical solution is obtained in section 5. To illustrate model application, a numerical example is given in section 6 and compare with traditional constant rotation cycle policy. Finally, the conclusion and some future research recommendations have presented in section 7.

1.1 Assumptions

- Demand is constant over time.
- There is a machine that can produce different products.
- The machine can produce only one item at a time.
- The start of the production of each product requires time and cost of machine setup.
- The setup cost of the machine and holding cost of the products are considered. Of course, the cost of production is constant and not considered.
- The machine setup costs are independent of the production sequences.
- In the entire period, all products are consumed at all times, and the shortage is not allowed at any time during the period.

1.2 Notations

n: The number of different type of products

D_j: The demand of product j per unit time

P_j : The production rate of product j per unit time

A_j: The machine setup cost for production j^{th} product

h_j: The carrying cost of a unit of product j per unit time

s_j: The required time for machine Setup time to production j^{th} product

N_j: The number of machine setups for product j

X_j : The lot size of product j^{th} after each setup

TP_j: The length of time to produce the j^{th} product after each machine setup

2. Mathematical model

A machine is intended to produce the total demands for different products, and shortages are not allowed. In other words, any product is in stock at any time, and as soon as an item is out of stock, its production will resume (Figure 1). Contrary to the constant rotation cycle method, there is no requirement for the same number of production times for products. And as a result, products that their setup costs are high compared to the holding cost, fewer runs will be launched and Instead, more lot size of production for them is determined.

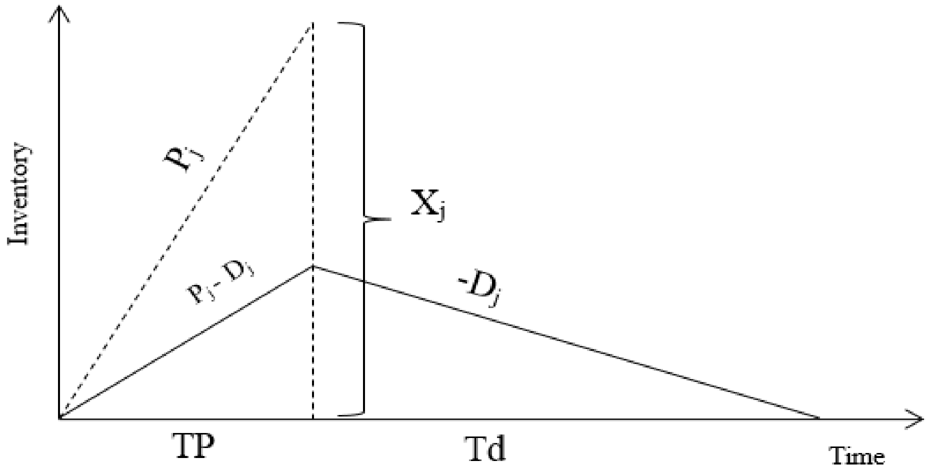

Figure1. Inventory behavior in a cycle

The total cost includes setup and holding costs. Because the total variable cost of production is constant, in determining the lot size of the production, it does not affect and is ignored.
The cost of preparation of each product is equal to the number of times that the product is setup multiplied by the cost of setting up the machine for each time the product is going to be produced. The number of times the jth product is launched is calculated as follows:

$$N_j = \frac{D_j}{X_j} \qquad (1)$$

Therefore, the costs of machine preparation for the production of jth product is $\frac{A_j D_j}{X_j}$. Each time the machine is launched for the product j, the number of X_j units of this product are produced. But at the same time as the production, this product is consumed; therefore, the whole inventory will not accumulate in the warehouse, and the maximum visible inventory in the warehouse will be$(P_j - D_j)TP_j$. Because$TP_j = X_j/P_j$, maximum warehouse inventory is equal to $\frac{(P_j - D_j)X_j}{P_j} = X_j(1 - D_j/P_j)$, And since the minimum inventory of this product is zero, the average inventory of product j will be$X_j(1 - D_j/P_j)/2$. And the total cost of all whole products is calculated as follows:

$$TC = \sum_{j=1}^{n} \left[\frac{A_j D_j}{X_j} + h_j X_j (1 - D_j/P_j)/2\right] \qquad (2)$$

On the other hand, the production capacity of the machine is limited. Therefore, the required time to produce X_j unit of jth product is as follows:

$$TP_j = X_j/P_j \qquad (3)$$

Due to the number of machine setups for the production of j'th product, and as well as that at each time production, set up time of s_j is necessary, the total time of production and setup of product j will be as follows:

$$\left(\frac{X_j}{P_j} + s_j\right) = \frac{D_j}{X_j}\left(\frac{X_j}{P_j} + s_j\right) = \frac{D_j}{P_j} + \frac{s_j D_j}{X_j} \qquad (4)$$

And thereupon, considering that the total production and setup times for products should not exceed one period, the following constraint is defined:

$$\sum_{j=1}^{n}\left(\frac{D_j}{P_j}+\frac{s_j D_j}{X_j}\right) \leq 1 \qquad (5)$$

Therefore, according to the obtained equations, the following model is proposed to minimize costs:

$$Min \quad TC = \sum_{j=1}^{n}\left[\frac{A_j D_j}{X_j}+\frac{h_j X_j\left(1-\frac{D_j}{P_j}\right)}{2}\right]$$

s.t.

$$\sum_{j=1}^{n}\left(\frac{D_j}{P_j}+\frac{s_j D_j}{X_j}\right) \leq 1$$

$$X_j \geq 0$$

3. Solving the model

The proposed mathematical model is nonlinear, and Lagrange method is used to solve it. To this end, the objective function is multiplied by -1 and the maximum of concave function is obtained by Lagrange method, which is equivalent to minimum point of previous function. The Lagrange function is as follows:

$$L(X_1, X_2, \dots, X_n, \lambda) = -\sum_{j=1}^{n}\left[\frac{A_j D_j}{X_j}+\frac{h_j X_j\left(1-\frac{D_j}{P_j}\right)}{2}\right] + \lambda\left[\sum_{j=1}^{n}\left(\frac{D_j}{P_j}+\frac{s_j D_j}{X_j}\right)-1\right] \qquad (6)$$

To minimize the function above, the partial derivative is taken to X_j and λ and set it to zero:

$$\frac{\delta L}{\delta X_j} = \frac{\delta L}{\delta \lambda} = 0 \qquad (7)$$

Therefore, the following relationships are obtained:

$$\frac{(A_j-\lambda s_j)D_j}{X_j^2} - \frac{h_j\left(1-\frac{D_j}{P_j}\right)}{2} = 0 \qquad (8)$$

$$\sum_{j=1}^{n}\left(\frac{D_j}{P_j}+\frac{s_j D_j}{X_j}\right) = 1 \qquad (9)$$

The Hessian function is as follows:

$$H = \begin{bmatrix} \frac{-2(A_1-\lambda S_1)D_1}{X_1^3} & \cdots & \frac{-S_1D_1}{X_1^2} \\ \vdots & \ddots & \vdots \\ \frac{-S_1D_1}{X_1^2} & \cdots & 0 \end{bmatrix} \qquad (10)$$

In the Hessian matrix, except the last row of the last column, only the values of the main diameter are non-zero. The determinant of the Hessian matrix calculated as follows:

$$|H| = (-1)^{2n+1}\prod_{i=1}^{n-1}\left[\frac{-2(A_i-\lambda S_i)D_i}{X_i^3}\right]\left(\frac{-S_iD_i}{X_i^2}\right) = (-1)^{3n+1}\prod_{i=1}^{n-1}\left[\frac{2(A_i-\lambda S_i)D_i}{X_i^3}\right]\left(\frac{S_iD_i}{X_i^2}\right) \qquad (11)$$

The sign of $|H|$ has assigned as $(-1)^{n+1}$. Therefore, it can be concluded that the function is concave, and the obtained point is the maximum point, so it is the minimum point for model 1.

To get the solution, X_j is obtained from relation (8) and put it in relation (9), as follows:

$$\sum_{j=1}^{n} D_j s_j\sqrt{\frac{h_j(1-\frac{D_j}{P_j})}{2D_j(A_j-\lambda s_j)}} = 1-\sum_{j=1}^{n}\frac{D_j}{P_j} \qquad (12)$$

The equation (12) is a single-variable equation, and after obtaining λ, X_j can be calculated from equation (8). It should be noted that solving the equation (12) is complicated and it should be solved using numerical methods. After obtaining λ, the lot size of jth product calculated as:

$$X_j = \sqrt{\frac{2D_j(A_j-\lambda s_j)}{h_j\left(1-\frac{D_j}{P_j}\right)}} \qquad (13)$$

4. Numerical example

Suppose a machine that can produce four different products, and other required information given in Table 1[22].

Table1. The data of the numerical example

Holding cost of each unit per year	Setup time (year)	Setup Cost	Production rate (unit/year)	Annual Demand	Product
2	0.001	50	10,000	3000	A
3	0.002	70	5,000	2000	B
1	0.005	120	50,000	5000	C
4	0.003	80	10,000	1000	D

First, the problem is solved using the constant rotation cycle method, and then it is solved using the obtained formula, and the results are compared with each other. In the constant rotation cycle method, $T = 0.20$ was obtained and the lot size and other items were calculated (Table 2). In solving by obtained formula, λ measured 0.002118 and other items calculated due to it.

Table2. Solving of the problem using the Constant Rotation Cycle and the Optimal metho

Optimal method				Rotation cycle method				Product
TC_j	TP_j	X_j	T_j	TC_j	TP_j	X_j	T	
648	0.05	463	0.15	670	0.06	600	0.20	A
710	0.08	394	0.20	710	0.08	400	0.20	B
1039	0.02	1155	0.23	1050	0.02	1000	0.20	C
759	0.02	211	0.21	760	0.02	200	0.20	D
3156				3190				Total

5. Conclusion

In this research, the assumption of constant rotation cycle for all products is ignored and studies a case that each product has its special cycle time. According to the removal of this condition, the solution to the problem is improved, and fortunately, a mathematically exact solution has been obtained. In the presented optimal solution, the total cost was reduced in proportion to the Constant Rotation Cycle method, and this show model improvement. Due to different product demands, Different production cycles, and thereupon different the number of product machine setup is quite natural. Notable point is that, in the presented optimal model, for each product, the total holding cost equals the total setup cost, which is the same as Wilson's basic inventory formula, but they aren't equal in the rotation cycle method. The demand for one product may be very little, or the setup cost is high than others, and therefore it will be not logical that has the number of setups as same as other products.

To avoid the error of rounding of numbers, the model can be solved considering integer variables with an OR software like LINGO. Furthermore, the probability and uncertainty of considering demand can be studied in future studies. And multi-machine problems can be also considered.

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