Robust DEA Models for Performance Evaluation of Systems with Continuous Uncertain Data under CRS and VRS Conditions

Mohammad Amirkhan*

Abstract – One of the most appropriate and efficient methods for evaluating the performance of homogenous decision-making units (DMU) is data envelopment analysis (DEA). Traditional DEA models are only able to evaluate DMUs with deterministic inputs and outputs, while in real-world problems, data are usually uncertain. So far, various approaches have been introduced to overcome the uncertainty of data. In this paper, two robust DEA models is presented to evaluate the performance of systems with continuous uncertain data under constant return to scale (CRS) and variable return to scale (VRS) conditions. The main advantage of the proposed robust DEA models over the previous robust DEA models is that they are able to formulate uncertainty in both input and output data. Moreover, these models are also developed directly on basic traditional DEA models (not alternative models). To demonstrate the applicability of two developed robust models, a numerical example is presented and the efficacy of models is exhibited.

Keywords: Performance Evaluation, Ranking, Efficiency, Data Envelopment Analysis, Robust Optimization

1. Introduction

Today, in order to outdo competitors, it is necessary for every organization to be fully aware of its true position. To determine their position, organizations must carefully evaluate the current situation and their performance. Many methods for measuring efficiency have been proposed in previous studies. But compared to all models, data envelopment analysis (DEA) is a better way to organize and analyze data. Because it allows the performance changes over time and does not require any assumptions about the efficiency frontier. It is also possible to include multiple inputs and multiple outputs for each decision-making unit (DMUs)[1]-[2]. DEA is one of the most important non-parametric techniques in measuring performance. It uses the mathematical modeling to measure the relative efficiency of DMUs under evaluation.

The efficiency frontiers obtained by DEA is sensitive to perturbations as well as their uncertainties of data, and if the data is uncertain, the efficiency frontiers may be shifted and the DEA results may be invalid. The ambiguity of the data can be due to various reasons such as uncertainty, unmeasurable information, incomplete and unattainable information, inconsistent and contradictory information, partial and incomplete truth, etc. [3]. This problem has led to different approaches to control the uncertainty of the DEA models under diverse situations. Robust optimization, stochastic programming and fuzzy set theory are the most important approaches in this field.

Robust optimization is a way to deal with the uncertain parameters of an optimization problem. In fact, uncertainty sets are the space for changing the uncertain parameters that contain all possible values for the uncertain parameters. The robust optimization approach optimizes the worst-case scenarios that may occur for the uncertain parameters. This approach gives equal importance to all points of the space of the uncertain parameters and produces the solutions that maintain optimality and feasibility of each member belonging to the sets of uncertainty. The concept of robustness means that the model outputs are not very sensitive to the exact values of parameters and inputs [4]-[5].

Robust optimization is an alternative approach to sensitivity analysis and stochastic programming that is more practical and flexible than the latter two approaches.

***Corresponding Author:** Department of Industrial Engineering, Aliabad Katoul Branch, Islamic Azad University, Aliabad Katoul, Iran. Email: m.amirkhan.ie@gmail.com

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If the robust optimization approach is used in DEA models, the ranking of DMUs will not change due to small noises in the problem inputs and outputs [6].

Robust optimization approaches are generally divided into two categories: discrete and continuous. In the discrete approach, the data are scenario-based, while in the continuous mode, data is noisy.

When in mathematical programming models the parameters are as stochastic variables, probabilistic programming is used to solve the problems. In the traditional DEA model, all inputs and outputs are assumed to be accurately measured, whereas, due to the inherent uncertainty of many real-world problems, the inputs and outputs of DMUs may be stochastic in nature, and the performance appraisal process may sometimes include stochastic estimates. Therefore, a model should be presented that, while calculating the efficiency of the system, also takes into account the stochastic nature of the problem parameters. In the random approach, the uncertain data of the problem are considered as random variables, and it is assumed that the probability distribution function of the stochastic processes is known [7]. Also, when one or more of the limiting equations in the linear programming model can be satisfied for a certain amount of probability, the chance-constraint programming can be used as an effectual approach [8].

Fuzzy set theory [9] is another approach to dealing with uncertainty. If the input data of the problem is qualitative, linguistic data, etc., the use of this approach is efficient [3]. The fuzzy DEA model combines the concept of fuzzy set theory with the conventional DEA model. There are the different approaches to apply fuzzy set theory to the DEA models, so that most of the obtained fuzzy DEA models are as mathematical linear programming. In the DEA models, the use of the fuzzy approach is important from two aspects. In the first aspect, the parameters of the problem are considered fuzzy and in the second aspect, the tolerance level is defined on the objective function and the violation of constraints.

In general, if input data of the problem is not accurate and is as qualitative, linguistic, etc., the fuzzy approach is utilized and also, if the parameters of the problem are random variables and the possible distribution of these parameters is known, Probabilistic programming is used and moreover, if the possible distribution of the random parameters of the problem is not available and the decision-maker seeks the model outputs that are not very sensitive to the exact values of the parameters and inputs, the robust optimization approach is employed.

In the classical DEA models, data is assumed to be accurate. Therefore, these models were not able to deal with the uncertain data. Given that in real-world problems, some data are inherently inaccurate, ambiguous, and uncertain, approaches are needed to control this uncertainty in the modeling process. Many methods have been proposed to control the uncertainty in optimization problems. Fuzzy set theory, stochastic programming and robust optimization are the most important approaches to deal with this type of problems that have attracted much attention in recent years.

One of the main challenges in applying the traditional DEA approach to real-world problems is the existence of interference and uncertainty in some input/output data of DMUs. In other words, the traditional DEA approach is a data-driven one that calculates the performance of DMUs using the efficiency frontier generated by the inputs and outputs of DMUs. Hence, any noise in the input/output data of DMUs causes the performance frontier to shift, and consequently the performance values of the DMUs may change.

Much research has been done to develop robust optimization methods to consider uncertainty corresponding to data in mathematical models. Researchers believe that uncertainty can affect both the optimality and the feasibility of solutions. Usually in the various problems, the best data estimation, called nominal data, is used for the mathematical models. Robust optimization was first introduced by Soyster [10] in 1973. This model is based on a pessimistic approach and is very conservative, so that to ensure the robustness of the solution, the optimal solution of the problem is very far from its optimality.

The model presented by Soyster [10] offers the best feasible solution for all input data, so that each input data can adopt any value of the certain interval. After the Soyster's paper [10], a great deal of research has been done to provide the robust models for dealing with the optimization problems with the uncertain data. El ghaoui et al. [11] proposed a robust optimization model that had two main problems. First, it increased the computational complexity of the problem compared to the original model, and second, it did not guarantee the probability of feasibility for the robust model. Ben-Tal and Nemirovski [4] proposed a quadratic conical model as a robust counterpart of a linear programming model for data under conditions of elliptical uncertainty. This model is less conservative than the Soyster's model [10] and offers more acceptable solutions. Ben-Tal and Nemirovski [12] introduced a robust optimization model. Because this model was nonlinear, it was difficult to find optimal solutions. Bertsimas and Sim [5] presented a model in which there was an interaction between optimality and robustness. Their model was a linear one that moderated the level of conservatism of the robust solutions. Linearity, the ability to control the level of conservatism of robust solutions as well as usability in integer problems are the main features of the Bertsimas and Sim's model [5]. The main advantage of the robust optimization approach developed by Bertsimas and Sim [5] is that when used on linear models, the final robust model still retains its linearity. In addition, this approach is not very conservative, unlike Soyster's approach [10], and the degree of conservatism can be determined by the decision-maker. The application of the robust optimization approach in DEA was first done by Sadjadi and Omrani [6]. They presented two robust DEA models to evaluate the performance of electricity distribution companies in Iran. The first model is based on the Ben-Tal and Nemirovski's [4] approach, and the second model is according to Bertsimas and Sim's [5] approach. It should be noted that in the models presented by them, the input data of DMUs are considered as certain and only the output data of DMUs are considered as uncertain. Sadjadi and Omrani [13] developed a bootstrap robust DEA model to assess and rank the telecommunications companies. In their model, only the uncertainty of the output data is considered. They first obtained the performance values of DMUs with noisy outputs and then, using the bootstrap technique, calculated the modified efficiency scores. Sadjadi et al. [14] presented a robust super-efficiency DEA model for ranking provincial gas companies. Using the Ben-Tal and Nemirovski' [4] approach, they developed a nonlinear programming model in the envelopment form. Based on the robust scenario-based optimization approach introduced by Mulvey et al. [15], Hafezalkotob et al. [16] presented a DEA model with discrete inputs and outputs to calculate the relative efficiency of 38 electricity distribution companies in Iran. They considered three scenarios (pessimistic, most likely, and optimistic) for the data and then, determined a coefficient for the probability of occurrence of each scenario. Using the approaches of Bertsimas et al. [17] and Bertsimas and Sim [5], Peykani et al. [18] developed three robust DEA models and then implemented these models in a data set derived from the stock market. The results of their research showed that in all three models, the efficiency values of the robust models are less than or equal to the efficiency values of the equivalent deterministic models. In addition, with increasing the level of conservatism, the number of efficient DMUs has decreased. Esfandiari et al. [19] examined the uncertainty of input, intermediate and output data of two-stage processes. They declared that a small disturbance in the problem parameters severely affects the efficiency values of the whole process as well as each of the sub-processes. Using the robust optimization approach of Mulvey et al. [15], they were able to develop a two-stage DEA models in the cooperative and non-cooperative modes under uncertainty conditions and calculate the values of the process efficiency for different scenarios. Yousefi et al. [20] used a hybrid approach including robust scenario-based optimization, dynamic DEA, and the ideal DMU to evaluate sustainable suppliers. Their proposed approach was able to provide a suitable model for efficient and inefficient DMUs and also allows future planning for inefficient DMUs. Rabbani et al. [21] used two models, robust DEA and robust DEA with common weight, to evaluate the performance and rank of 56 mines. Amirkhan et al. [22] introduced a hybrid approach based on fuzzy and discrete robust optimization to cope with the uncertainty in the CCR and BCC DEA models. Peykani et al. [23] reviewed 73 studies (from 2008 to 2019) in the field of robust DEA. They surveyed and classified the robust approaches used in DEA models. Omrani et al. [24] introduced a model based on the best worst method and robust DEA for incorporating decision-makers' preferences into the classic DEA model. This model was bi-objective and formulated under uncertainty condition. Tavana et al. [25] presented a robust interval cross-efficiency model to rank DMUs. They declared that the proposed model avoids problems of non-unique optimal weights and uncertain data.

Hitherto, researchers have developed many robust DEA models using the Bertsimas and Sim's approach [5]. It is noteworthy that in the previous proposed models, none of the previous models directly used the Bertsimas and Sim's approach [5] on the basic DEA models, so that both input and output data are uncertain. It is worth noting that since the basic DEA models including the CCR and BCC ones have equality constraints, the Bertsimas and Sim's approach [5] cannot be used. A review of previous research shows that researchers have adopted two approaches to overcome this issue. The first approach involves research in which researchers used alternative models instead of the basic DEA models. The second approach involves research in which researchers have not considered uncertainty in input and output data simultaneously. In other words, in this type of research, either input or output data are uncertain and both of them do not meet these conditions at the same time.

In the present study, using the Bertsimas and Sim's approach [5], two robust DEA models have been developed under constant return to scale (CRS) and variable return to scale (VRS) conditions. In the proposed models, both input and output data have uncertainties. In addition, an efficient approach to overcome the issue of the equality constraint in the basic DEA models is proposed. It should be noted that in the present study, the Bertsimas and Sim's approach [5] is been implemented directly on the basic DEA models (and not alternative models). Meanwhile, the proposed models are able to take into account the uncertainty in both input and output data simultaneously.

The rest of the research is organized as follows. Section 2 describes the research background including DEA and Robust Optimization, and the proposed robust models are presented in Chapter 3. Chapter 4 presents a numerical example and then, analyzes the results of implementing the developed robust models on the example. Finally, the results of the research are presented in Chapter 5.

2. Background

2.1. Data Envelopment Analysis

Based on the returns to scale conditions, the basic DEA models are divided into two categories: CRS models and VRS models. In the following, the basic DEA models related to CRS and VRS conditions are briefly described.

If the goal is to evaluate the efficiency of *n* DMUs with *m* inputs and *s* outputs (as shown in Figure 1), the efficiency of DMU_j is calculated as Equation (1) [26]:

$$\text{Efficiency of } DMU_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \tag{1}$$

Where x_{ij} and y_{rj}, represent the i^{th} input $(i = 1,...,m)$ and the r^{th} output $(r = 1,...,s)$ of DMU_j $(j = 1,...,n)$, respectively. Also, v_i and u_r are the weighted variables of the i^{th} input and the r^{th} output of DMU_j, respectively.

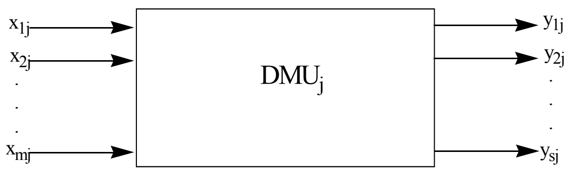

Figure 1. DMU_j with *m* inputs and *s* outputs

Based on the above concept, Charnes, Cooper, and Rhodes [26] proposed the CCR Model (2), derived from the initials of their names, to calculate performance of DMU_o. Model (2) is under the CRS conditions and is known as the multiplier input-oriented CCR model. In model (2), E_o denote the efficiency value of DMU_o.

$$\begin{aligned} Max \quad & E_o = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j \\ & v_i, u_r \ge 0, \quad \forall i, j \end{aligned} \tag{2}$$

The multiplier input-oriented BCC model proposed by Banker, Charnes, and Cooper [27] is as (3).

$$\begin{aligned} Max \quad & E_o = \sum_{r=1}^{s} u_r y_{ro} + w \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} + w - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j \\ & v_i, u_r \ge 0, \quad \forall i, j \\ & w, \quad \text{is free in sign} \end{aligned} \tag{3}$$

Model (3) is presented under the VRS conditions.

2.2. Robust Optimization

In mathematical programming, problems are solved based on default that data is crisp and certain, while in the real-world problems most data are usually uncertain. In these cases, due to the very small change in one of the data, a large number of constraints may be violated and even the solutions obtained may be non-optimal or even infeasible. Robust optimization is one of the approaches that operates very effectively in such situations. Robust optimization is an alternative approach to the sensitivity analysis and the stochastic programming methods. The main advantage of this approach is its practicality and flexibility.

Ben-Tal and Nemirovski ([4], [12], [17]), El Ghaoui and Lebert [11] and Mulvey et al. [15] presented various models in the field of robust optimization. Ben-Tal and Nemirovski [12] by studying and examining a sample problem showed that a small disturbance in the input data causes the previous optimal solution of the corresponding problem to be infeasible. Bertsimas and Sim [5] proposed an approach in which the final robust model still retains its linearity. Unlike the Soyster's model [10], this approach is not very conservative and also, the conservative level can be determined by the decision-maker.

To better describe the Bertsimas and Sim's [5] approach, consider the linear programming Model (4):

$$\begin{aligned} \max \quad & z = cx \\ & \tilde{a}_i x \le b_i, \qquad \forall i \\ & l_j \le x_j \le u_j, \qquad \forall j \end{aligned} \tag{4}$$

In Model (4), it is assumed that only the elements of matrix *A* are uncertain. Suppose that *i* is a row of matrix *A* and J_i is a set of coefficients with uncertainty related to row *i*.

for each $j \in J_i$, a_{ij} is defined as a bounded and symmetric stochastic variable $\tilde{a}_{ij}$ in the interval $\left[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}\right]$. Corresponding to each variable $\tilde{a}_{ij}$, a random variable $\eta_{ij} = (\tilde{a}_{ij} - a_{ij}) / \hat{a}_{ij}$ is defined by an unknown but asymmetric distribution in the interval $[0,1]$.

Consider the constraint *i* of Model (4) as $\tilde{a}_i x \le b_i$. For each *i*, the parameter Γ_i, which is not necessarily an integer number and is in the interval $\left[0, |J_i|\right]$, is defined. This parameter sets the robustness rate of the proposed method against the conservatism level of the solution. The proposed counterpart robust model of Bertsimas and Sim [5] is as (5).

$$\begin{aligned} \max \quad & z = cx \\ & a_i x + \Gamma_i p_i + \sum_{j \in J_i} q_{ij} \le b_i, \qquad \forall i \\ & p_i + q_{ij} \ge \hat{a}_{ij} y_j, \qquad \forall i, j \\ & -y_j \le x_j \le y_j, \qquad \forall j \\ & l_j \le x_j \le u_j, \qquad \forall j \\ & p_i, q_{ij}, y_j \ge 0, \qquad \forall i, j \end{aligned} \tag{5}$$

3. Proposed Robust DEA Models

In this section, the approach introduced by Bertsimas and Sim [5] is used to develop two robust DEA models under the CRS and VRS conditions. The proposed models are able to take into account uncertainty in both input and output data.

The main problem in applying the Bertsimas and Sim's [5] approach for CCR and BCC models is the existence of equality constraints in these models. It should be noted that the Bertsimas and Sim's [5] approach cannot be used for equality constraints. To solve this problem, Lemma 1 and Lemma 2 are presented.

Lemma 1. Model (2) is equivalent to Model (6).

$$\begin{aligned} Max \quad & E_o = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{i=1}^{m} v_i x_{io} \le 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad \forall j \\ & v_i, u_r \ge 0, \qquad \forall i, j \end{aligned} \tag{6}$$

Proof. First, it is necessary to write the dual model corresponding to Model (2). This model which is called the envelopment input-oriented CCR one, is as (7).

$$\begin{aligned} Min \quad & E_o = \theta \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \qquad \forall i \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \qquad \forall r \\ & \lambda_j \ge 0, \qquad \forall j \\ & \theta \text{ is free in sign} \end{aligned} \tag{7}$$

The parameter x_{ij} ($\forall i, j$) is always positive and in addition, the variable λ_j ($\forall j$) is always greater than or equal to zero. considering the first constraint of Model (7), it can be concluded that $\theta \ge 0$. Therefore, Model (7) is equivalent to Model (8).

$$\begin{aligned} Min \quad & E_o = \theta \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \qquad \forall i \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \qquad \forall r \\ & \lambda_j \ge 0, \qquad \forall j \\ & \theta \ge 0 \end{aligned} \tag{8}$$

Now, if the dual form corresponding to Model (8) is rewritten, Model (9) is obtained.

$$\begin{aligned} Max \quad & E_o = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{i=1}^{m} v_i x_{io} \le 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad \forall j \\ & v_i, u_r \ge 0, \qquad \forall i, j \end{aligned} \tag{9}$$

Lemma 2. Model (3) is equivalent to Model (10).

$$
\begin{aligned}
Max \quad & E_o = \sum_{r=1}^{s} u_r y_{ro} + w \\
& \sum_{i=1}^{m} v_i x_{io} \le 1 \\
& \sum_{r=1}^{s} u_r y_{rj} + w - \sum_{i=1}^{m} v_i x_{ij} \le 0, && \forall j \\
& v_i, u_r \ge 0, && \forall i, j \\
& w, && \text{is free in sign}
\end{aligned} \tag{10}
$$

Proof. Such as the proof of lemma 1.

Since the problem of equality constraints in CCR and BCC models has been solved, using the Bertsimas and Sim's [5] approach, the robust CCR Model (11) and the robust BCC Model (12) can be developed.

$$
\begin{aligned}
Max \quad & z = \sum_{r=1}^{s} u_r y_{ro} - \Gamma^0 z^0 - \sum_{r \in J_r} p_r^0 \\
& \sum_{i}^{m} v_i x_{io} + \Gamma^1 z^1 + \sum_{i \in J_i} p_i^1 \le 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + z_j^2(\Gamma_j^2 + \Gamma_j^3) + \sum_{r \in J_r} p_{rj}^2 + \sum_{i \in J_i} p_{ij}^3 \le 0, && \forall j \\
& z^0 + p_r^0 \ge \varepsilon_{ro}^y y_{ro} q_r^0, && \forall r \in J_r \\
& -q_r^0 \le u_r \le q_r^0, && \forall r \in J_r \\
& z^1 + p_i^1 \ge \varepsilon_{io}^x x_{io} q_i^1, && \forall i \in J_i \\
& -q_i^1 \le v_i \le q_i^1, && \forall i \in J_i \\
& z^2 + p_{rj}^2 \ge \varepsilon_{rj}^y y_{rj} q_{rj}^2, && \forall r \in J_r \\
& -q_{r,j}^2 \le u_r \le q_{r,j}^2, && \forall r \in J_r \\
& z_j^2 + p_{ij}^3 \ge \varepsilon_{ij}^x x_{ij} q_{ij}^3, && \forall i \in J_i \\
& -q_{ij}^3 \le u_r \le q_{ij}^3, && \forall i \in J_i \\
& v_i, u_r, z^0, z^1, z_j^2, p_r^0, p_i^1, p_{rj}^2, p_{ij}^3, q_r^0, q_i^1, q_{rj}^2, q_{ij}^3 \ge 0
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
Max \quad & z = \sum_{r=1}^{s} u_r y_{ro} + w - \Gamma^0 z^0 - \sum_{r \in J_r} p_r^0 \\
& \sum_{i}^{m} v_i x_{io} + \Gamma^1 z^1 + \sum_{i \in J_i} p_i^1 \le 1 \\
& \sum_{r=1}^{s} u_r y_{rj} + w - \sum_{i=1}^{m} v_i x_{ij} + z_j^2(\Gamma_j^2 + \Gamma_j^3) + \sum_{r \in R_j} p_{rj}^2 + \sum_{i \in R_i} p_{ij}^3 \le 0, && \forall j \\
& z^0 + p_r^0 \ge \varepsilon_{ro}^y y_{ro} q_r^0, && \forall r \in J_r \\
& -q_r^0 \le u_r \le q_r^0, && \forall r \in J_r \\
& z^1 + p_i^1 \ge \varepsilon_{io}^x x_{io} q_i^1, && \forall i \in J_i \\
& -q_i^1 \le v_i \le q_i^1, && \forall i \in J_i \\
& z^2 + p_{rj}^2 \ge \varepsilon_{rj}^y y_{rj} q_{rj}^2, && \forall r \in J_r \\
& -q_{r,j}^2 \le u_r \le q_{r,j}^2, && \forall r \in J_r \\
& z_j^2 + p_{ij}^3 \ge \varepsilon_{ij}^x x_{ij} q_{ij}^3, && \forall i \in J_i \\
& -q_{ij}^3 \le u_r \le q_{ij}^3, && \forall i \in J_i \\
& v_i, u_r, z^0, z^1, z_j^2, p_r^0, p_i^1, p_{rj}^2, p_{ij}^3, q_r^0, q_i^1, q_{rj}^2, q_{ij}^3 \ge 0 \\
& w \quad \text{is free in sign}
\end{aligned} \tag{12}
$$

In Models (11) and (12), ε_{ij}^x and ε_{rj}^y represent the disturbances percentage of parameters x_{ij} and y_{rj}, respectively and also, Γ^0 and Γ^1 denote the robustness moderator parameters of the objective function and the first constraint against the conservatism level, respectively. Also, in the second constraint, Γ_j^2 and Γ_j^3 represent the robustness modulators of the parameters y_{rj} and x_{ij} against the conservatism level of solution, respectively.

4. Numerical Example

In this section, to show the applicability and efficacy of the proposed models, a numerical example is presented and then, the example is solved using the two presented robust models. In this example, 20 DMUs with 3 inputs and 2 outputs are considered. The data are listed in Table 1.

Table 1. Data of Numerical Example

DMU	Inputs			Outputs	
	i_1	i_2	i_3	r_1	r_2
1	37	133	405	444	67
2	72	988	698	283	92
3	16	766	450	961	33
4	86	440	251	386	97
5	69	301	801	586	82
6	54	513	223	629	87
7	41	923	425	208	66
8	46	596	258	981	95
9	12	101	122	827	94
10	75	767	647	858	66
11	20	863	942	738	43
12	75	249	575	926	21
13	25	928	179	242	24
14	40	857	176	239	13
15	94	945	998	446	11
16	48	168	194	785	75

17	61	772	364	893	76
18	72	188	311	166	61
19	66	684	603	990	95
20	49	805	517	774	36

All models in the current paper have been coded in GAMS 24.0.1 solver software. Moreover, Excel 2019 has been used for drawing pictures. In this example, the values of all the ε^x s and ε^y s are set to 0.05. In addition, since all the input and output data are uncertain, the values Γ^0, Γ^1, Γ^2, and Γ^3 are set to 3, 2, 3, and 2, respectively.

The solutions obtained from solving the models presented in the previous section are summarized in Table 2.

Table 2. Results of DEA Models (2)-(3) v.s. Robust DEA Model (11)-(12)

DMU	Constant Return to Scale		Variable Return to Scale	
	CCR	Robust CCR	BCC	Robust BCC
j01	0.54	0.44	0.76	0.69
j02	0.17	0.14	0.17	0.16
j03	0.87	0.71	1	0.75
j04	0.50	0.41	1	0.44
j05	0.29	0.24	0.34	0.30
j06	0.51	0.41	0.55	0.49
j07	0.21	0.17	0.29	0.26
j08	0.56	0.46	1	0.62
j09	1	0.82	1	0.90
j10	0.20	0.16	0.26	0.17
j11	0.54	0.44	0.60	0.54
j12	0.45	0.37	1	0.43
j13	0.20	0.16	0.68	0.62
j14	0.20	0.16	0.69	0.63
j15	0.07	0.06	0.13	0.12
j16	0.60	0.49	0.63	0.57
j17	0.36	0.30	0.50	0.3
j18	0.35	0.29	0.54	0.49
j19	0.24	0.20	1	0.41
j20	0.23	0.19	0.24	0.22

The second and fourth columns of Table 2 show the efficiency values of the CCR and BCC models, respectively. Applying Models (11) and (12) for the data presented in Table 1, the efficiency values of DMUs can be calculated for the two robust CCR and robust BCC models. The third and fifth columns of Table 2 display these values.

Table 2 as well as Figures 2 and 3 show that in both the CCR model and the BCC model, the efficiency values of the robust models are lower than or equal to the deterministic model. Because the efficiency frontier of the robust model is different from the efficiency frontier in the deterministic model, this condition is not always true and, in some cases, may be violated. A noteworthy point from Table 2 is that in deterministic models at least one DMU always has an efficiency score of 1, whereas in robust models this condition is not necessarily met.

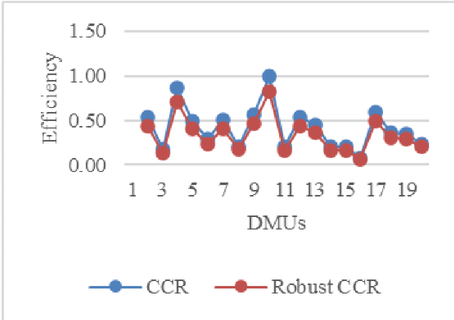

Figure 2. Efficiency Scores of CCR Model (2) and Robust CCR Model (11)

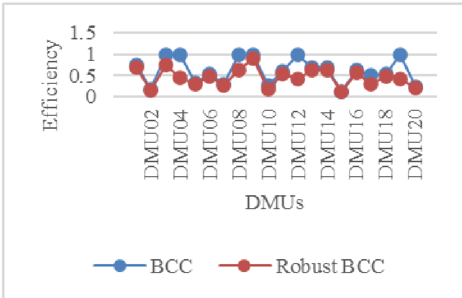

Figure 3. Efficiency Scores of BCC Model (3) and Robust BCC Model (12)

5. Conclusion

Methods and tools of performance appraisal have always been one of the important topics in organizational and academic research. On the other hand, the ability of DEA models in performance appraisal has led to extensive research in various scientific fields. One of the main challenges in applying the traditional DEA models is the existence of uncertainty in the data related to inputs and outputs. In the present study, two robust DEA models for calculating the efficiency of DMUs with uncertain data are presented. The first proposed robust model is based on multiplier input-oriented CCR model and used under CRS conditions. Also, the second proposed robust model is based on multiplier input-oriented BCC model and used under VRS conditions. Both robust models are linear and can be used for issues in which both input and output data are uncertain. Unlike previous robust DEA models, both robust DEA models presented in this study are developed directly on the traditional and basic DEA models and are also able to consider uncertainty for both input and output data. To show the applicability of the developed DEA models, a numerical example with three inputs and two outputs is presented. For this example, where all the data is considered uncertain, the efficiency values are calculated for both the CRS and VRS condition. The results show that both under CRS and VRS conditions, the efficiency values of the robust DEA models are less than or equal to the traditional DEA models.

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