Gautama and Almost Gautama Algebras and their associated logics

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Abstract. Recently, Gautama algebras were defined and investigated as a common generalization of the variety $\mathbb{RDBLS}$t of regular double Stone algebras and the variety $\mathbb{RKLS}$t of regular Kleene Stone algebras, both of which are, in turn, generalizations of Boolean algebras. Those algebras were named in honor and memory of the two founders of Indian Logic–**Akshapada Gautama** and **Medhatithi Gautama**. The purpose of this paper is to define and investigate a generalization of Gautama algebras, called "Almost Gautama algebras ($\mathbb{AG}$, for short)." More precisely, we give an explicit description of subdirectly irreducible Almost Gautama algebras. As consequences, explicit description of the lattice of subvarieties of $\mathbb{AG}$ and the equational bases for all its subvarieties are given. It is also shown that the variety $\mathbb{AG}$ is a discriminator variety. Next, we consider logicizing $\mathbb{AG}$; but the variety $\mathbb{AG}$ lacks an implication operation. We, therefore, introduce another variety of algebras called "Almost Gautama Heyting algebras" ($\mathbb{AGH}$, for short) and show that the variety $\mathbb{AGH}$ is term-equivalent to that of $\mathbb{AG}$. Next, a propositional logic, called $\mathcal{AG}$ (or $\mathcal{AGH}$), is defined and shown to be algebraizable (in the sense of Blok and Pigozzi) with the variety $\mathbb{AG}$, via $\mathbb{AGH}$, as its equivalent algebraic semantics (up to term equivalence). All axiomatic extensions of the logic $\mathcal{AG}$, corresponding to all the subvarieties of $\mathbb{AG}$ are given. They include the axiomatic extensions $\mathcal{RDBLS}t$, $\mathcal{RKLS}t$ and $\mathcal{G}$ of the logic $\mathcal{AG}$ corresponding to the varieties $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t, and $\mathbb{G}$ (of Gautama algebras), respectively. It is also deduced that none of the axiomatic extensions of $\mathcal{AG}$ has the Disjunction Property. Finally, We revisit the classical logic with strong negation $\mathcal{CN}$ and classical Nelson algebras $\mathbb{CN}$ introduced by Vakarelov in 1977 and improve his results by showing that $\mathcal{CN}$ is algebraizable with $\mathbb{CN}$ as its algebraic semantics and that the logics $\mathcal{RKLS}t$, $\mathcal{RKLS}t\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent.

1 Introduction

Boolean algebras have been a springboard for many new classes of algebras. Recall that an algebra $\mathbf{A} = \langle A, \vee, \wedge, {}^c, 0, 1 \rangle$ is a Boolean algebra if $\mathbf{A}$ is a complemented distributive lattice. The following 2-element algebra with universe $\{0, 1\}$, denoted by $\mathbf{2}$, is the smallest nontrivial Boolean algebra, up to isomorphism.

This paper is an expanded version of an invited lecture with the same title, presented at the 10th International Conference on Non-Classical Logics: Theory and Applications, held at Łodz, Poland, during March 14-18, 2022.

$$
\mathbf{2}: \quad
\begin{array}{c|cc}
\vee & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1
\end{array}
\qquad
\begin{array}{c|cc}
\wedge & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
\qquad
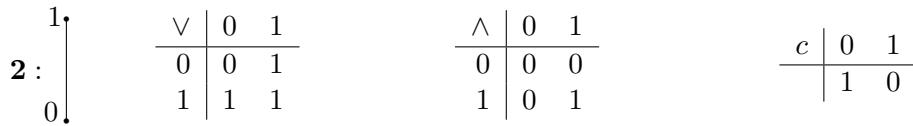\begin{array}{c|cc}
c & 0 & 1 \\
\hline
 & 1 & 0
\end{array}
$$

Figure 1

Let $\mathbb{BA}$ denote the variety of Boolean algebras. It is well-known that $\mathbb{BA} = \mathbb{V}(\mathbf{2})$ (i.e. the variety generated by $\{\mathbf{2}\}$). In what follows, the symbol $\mathbf{2}$ denotes a two-element Boolean algebra whose signature, though varies, will be clear from the context where it appears. It is well-known that the Boolean complement has led to several weaker notions; among them are the following three:

(1) the pseudocomplement *, (2) the dual pseudocomplement $^+$, and (3) the De Morgan complement $'$.

Algebras based on the 3-element chain:

It was only natural to consider the above-mentioned operations on a 3-element chain (viewed as a bounded distributive lattice) denoted by $\mathbf{3}$. The three-element chain and the three operations mentioned above are shown below in Figure 2.

$$
\mathbf{3}: \quad
\begin{array}{c|ccc}
^* & 0 & a & 1 \\
\hline
 & 1 & 0 & 0
\end{array}
\qquad
\begin{array}{c|ccc}
^+ & 0 & a & 1 \\
\hline
 & 1 & 1 & 0
\end{array}
\qquad
\begin{array}{c|ccc}
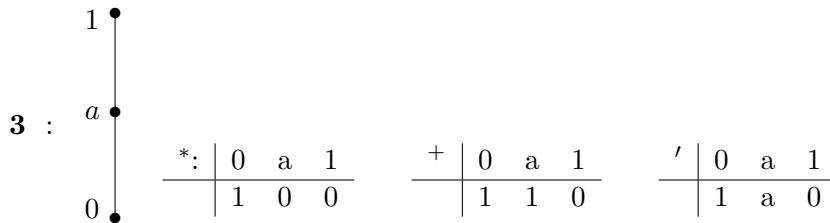' & 0 & a & 1 \\
\hline
 & 1 & a & 0
\end{array}
$$

Figure 2

Let us, therefore, expand the language $\langle \vee, \wedge, 0, 1 \rangle$ of bounded distributive lattices to $\langle \vee, \wedge, f, 0, 1 \rangle$ by adding one unary operation symbol f and interpret f, on the chain $\mathbf{3}$, as the operation *, $^+$, or $'$, with the added restriction that $f(0) := 1$ and $f(1) := 0$. Then we get the following three algebras:

(1) $\mathbf{3_{st}} := \langle 3, \vee, \wedge, ^*, 0, 1 \rangle$,
(2) $\mathbf{3_{dst}} := \langle 3, \vee, \wedge, ^+, 0, 1 \rangle$,
(3) $\mathbf{3_{kl}} = \langle 3, \vee, \wedge, ', 0, 1 \rangle$.

The varieties generated by $\mathbf{3_{st}}$, $\mathbf{3_{dst}}$ and $\mathbf{3_{kl}}$ are well-known, respectively, as those of Stone algebras, dual Stone algebras and Kleene algebras. We will denote these varieties by $\mathbb{St}$, $\mathbb{DSt}$ and $\mathbb{KL}$, respectively. $\mathbb{St}$ and $\mathbb{KL}$ have been researched well; as such, there is a fair amount of literature on them (see, for example, [7, 21].

In order to define Stone algebras, we need the notion of a pseudocomplemented lattice which was first introduced by Skolem [66] (see also [70]). It is clear that the usual definition of pseudocomplement (namely, $a \wedge x = 0$ iff $x \le a^*$) is not equational. However, in 1949, Ribenboim [41] proved that the class of pseudocomplemented lattices is a variety. For our purpose here, the following axiomatization given in [49, Corollary 2.8] is more suitable.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, 0, 1 \rangle$ is a distributive pseudocomplemented lattice (p-algebra for short) if $\mathbf{A}$ satisfies the following:

(1) $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice,
(2) the operation * satisfies the identities:
   (a) $0^* \approx 1$,
   (b) $1^* \approx 0$,

(c) $(x \lor y)^* \approx x^* \land y^*$,

(d) $(x \land y)^{**} \approx x^{**} \land y^{**}$,

(e) $x \leq x^{**}$,

(f) $x^* \land x^{**} \approx 0$.

Note that the identity (f) can be replaced by the identity: $x \land x^* \approx 0$.

A *p*-algebra **A** is a Stone algebra if **A** satisfies the identity:

(3) $x^* \lor x^{**} \approx 1$ (Stone identity).

Stone algebras have an extensive literature (for example, see [7, 21, 22] and the references therein).

It is also well-known that the variety $\mathbb{S}t = \mathbb{V}(\mathbf{3_{st}})$ (the variety generated by $\mathbf{3_{st}}$). Dual Stone algebras are, of course, defined dually.

Kleene algebras are well-known too. The variety of Kleene algebras is a subvariety of that of De Morgan algebras, first introduced by Moisil [28] in 1935 (see also [29, 30]). They were further investigated later in [8, 24, 44]. They are generalized to semi-De Morgan algebras in [49], and further studied in [23, 37, 35, 36, 38, 53, 55].

An algebra $\langle A, \lor, \land, ', 0, 1 \rangle$ is a De Morgan algebra if

(1) $\langle A, \lor, \land, 0, 1 \rangle$ is a bounded distributive lattice,

(2) $0' \approx 1$ and $1' \approx 0$,

(3) $(x \land y)' \approx x' \lor y'$ ($\land$-De Morgan law),

(4) $x'' \approx x$ (Involution).

A De Morgan algebra is a Kleene algebra if it satisfies:

(5) $x \land x' \leq y \lor y'$ (Kleene identity).

It is also well-known that the variety $\mathbb{KL} = \mathbb{V}(\mathbf{3_{kl}})$.

Algebras on the 3-element chain with two additional unary operations:

The next natural step in this development was to consider the expansion of the language $\langle \lor, \land, 0, 1 \rangle$ by adding two unary operation symbols corresponding to two of the above three unary operations on the 3-element chain, leading to the following three algebras on the 3-element chain:

(a) $\mathbf{3_{dblst}} = \langle 3, \lor, \land, ^*, ^+, 0, 1 \rangle$: This is known as a "double Stone algebra." It was observed in [68] and [25] that $\mathbf{3_{dblst}}$ also satisfies an additional identity, called a "regular identity":

(R) $x \land x^+ \leq y \lor y^*$.

So, $\mathbf{3_{dblst}}$ is a "regular double Stone algebra."

(b) $\mathbf{3_{klst}} = \langle 3, \lor, \land, ^*, ', 0, 1 \rangle$: This is a Kleene Stone algebra (see [43] and [48]). This algebra also satisfies an interesting identity (see [48]), also called "regular identity":

(R1) $x \land x'^{*'} \leq y \lor y^*$

So, $\mathbf{3_{klst}}$ is a "regular Kleene Stone algebra".

(c) $\mathbf{3_{klst}} = \langle 3, \lor, \land, ^+, ', 0, 1 \rangle$: This, being the dual of (b), would not be of much interest to us in this paper. Thus, (a) and (b) yield the well-known varieties of regular double Stone algebras and regular Kleene Stone algebras, respectively.

An algebra $\mathbf{A} = \langle A, \lor, \land, ^*, ^+, 0, 1 \rangle$ is a regular double Stone algebra if

(1) $\langle A, \lor, \land, ^*, 0, 1 \rangle$ is a Stone algebra,

(2) $\langle A, \lor, \land, ^+, 0, 1 \rangle$ is a dual Stone algebra,

(3) **A** satisfies the identity:

(R) $x \land x^+ \leq y \lor y^*$.

The variety of regular double Stone algebras is denoted by $\mathbb{RDBLSt}$. For the more general variety of double *p*-algebras, of which $\mathbb{RDBLSt}$ is a subvariety, see, for example, [68, 25, 45, 54, 58, 64, 5, 12, 17] and references therein.

We now pause briefly to recall some universal algebraic notions (see, for example, [10, 72]).

Definition 1.1. Let $\mathbf{A}$ be an algebra. An n-ary function $f : A^n \to A$ is *representable by a term* if there is a term p such that $f(a_1, \ldots, a_n) = p^{\mathbf{A}}(a_1, \ldots, a_n)$, for $a_1, \ldots, a_n \in A$. A finite algebra $\mathbf{A}$ is *primal* if every n-ary function on A, for every $n \geq 1$, is representable by a term.

The *discrimination function* on a set A is the function $t : A^3 \to A$ defined by

$$t(a, b, c) := \begin{cases} a, & \text{if } a \neq b \\ c, & \text{if } a = b. \end{cases}$$

A ternary term $t(x, y, z)$ representing the discriminator on A is called a *discriminator term* for the algebra $\mathbf{A}$. If a class $\mathbb{K}$ of algebras has a common discriminator term $t(x, y, z)$, then $\mathbb{V}(\mathbb{K})$ is called a discriminator variety. A finite algebra $\mathbf{A}$ with a discriminator term is called *quasiprimal*.

Discriminator varieties have been of great interest for a few decades now. For readers interested in this area, we recommend the books [72] and [10].

Returning to regular double Stone algebras, the following theorem is also well-known.

Theorem 1.2.

(i) $\mathbf{2}$ *and* $\mathbf{3_{dblst}}$, *up to isomorphism, are the only subdirectly irreducible (equiv. simple) algebras in* $\mathbb{RDBLSt}$

(ii) *The variety* $\mathbb{RDBLSt} = \mathbf{V}(\mathbf{3_{dblst}})$,

(iii) *The variety* $\mathbb{RDBLSt}$ *is a discriminator variety* ([45]),

(iv) $\mathbf{3_{dblst}}$ *is quasiprimal* ([45]),

(v) $\mathbb{BA}$ *is the only nontrivial proper subvariety of* $\mathbb{RDBLSt}$.

Regular Kleene Stone algebras are also well-known.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1 \rangle$ is a regular Kleene Stone algebra if

(1) $\langle A, \vee, \wedge, ^*, 0, 1 \rangle$ is a Stone algebra,

(2) $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is a Kleene algebra,

(3) $\mathbf{A}$ satisfies the identity:

    (R1) $x \wedge x'^{*'} \leq y \vee y^*$ (Regularity).

The variety of regular Kleene Stone algebras is denoted by $\mathbb{RKLSt}$. For the more general variety of pseudocomplemented De Morgan and Ockham algebras, of which $\mathbb{RKLSt}$ is a subvariety, see [43, 48, 50, 46, 56, 58, 65, 6] and references therein.

The following theorem lists some of the known properties of the variety $\mathbb{RKLSt}$.

Theorem 1.3. [58]

(i) $\mathbf{2}$ *and* $\mathbf{3_{klst}}$, *up to isomorphism, are the only subdirectly irreducible (equiv. simple) algebras in* $\mathbb{RKLSt}$.

(ii) *The variety* $\mathbb{RKLSt} = \mathbb{V}(\mathbf{3_{klst}})$,

(iii) *The variety* $\mathbb{RKLSt}$ *is a discriminator variety,*

(iv) $\mathbf{3_{klst}}$ *is quasiprimal,*

(v) $\mathbb{BA}$ *is the only nontrivial proper subvariety of* $\mathbb{RKLSt}$.

Remark 1.4. It is easy to verify that the algebra $\mathbf{3_{dblst}}$ also satisfies the identity (R1) and hence the variety $\mathbb{RDBLSt}$ also satisfies (R1).

In view of the amazing similarities of $\mathbb{RDBLSt}$ and $\mathbb{RKLSt}$, as seen in their definitions, as well as in Theorem 1.2 and in Theorem 1.3, it was only natural to ask for a common generalization of $\mathbb{RDBLSt}$ and $\mathbb{RKLSt}$. To find such a common generalization, it was essential, first, to have a common generalization of dually Stone algebras and Kleene algebras, which luckily was already present since 1987, as the notion of a "dually quasi-De Morgan algebra." In 1987, the second author had introduced the variety of "upper quasi-De Morgan algebras," as a subvariety of the variety of semi-De Morgan algebras in [49]. (We drop the word "upper." here.) Actually, for our purpose here, we need the dual notion of "dually quasi-De Morgan algebra."

Definition 1.5. An algebra $\mathbf{A} = \langle A, \vee, \wedge,', 0, 1 \rangle$ is a dually quasi-De Morgan algebra if the following conditions hold:

(a) $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice,
(b) The operation $'$ is a dual quasi-De Morgan operation; that is, $'$ satisfies:

(i) $0' \approx 1$ and $1' \approx 0$,

(ii) $(x \wedge y)' \approx x' \vee y'$,

(iii) $(x \vee y)'' \approx x'' \vee y''$,

(iv) $x'' \leq x$.

The variety of dually quasi-De Morgan algebras is denoted by $\mathbb{DQD}$.

THE VARIETY OF GAUTAMA ALGEBRAS

The problem of finding a common generalization of $\mathbb{RDBLSt}$ and $\mathbb{RKLSt}$, mentioned above, led the second author, to define, in [64], the variety of Gautama algebras, named in honor and memory of Medhatithi Gautama and Aksapada Gautama, the founders of Indian Logic.

Definition 1.6. An algebra $\mathbf{A} = \langle A, \vee, \wedge,^*,', 0, 1 \rangle$ is a Gautama algebra if the following conditions hold:

(a) $\langle A, \vee, \wedge,^*, 0, 1 \rangle$ is a Stone algebra,
(b) $\langle A, \vee, \wedge,', 0, 1 \rangle$ is a dually quasi-De Morgan algebra,
(c) $\mathbf{A}$ is regular; i.e., $\mathbf{A}$ satisfies the identity:
　(R1)　$x \wedge x'^{*'} \leq y \vee y^*$,
(d) $\mathbf{A}$ is star-regular; i.e., $\mathbf{A}$ satisfies the identity:
　(*)　$x^{*'} \approx x^{**}$.

Let $\mathbb{G}$ denote the variety of Gautama algebras.

Clearly, $\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}$ are algebras in $\mathbb{G}$; and so, the varieties $\mathbb{BA}$, $\mathbb{RDBLSt}$, and $\mathbb{RKLSt}$ are subvarieties of the variety $\mathbb{G}$.

The following theorem, proved in [64], gives a concrete description of the subdirectly irreducible algebras in the variety $\mathbb{G}$.

Theorem 1.7. [64] *Let* $\mathbf{A} \in \mathbb{G}$. *Then the following are equivalent:*

(1) $\mathbf{A}$ is simple;
(2) $\mathbf{A}$ is subdirectly irreducible;
(3) $\mathbf{A}$ is directly indecomposable;
(4) For every $x \in A$, $x \vee x^* = 1$ implies $x = 0$ or $x = 1$;
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}\}$, up to isomorphism.

In this paper, we introduce and investigate a generalization of Gautam algebras, called "Almost Gautama algebras" ($\mathbb{AG}$ for short). We describe the subdirectly irreducible algebras in $\mathbb{AG}$ and then give several consequences, including the description of the lattice of subvarieties of $\mathbb{AG}$ and equational bases for all the subvarieties of $\mathbb{AG}$. It is also shown that the variety AG is a discriminator variety. Next, we consider the problem of logicizing the varety $\mathbb{AG}$, Unfortunately, $\mathbb{AG}$ lacks an implication operation. So, we introduce another variety called "Almost Gautama Heyting algebras ($\mathbb{AGH}$ for short) such that the language of $\mathbb{AGH}$ contains an implication operation symbol $\rightarrow$ and $\mathbb{AGH}$ is term-equivalent to $\mathbb{AG}$. We then consider $\mathbb{AG}$ from a logical point of view, via $\mathbb{AGH}$. More explicitly, we define a new propositional logic called $\mathcal{AG}$ (or $\mathcal{AGH}$) as an axiomatic extension of the logic $\mathcal{DHMH}$ which was introduced in [15] and show that $\mathcal{AG}$ is algebraizable with $\mathbb{AGH}$ as its equivalent algebraic semantics. Since $\mathbb{AGH}$ is term-equivalent to $\mathbb{AG}$, it can be viewed that the logic $\mathcal{AG}$ is the logic corresponding to $\mathbb{AG}$. It is also shown that the logic $\mathcal{AG}$ is decidable. Finally, all axiomatic extensions of the logic $\mathcal{AG}$, corresponding to all subvarieties of $\mathbb{AG}$ are determined. They include the axiomatic extensions $\mathcal{RDBLS}$t, $\mathcal{RKLS}$t and $\mathcal{G}$ of the logic $\mathcal{AG}$ corresponding to the varieties $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{G}$, respectively. It is also deduced that none of the axiomatic extensions of $\mathcal{AG}$ has the Disjunction Property. The paper concludes with a few open problems for further research and with a fairly extensive (though not complete) bibliography.

It is assumed that the reader has had some familiarity with lattice theory and universal algebra (see [7, 21, 10], for example). As such, for notions, notations and results assumed here, the reader can refer to these or other relevant books.

2   The variety of Almost Gautama algebras

The purpose of this section is to introduce and investigate a new variety of algebras, called "Almost Gautama algebras" which, as mentioned earlier, is a generalization of Gautama algebras. The following lemma offers a hint for such a generalization.

Lemma 2.1. *Let $\mathbb{G}$ be the variety of Gautama algebras. Then*

(1) $\mathbb{G} \models x^{*\prime\prime} \approx x^*$ (Weak Star-Regular Identity),

(2) $\mathbb{G} \models (x \wedge x^{\prime *})^{\prime *} \approx x \wedge x^{\prime *}$ (L1).

Proof. Let $\mathbf{A} \in \mathbb{G}$. Let $a \in A$. Then, $a^{*\prime\prime} = a^{**\prime} = a^{***} = a^*$, proving (1), while it is routine to verify that (2) holds in $\mathbf{3}_{\mathbf{dblst}}$ and $\mathbf{3}_{\mathbf{klst}}$.   □

We are now ready to define the variety of Almost Gautama algebras.

Definition 2.2. An algebra $\mathbf{A} = \langle A, \vee, \wedge, {}^*, {}', 0, 1 \rangle$ is an Almost Gautama algebra if the following conditions hold:

(a) $\langle A, \vee, \wedge, {}^*, 0, 1 \rangle$ is a Stone algebra,

(b) $\langle A, \vee, \wedge, {}', 0, 1 \rangle$ is a dually quasi-De Morgan algebra,

(c) $\mathbf{A}$ is regular. That is, $\mathbf{A}$ satisfies the identity:

  (R1) $x \wedge x^{\prime *\prime} \leq y \vee y^*$ (Regularity),

(d) $\mathbf{A}$ is Weak Star-Regular. That is, $\mathbf{A}$ satisfies the identity:

  $(*)_{\mathrm{w}}$ $x^{*\prime\prime} \approx x^*$, (weak star-regularity),

(e) $\mathbf{A}$ satisfies the identity:

  $(x \wedge x^{\prime *})^{\prime *} \approx x \wedge x^{\prime *}$ (L1).

Let $\mathbb{AG}$ denote the variety of Almost Gautama algebras.

Clearly, in view of Lemma 2.1, every Gautama algebra is an Almost Gautama algebra. Hence, the varieties $\mathbb{BA}$ of Boolean algebras, $\mathbb{RDBLSt}$ of regular double Stone algebras, and $\mathbb{RKLSt}$ of regular Kleene Stone algebras and the variety $\mathbb{G}$ of Gautama algebras are all subvarieties of the variety $\mathbb{AG}$ of Almost Gautama algebras.

Consider the following 4-element algebra $\mathbf{4_{dmba}} := \langle \{0, a, b, 0, 1\}, \vee, \wedge, ^*, ', 0, 1 \rangle$ (see Figure 3), where $*$ is the Boolean complement with $a^* = b$, $b^* = a$; and $0' = 1$, $1' = 0$, $a' = a$ and $b' = b$. It is easy to see that $\mathbf{4_{dmba}}$ is an Almost Gautama algebra. Observe that $\mathbf{4_{dmba}}$ is not a Gautama algebra (e.g., take $x := a$ in $(*)_w$).
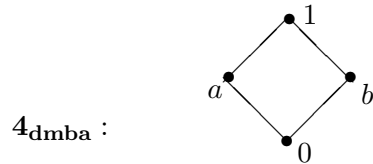
$\mathbf{4_{dmba}}$:

Figure 3

We close this section with a few concepts needed later in this paper.

The notion of "hemimorphic algebra" was implicit in [49] and was made explicit later in [58] (in its dual form).

Definition 2.3. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ', 0, 1 \rangle$ is a dually hemimorphic algebra if $\mathbf{A}$ satisfies the following conditions:

(H1) $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice,

(H2) $0' \approx 1$,

(H3) $1' \approx 0$,

(H4) $(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law).

The variety of dually hemimorphic algebras is denoted by $\mathbb{DHM}$.

We can recast the definition of a dually quasi-De Morgan algebra (see Definition 1.5) as follows:
    $\mathbf{A} \in \mathbb{DHM}$ is a dually quasi-De Morgan algebra if it satisfies:

(H5) $(x \vee y)'' \approx x'' \vee y''$,

(H6) $x'' \leq x$.

The variety of dually quasi-De Morgan algebras is denoted by $\mathbb{DQD}$.

We will now introduce a far-reaching generalization of Gautama algebras.

Definition 2.4. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1 \rangle$ is a dually hemimorphic p-algebra if it satisfies:
    (a) $\langle A, \vee, \wedge, ^*, 0, 1 \rangle$ is a p-algebra,
    (b) $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is a dually hemimorphic algebra.
The variety of dually hemimorphic p-algebras is denoted by $\mathbb{DHMP}$.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1 \rangle$ is a *dually quasi-De Morgan p-algebra* if :
    (a) $\langle A, \vee, \wedge, ^*, 0, 1 \rangle$ is a p-algebra,
    (b) $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is a dually quasi-De Morgan algebra.

The variety of dually quasi-De Morgan p-algebras is denoted by $\mathbb{DQDP}$. In fact, dually hemimorphic p-algebras are a common generalization of double p-algebras and pseudocomplemented Ockham algebras, which have been investigated in several papers, some of which are: [50, 51, 52, 54, 56, 65].

$\mathbf{A} \in \mathbb{DHMP}$ is regular if it satisfies (R1): $x \wedge x'^{*'} \leq y \vee y^*$. The variety of regular dually hemimorphic p-algebras is denoted by $\mathbb{RDHMP}$.

Clearly, $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{RDBLP} \subset \mathbb{RDQDP} \subset \mathbb{DQDP} \subset \mathbb{DHMP}$. Also, $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{RDMP} \subset \mathbb{RDQDP} \subset \mathbb{DQDP} \subset \mathbb{DHMP}$.

3   Subdirectly irreducible Almost Gautama Algebras

We now wish to characterize the subdirectly irreducible Almost Gautama algebras. To achieve this, we need some preliminary results. Recall the well-known fact (see [7, 21]) that if $\mathbf{A}$ is a p-algebra then $\mathbf{A} \models x \wedge (x \wedge y)^* \approx x \wedge y^*$.

Let $\mathbf{A} \in \mathbb{DQDP}$, $a \in A$ and let $(a] := \{x \in A : x \leq a\}$. Define the algebra $(\mathbf{a}]$ as follows:
$(\mathbf{a}] := \langle (a], \vee, \wedge, {}^{*a}, {}'^{a}, 0, a \rangle \in \mathbb{DQDP}$, where $x^{*a} := x^* \wedge a$ and $x'^a := x' \wedge a$, for $x \in (a]$. Similarly, the algebra $(\mathbf{a}^*]$ is defined.

Lemma 3.1. *Let* $\mathbf{A} \in \mathbb{DQDP}$ *satisfying* (L1): $(x \wedge x'^*)'^* \approx x \wedge x'^*$ *and let* $a \in A$ *such that* $a \wedge a' = 0$. *Then*

  (i) $(\mathbf{a}] \in \mathbb{DQDP}$,

  (ii) $a^* \wedge a^{*'} = 0$,

  (iii) $(\mathbf{a}^*] \in \mathbb{DQDP}$.

Proof. Let $x, y \in (a]$. Then $x \wedge (x \wedge y)^{*a} = x \wedge (x \wedge y)^* \wedge a = x \wedge y^* \wedge a = x \wedge y^{*a}$, since * is the pseudocomplement. Also, $a^{*a} = a^* \wedge a = 0$, and $0^{*a} = 0^* \wedge a = 1 \wedge a = a$. So, x^{*a} is the pseudocomplement of $x \in (a]$. Now,

$$
\begin{aligned}
(x \wedge y)'^a &= (x \wedge y)' \wedge a \\
&= (x' \vee y') \wedge a \\
&= (x' \wedge a) \vee (y' \wedge a) \\
&= x'^a \vee y'^a.
\end{aligned}
$$

Next,

$$
\begin{aligned}
(x \vee y)'^a{}'^a &= [(x \vee y)' \wedge a]' \wedge a \\
&= [(x \vee y)'' \vee a'] \wedge a \\
&= [x'' \vee y'' \vee a'] \wedge a \\
&= (x'' \wedge a) \vee (y'' \wedge a) \vee (a' \wedge a) \\
&= (x'' \wedge a) \vee (a' \wedge a) \vee (y'' \wedge a) \vee (a' \wedge a), \text{ as } a' \wedge a = 0 \\
&= [(x'' \vee a') \wedge a)] \vee [(y'' \vee a') \wedge a] \\
&= [(x' \wedge a)' \wedge a)] \vee [(y' \wedge a)' \wedge a] \\
&= x'^a{}'^a \vee y'^a{}'^a.
\end{aligned}
$$

Also,

$$
\begin{aligned}
x'^a{}'^a \vee x &= (x' \wedge a)' \wedge a) \vee x \\
&= (x'' \vee a') \wedge a) \vee x \\
&= [(x'' \wedge a) \vee (a' \wedge a)] \vee x \\
&= (x'' \wedge a) \vee x \\
&= (x'' \vee x) \wedge (a \vee x) \\
&= x \quad \text{as } x'' \leq x \leq a.
\end{aligned}
$$

Finally, $0'^a = 0' \wedge a = 1 \wedge a = a$, and $a'^a = a' \wedge a = 0$.

Thus, dual quasi-De Mogan identities hold in $(\mathbf{a}]$, proving (i).

For (ii), from $a \wedge a' = 0$, we get $a' \vee a'' = 1$ which implies $(a \wedge a') \vee (a \wedge a'') = a$. Hence $a \leq a''$ as $a \wedge a' = 0$. Thus we have

$$a'' = a. \tag{1}$$

From $a'' \wedge a''^* = 0$, we have $(a' \vee a'') \wedge (a' \vee a''^*) = a'$, whence $(a' \vee a^*) = a'$ as $a'' = a$ and $a \wedge a' = 0$; thus we have

$$a^* \leq a'. \tag{2}$$

Now, $a^* \wedge a^{*\prime *} = a''^* \wedge a''^{*\prime *} = (a' \wedge a''^*)'^* = a' \wedge a''^* = a' \wedge a^* = a^*$ by (1), (L1) and (2). Hence,

$$a^* \leq a^{*\prime *}. \tag{3}$$

So, in view of (3), we get $a^* \wedge a^{*\prime} \leq a^{*\prime *} \wedge a^{*\prime} = 0$, implying $a^* \wedge a^{*\prime} = 0$, which proves (ii). (iii) follows from (i) and (ii). $\square$

Recall the well-known result (see [7, 21]) that if $\mathbf{A}$ is a Stone algebra then $\mathbf{A} \models (x \wedge y)^* \approx x^* \vee y^*$.

Lemma 3.2. *Let* $\mathbf{A} \in \mathbb{AG}$ *and let* $a \in A$ *such that* $a \wedge a' = 0$ *Then*

(i) $(\mathbf{a}] = \langle (a], \vee, \wedge, ^{*a}, '^a, 0, a \rangle \in \mathbb{AG}$.

(ii) $(\mathbf{a}^*] = \langle (a^*], \vee, \wedge, ^{*a^*}, '^{a^*}, 0, a^* \rangle \in \mathbb{AG}$.

Proof. By Lemma 3.1, we already know that $(\mathbf{a}] \in \mathbb{DQDP}$. So, it suffices to prove (St), (R1), $(^*)_{\mathrm{w}}$ and (L1). Toward this end, let $x, y \in A$ such that $x \leq a$ and $y \leq a$.

Since $\mathbf{A}$ is a Stone algebra, we have
$x^{*a} \vee x^{*a*a} = (x^* \wedge a) \vee [(x^* \wedge a)^* \wedge a] = (x^* \wedge a) \vee [(x^{**} \vee a^*) \wedge a] = (x^* \wedge a) \vee (x^{**} \wedge a) = (x^* \vee x^{**}) \wedge a = 1 \wedge a = a$.
So, Stone identity holds in $(\mathbf{a}]$.

Now,

$$
\begin{aligned}
(x \wedge x'^{a*a'a}) &= x \wedge [(x' \wedge a)^* \wedge a]' \wedge a \\
&= x \wedge [(x' \wedge a)^* \wedge a]', \text{ as } x \leq a \\
&= x \wedge [(x' \wedge a)^{*\prime} \vee a'] \\
&= [x \wedge (x' \wedge a)^{*\prime}] \vee (x \wedge a'), \\
&= x \wedge (x' \wedge a)^{*\prime}, \text{ since } x \leq a \text{ and } a \wedge a' = 0.
\end{aligned}
$$

Thus we have

$$x \wedge x'^{a*a'a} = x \wedge (x' \wedge a)^{*\prime}. \tag{4}$$

Since $(x' \wedge a)^{*\prime} \leq x'^{*\prime}$, we have $x \wedge (x' \wedge a)^{*\prime} \leq x \wedge x'^{*\prime} \leq y \vee y^*$. Also, observe that $x \wedge (x' \wedge a)^{*\prime} \leq x \leq a$. Hence, it follows that $x \wedge (x' \wedge a)^{*\prime} \leq (y \vee y^*) \wedge a$. Therefore, from (4) we get

$$
\begin{aligned}
(x \wedge x'^{a*a'a}) \vee (y \vee y^{*a}) &= (y \vee y^*) \wedge a, \\
&= (y \wedge a) \vee (y^* \wedge a), \\
&= y \vee (y^* \wedge a), \\
&= y \vee y^{*a}.
\end{aligned}
$$

Hence (R1) holds in (**a**].
Next,

$$
\begin{aligned}
x^{*a'a'a} &= [(x^* \wedge a)' \wedge a]' \wedge a \\
&= [(x^{*'} \vee a') \wedge a]' \wedge a \\
&= [(x^{*'} \wedge a) \vee (a' \wedge a)]' \wedge a \\
&= (x^{*'} \wedge a)' \wedge a \\
&= (x^{*''} \vee a') \wedge a \\
&= x^{*''} \wedge a \\
&= x^* \wedge a, \text{ by the identity } (^*)_{\mathrm{w}} \\
&= x^{*a}.
\end{aligned}
$$

Hence the weak star regular identity $(^*)_{\mathrm{w}}$ holds in (**a**].
Finally,

$$
\begin{aligned}
(x \wedge x'^{a*a})'^{a*a} &= [x \wedge (x' \wedge a)^* \wedge a]'^{a*a} \\
&= [x \wedge (x' \wedge a)^*]'^{a*a} \text{ as } x \leq a \\
&= [\{x \wedge (x' \wedge a)^*\}' \wedge a]^* \wedge a \\
&= [\{x \wedge (x' \wedge a)^*\}'^* \vee a^*] \wedge a \quad \text{ as } \mathbf{A} \text{ is a Stone algebra} \\
&= [x \wedge (x' \wedge a)^*]'^* \wedge a \\
&= [x \wedge (x'^* \vee a^*)]'^* \wedge a \quad \text{ as } \mathbf{A} \text{ is a Stone algebra} \\
&= [(x \wedge x'^*) \vee (x \wedge a^*)]'^* \wedge a \\
&= (x \wedge x'^*)'^* \wedge a \quad \text{ since } x \wedge a^* = 0 \text{ as } x \leq a \\
&= x \wedge x'^* \wedge a \quad \text{ by (L1)} \\
&= x \wedge [(x'^* \wedge a) \vee (a^* \wedge a)] \\
&= x \wedge [(x'^* \vee a^*) \wedge a] \\
&= x \wedge (x' \wedge a)^* \wedge a \quad \text{ as } \mathbf{A} \text{ is a Stone algebra} \\
&= x \wedge x'^{a*a}.
\end{aligned}
$$

So, (L1) holds in (**a**], proving (i). The proof of (ii) is similar to (i).   □

Lemma 3.3. *Let* $\mathbf{A} \in \mathbb{DQDP}$ *satisfy (L1) and let* $a \in \mathbf{A}$ *such that* $a \vee a^* = 1$ *and* $a \wedge a' = 0$. *Let* $g : \mathbf{A} \to (\mathbf{a}] \times (\mathbf{a}^*]$ *be defined by* $g(x) = \langle x \wedge a, x \wedge a^* \rangle$. *Then* g *is an isomorphism from* $\mathbf{A}$ *onto* $(\mathbf{a}] \times (\mathbf{a}^*]$.

Proof. It is easy to see that g is a lattice-homomorhism. Now,

$$
\begin{aligned}
(g(x))^* &= (\langle x \wedge a, x \wedge a^* \rangle)^* \\
&= \langle (x \wedge a)^{*a}, (x \wedge a^*)^{*(a^*)} \rangle \\
&= \langle (x \wedge a)^* \wedge a, (x \wedge a^*)^* \wedge a^* \rangle \\
&= \langle x^* \wedge a, \ x^* \wedge a^* \rangle \text{ since } * \text{ is a pseudocomplement} \\
&= g(x^*).
\end{aligned}
$$

Next,

$$
\begin{aligned}
(g(x))' &= (\langle x \wedge a, x \wedge a^* \rangle)' \\
&= \langle (x \wedge a)'^a, (x \wedge a^*)'^{a^*} \rangle \\
&= \langle (x \wedge a')' \wedge a, (x \wedge a^*)' \wedge a^* \rangle \\
&= \langle (x' \vee a') \wedge a, (x' \vee a^{*'}) \wedge a^* \rangle \\
&= \langle x' \wedge a, (x' \wedge a^*) \vee (a^{*'} \wedge a^*) \rangle \text{ since } a \wedge a' = 0 \\
&= \langle x' \wedge a, x' \wedge a^* \rangle \text{ since } a^{*'} \wedge a^* = 0 \text{ by (ii) of Lemma 3.1} \\
&= g(x').
\end{aligned}
$$

Next, suppose $g(x) = g(y)$. Then, $\langle x \wedge a, x \wedge a^* \rangle = \langle y \wedge a, y \wedge a^* \rangle$, whence $x \wedge a = y \wedge a$ and $x \wedge a^* = y \wedge a^*$. Thus, in view of the hypothesis, $x = x \wedge (a \vee a^*) = (x \wedge a) \vee (x \wedge a^*) = (y \wedge a) \vee (y \wedge a^*) = y \wedge (a \vee a^*) = y$, implying g is one-one. It is clear that g is onto.   □

Lemma 3.4. *Let* $\mathbf{A} \in \mathbb{AG}$. *If* $a = a'^*$, *then* $a' = a^*$.

Proof. From $a = a'^*$, we get $a'' = a'^{*''} = a'^* = a$, in view of the axiom $(*)_{\mathrm{w}}$. Thus we have the following:

$$a'' = a. \tag{5}$$

Next, we have $a' \leq a'^{**} = a^*$, since $a'^* = a$. Thus,

$$a' \leq a^*. \tag{6}$$

Hence, we get

$$
\begin{aligned}
a' &= a' \wedge a^* \text{ by (6)} \\
&= a' \wedge a''^* \text{ by (5)} \\
&= (a' \wedge a''^*)'^* \text{ by (L1)} \\
&= (a'' \vee a''^{*\prime})^* \\
&= a''^* \wedge a''^{*\prime*} \\
&= a^* \wedge a^{*\prime*}, \text{ by (5)}.
\end{aligned}
$$

Thus,

$$a' = a^* \wedge a^{*\prime*}. \tag{7}$$

By (6), we have $a' = a' \wedge a^*$, whence $a'' = a'' \vee a^{*\prime}$, implying $a = a \vee a^{*\prime}$ in view of (5). Hence we get $a^* = a^{*\prime*} \wedge a^*$. Hence, in view of (7), we conclude $a' = a^*$.   □

Lemma 3.5. *Let* $\mathbf{A} \in \mathbb{DHMP}$ *satisfying* $x'' \leq x$ *and let* $a \in A$ *such that* $a' = a^*$. *Then* $a \vee a^* = 1$.

Proof. From $a \wedge a^* = 0$, we get $a' \vee a^{*\prime} = 1$, implying $a' \vee a'' = 1$ as $a' = a^*$. Hence $a' \vee a = 1$ since $a'' \leq a$, whence $a \vee a^* = 1$, as $a' = a^*$.   □

Corollary 3.6. *Let* $\mathbf{A} \in \mathbb{AG}$ *and let* $a \in A$ *such that* $a' = a^*$. *Then* $\mathbf{A} \cong (a] \times (a^*]$.

Proof. Let $\mathbf{A}$ and a be as in the hypothesis. Then by Lemma 3.5 we have $a \vee a^* = 1$. Also, since $a' = a^*$ by hypothesis, it is clear that $a \wedge a' = 0$. Hence, in view of Lemmma 3.3 we conclude that $\mathbf{A} \cong (a] \times (a^*]$.   □

Lemma 3.7. *Let* $\mathbf{A} \in \mathbb{DHMP}$ *such that*
   (1) $\mathbf{A} \models x'' \leq x$ *and*
   (2) $\mathbf{A} \models x \wedge x'^* \approx (x \wedge x'^*)'^*$ (L1).
Let $y \in A$. Then $(y \wedge y'^*) \vee (y \wedge y'^*)^* = 1$.

Proof. Observe $y \vee (y \wedge y'^*)^* \geq y'' \vee (y \wedge y'^*)^* = y'' \vee (y \wedge y'^*)'^{**} = y'' \vee (y' \vee y'^{*\prime})^{**} = y'' \vee (y'^* \wedge y'^{*\prime*})^* \geq y'' \vee y'^{*\prime**} \geq y'' \vee y'^{*\prime} = (y' \wedge y'^*)' = 0' = 1$, Thus,

$$y \vee (y \wedge y'^*)^* \approx 1, . \tag{8}$$

Now

$$
\begin{aligned}
(y \wedge y'^*) \vee (y \wedge y'^*)^* &= [y \vee (y \wedge y'^*)^*] \wedge [y'^* \vee (y \wedge y'^*)^*] \\
&= 1 \wedge [y'^* \vee (y \wedge y'^*)^*] \text{ by (8)} \\
&= [y'^* \vee (y \wedge y'^*)^*] \\
&= [y'^* \vee (y \wedge y'^*)'^{**}] \text{ by (L1)} \\
&= [y'^* \vee (y'^* \wedge y'^{*\prime*})^*] \\
&= 1, \text{ by (8)},
\end{aligned}
$$

which completes the proof.   □

Now we will introduce a condition for an algebra $\mathbf{A} \in \mathbb{AG}$ that will be used in the rest of the paper.

$$(\mathrm{SC}) \qquad x \neq 1 \quad \text{then} \quad x \wedge x'^* = 0.$$

Lemma 3.8. *Let* $\mathbf{A} \in \mathbb{AG}$ *be directly indecomposable. Then* $\mathbf{A}$ *satisfies* (SC).

Proof. Suppose $\mathbf{A}$ does not satisfy (SC). Then there exists a $b \in A$ such that $b \neq 1$ and $b \wedge b'^* \neq 0$. Since $b \wedge b'^* = (b \wedge b'^*)'^*$ by (L1), we have $(b \wedge b'^*)^* = (b \wedge b'^*)'$ by Lemma 3.4, which, in view of Lemma 3.7 (b) implies $(b \wedge b'^*) \vee (b \wedge b'^*)^* = 1$. Hence by Corollary 3.6, we get $\mathbf{A} \cong (b \wedge b'^*] \times ((b \wedge b'^*)^*]$. Since $b \wedge b'^* \notin \{0, 1\}$, $\mathbf{A}$ is expressed as a nontrivial direct product, completing the proof. $\square$

Lemma 3.9. *Let* $\mathbf{A} \in \mathbb{AG}$ *satisfying* (SC) *and* $a \in A \setminus \{0\}$. *Then* $a \vee a^{*\prime} = 1$.

Proof. Let $x \in A$. Then, since $x' \leq x \vee x'$, we get $x \geq x'' \geq (x' \vee x)'$. Hence, $x \vee (x' \vee x)' = x$, implying $x^* \wedge (x' \vee x)'^* = x^*$. Thus, we have

$$\text{For } x \in A, \ x^* \leq (x \vee x')'^*. \tag{9}$$

Replacing x by x^* in (9), we get $x \leq x^{**} \leq (x \vee x^{*\prime})'^*$. Thus,

$$\text{For } x \in A, \ x \leq (x \vee x^{*\prime})'^*. \tag{10}$$

Now, suppose the conclusion of the lemma is false. Then $a \vee a^{*\prime} \neq 1$. Therefore, $(a \vee a^{*\prime}) \wedge (a \vee a^{*\prime})'^* = 0$ by (SC), from which we get $a \wedge (a \vee a^{*\prime}) \wedge (a \vee a^{*\prime})'^* = 0$, which simplifies to $a \wedge (a \vee a^{*\prime})'^* = 0$. It follows, in view of (10), that $a = 0$, which is a contradiction to $a \neq 0$., proving the lemma. $\square$

Lemma 3.10. *Let* $\mathbf{A} \in \mathbb{AG}$ *and* $a, b \in A$ *such that*
    (i) $\mathbf{A}$ *satisfies* (SC);
    (ii) $0 < a < b < 1$.
Then, $a^{**} = 1$.

Proof. Assume $\mathbf{A}$ and a, b satisfy the hypothesis, and further suppose $a^{**} \neq 1$. We wish to arrive at a contradiction.
    CLAIM 1: $a^{**} = a$.
For, from the supposition that $a^{**} \neq 1$, it is clear that $a^* \neq 0$ and so, $a^{*\prime} \neq 1$, in view of the axiom $(*)_w$. Hence, $a^{*\prime} \wedge a^{*\prime\prime *} = 0$ by (SC). So,

$$
\begin{aligned}
a &= a \vee (a^{*\prime} \wedge a^{*\prime\prime *}) \\
&= (a \vee a^{*\prime}) \wedge (a \vee a^{*\prime\prime *}) \\
&= a \vee a^{*\prime\prime *}, \text{ since } a \vee a^{*\prime} = 1 \text{ by Lemma } 3.9 \\
&= a \vee a^{**} \text{ since } a^{*\prime\prime} = a^* \text{ by the axiom } (*)_w \\
&= a^{**}, \text{ as } a \leq a^{**},
\end{aligned}
$$

proving the claim.

    CLAIM 2: $a'' = a$.
For,

$$
\begin{aligned}
a'' &= a^{**\prime\prime} \text{ by } \mathbf{CLAIM \ 1} \\
&= a^{**} \text{ by the axiom } (*)_w \\
&= a \text{ by } \mathbf{CLAIM \ 1}.
\end{aligned}
$$

    CLAIM 3: $b \leq a'$.
From (R1) we have $b \wedge (a' \vee a'^*) = b \wedge [(b \wedge b'^{*\prime}) \vee (a' \vee a'^*)] = (b \wedge b'^{*\prime}) \vee [b \wedge (a' \vee a'^*)] = b \wedge [b'^{*\prime} \vee (a' \vee a'^*)]$. Thus we have

$$b \wedge (a' \vee a'^*) = b \wedge [b'^{*\prime} \vee a' \vee a'^*]. \tag{11}$$

As $b \neq 1$, we have $b \wedge b'^* = 0$ by (SC). Also, as $a \leq b$, we have $a'^* \leq b'^*$ implying $b \wedge a'^* = 0$. So $a' = a' \vee (b \wedge a'^*)$, which implies $a' = (a' \vee b) \wedge (a' \vee a'^*)$, whence, $b \wedge a' = b \wedge (a' \vee a'^*)$. But we know $b \wedge (a' \vee a'^*) = b \wedge [b'^{*'} \vee (a' \vee a'^*)]$ from (11), whence we have the following:

$$b \wedge a' = b \wedge (b'^{*'} \vee a' \vee a'^*). \tag{12}$$

In view of (SC), as $b \neq 1$, we get $b \wedge b'^* = 0$. So, $b' \vee b'^{*'} = 1$, implying $a' \vee b'^{*'} = 1$, as $a \leq b$. Hence, from (12) we have $b \wedge a' = b \wedge (1 \vee a'^*)$. Thus we get $b \wedge a' = b$.

CLAIM 4: $a' = 1$.

For, suppose $a' \neq 1$. Then $a' \wedge a'^* = 0$. Hence $(a' \wedge a'^*) \vee (a' \wedge a''^*) = a' \wedge a''^*$, implying $a' \wedge (a'^* \vee a''^*) = a' \wedge a''^*$. Since $a'^* \vee a''^* = 1$, we have $a' = a' \wedge a''^*$. Hence $b \wedge a' = b \wedge a''^*$. But, from CLAIM 3, we have $b = b \wedge a'$. Hence, we get $b \leq a''^* \leq a^{**}$.

But a"=a by CLAIM 2. Hence we have $b \leq a^{**}$. Also we know $a^{**} = a$ by CLAIM 1. Thus $b \leq a$, which is a contradiction to $a < b$. Hence we conclude $a' = 1$, proving the claim.

Now, in view of CLAIM 2 and CLAIM 4, we get $a = a'' = 0$, which is a contradiction to the hypothesis that $a > 0$. This contradiction proves that our initial supposition is false. Thus the conclusion $a^{**} = 1$ holds, proving the lemma.   □

Let $h(\mathbf{A})$ denote the height of $\mathbf{A} \in \mathbb{AG}$.

Lemma 3.11. *Let $\mathbf{A} \in \mathbb{AG}$ satisfy* (SC). *Then $h(A) \leq 2$.*

Proof. Suppose $h(A) > 2$. Hence, there exist elements $a, b \in A$ such that $0 < a < b < 1$. Since $b \neq 1$, we have $b \wedge b'^* = 0$ by (SC), implying that $a \wedge b'^* = 0$ since $a < b$. So, $a^{**} \wedge b'^{***} = 0$ which implies $b'^* = 0$, since $a^{**} = 1$ by Lemma 3.10. Thus we have

$$b'^* = 0. \tag{13}$$

From regularity it follows that $b \wedge b'^{*'} \leq a \vee a^*$, which implies $b \leq a$ in view of (13) and Lemma 3.10. Hence we have arrived at a contradiction, proving the lemma.   □

The following theorem gives an explicit description of subdirectly irreducible Almost Gautama algebras.

Theorem 3.12. *Let $\mathbf{A} \in \mathbb{AG}$. Then the following are equivalent:*

(1) $\mathbf{A}$ *is simple;*
(2) $\mathbf{A}$ *is subdirectly irreducible;*
(3) $\mathbf{A}$ *is directly indecomposable;*
(4) $\mathbf{A}$ *satisfies (SC);*
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{dmst}}, \mathbf{4_{dmba}}\}$, *where $\mathbf{4_{dmba}}$ is the algebra in Figure 3.*

Proof. It is well-known that $(1) \Rightarrow (2) \Rightarrow (3)$. $(3) \Rightarrow (4)$ is proved in Lemma 3.8. We now prove $(4) \Rightarrow (5)$; so, we assume (4). We know from Lemma 3.11 that the height of the lattice reduct of $\mathbf{A}$ is ≤ 2. Now it is easy to see that the only nontrivial algebras in $\mathbb{AG}$ of height ≤ 2, up to isomorphism, are $\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}$, $\mathbf{4_{dmba}}$ and $\mathbf{2} \times \mathbf{2}$. But, it is easily seen that the algebra $\mathbf{2} \times \mathbf{2}$ does not satisfy (4); thus, (5) holds. Finally, it is routine to verify that $\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}$, and $\mathbf{4}_{dmba}$ are indeed simple, thus, $(5) \Rightarrow (1)$. Hence, the proof of the theorem is complete.   □

In the rest of this section we present several consequences of Theorem 3.12.

Corollary 3.13. *The variety $\mathbb{AG}$ is generated by $\{\mathbf{3}_{dblst}, \mathbf{3}_{dmst}, \mathbf{4_{dmba}}\}$. Hence, every algebra $A \in \mathbb{AG}$ is a subdirect product of $\mathbf{2}$, $\mathbf{3}_{dblst}$, $\mathbf{3}_{dmst}$, and $\mathbf{4_{dmba}}$.*

Corollary 3.13 will be improved further in Corollary 5.4.

3.1   Equational Bases for subvarieties of $\mathbb{AG}$

We now give equational bases for all subvarieties of the variety $\mathbb{AG}$. In view of Theorem 3.12, the proofs of the following theorems are easy and hence are left to the reader.

Corollary 3.14. *The variety* $\mathbb{V}(\mathbf{2})(= \mathbb{BA})$ *is defined, modulo* $\mathbb{AG}$, *by the identity:* $x^* \approx x'$.

Corollary 3.15. *The variety* $\mathbb{V}(\mathbf{3_{dblst}})$ *is defined, modulo* $\mathbb{AG}$, *by*
(i) the identity: $x \vee x' \approx 1$
or by
(ii) the identity: $x' \wedge x'' \approx 0$,
or by
(iii) the identity: $x'^{*'} \approx x'$.

Corollary 3.16. *The variety* $\mathbb{V}(\mathbf{3_{klst}})$ *is defined, modulo* $\mathbb{AG}$, *by the identities:* $x^{*'} \approx x^{**}$, *and* $x'' \approx x$.

Corollary 3.17. *The variety* $\mathbb{V}(\mathbf{4_{dmba}})$ *is defined, modulo* $\mathbb{AG}$, *by*
(i) the identity: $x \vee x^* \approx 1$.
or by
(ii) the identity: $x^{**} \approx x$,
or by
(iii) the identity: $x'^{*'} \approx x^*$.
or by
(iv) $x^{**} \approx x''$.

Corollary 3.18. *The variety* $\mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{3_{klst}}\})$ *is defined, modulo* $\mathbb{AG}$, *by:*
$x^{*'} \approx x^{**}$.

Since $\mathbf{3_{dblst}}$ and $\mathbf{3_{klst}}$ are Gautama algebras and $\mathbf{4_{dmba}}$ is not, the following corollary, which was first proved in [64], is immediate.

Corollary 3.19. [64] *The variety* $\mathbb{G}$ *of Gautama algebras is generated by* $\{\mathbf{3_{dblst}}, \mathbf{3_{klst}}\}$ *(i.e.,* $\mathbb{G} = \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{3_{klst}})$*)*.

Corollary 3.20. *The variety* $\mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{4_{dmba}}\})$ *is defined, modulo* $\mathbb{AG}$, *by the identity:*
    $x' \vee (y^* \vee z) \approx (x' \vee y)^* \vee (x' \vee z)$ (A version of (JID),
    or by the identity:
    $(x' \vee y)^* \vee x' \approx x' \vee y^*$.

Corollary 3.21. *The variety* $\mathbb{V}(\{\mathbf{3_{klst}}, \mathbf{4_{dmba}}\})$ *is defined, modulo* $\mathbb{AG}$, *by the identity:* $x'' \approx x$.

3.2   The Lattice of Subvarieties of $\mathbb{AG}$

We give more applications of Theorem 3.12. The proof of the following corollary of Theorem 3.12 is easy.

Corollary 3.22.

(1) *The lattice of nontrivial subvarieties of* $\mathbb{AG}$ *is isomorphic to the 8-element Boolean lattice with* $\mathbb{V}(\mathbf{2})$ *(i.e., the variety of Boolean allgebras) as the least element. The Hasse diagram of this lattice is given in Figure 4.*
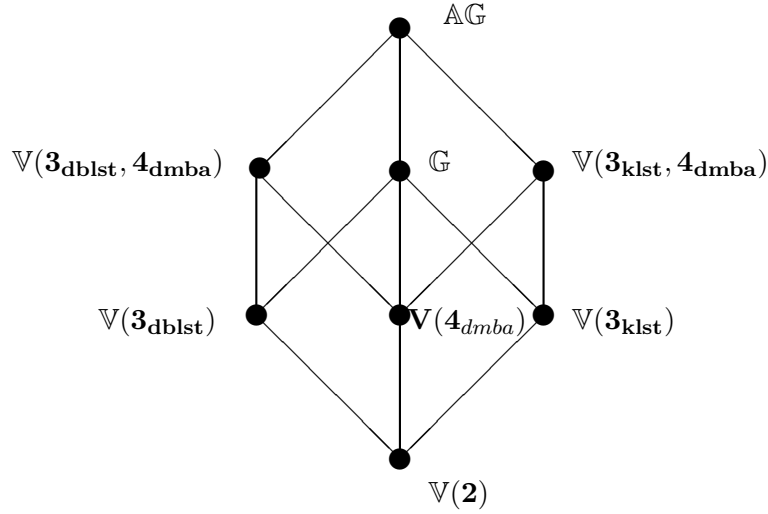
Figure 4

(2) *The lattice of nontrivial subvarieties of $\mathbb{G}$ is isomorphic to the 4-element Boolean lattice with $\mathbb{V}(\mathbf{2})$ as the least element.*

Since the variety $\mathbb{AG}$ is finitely generated, the following corollary is immediate.

Corollary 3.23. *The equational theories of $\mathbb{AG}$ and all its subvarieties are decidable.*

In fact, a much stronger result is true (see Corollary 5.5).

In passing, we mention two new axiomatizations for the variety $\mathbb{G}$, whose proofs are left to the reader. The variety of regular dually quasi-De Morgan p-algebras of level 1 (i.e., satisfying $x \wedge x'^* \wedge x'^{*'*} \approx x \wedge x'^*$) is denoted by $\mathbb{RDQDP}_1$.

Theorem 3.24. *The variety $\mathbb{G}$ is also defined, modulo $\mathbb{RDQDP}_1$, by*
$$x^{*'*} = x^*.$$

Theorem 3.25. *The variety $\mathbb{G}$ is also defined, modulo $\mathbb{RDQDP}_1$, by*
$$x'^{*'*} = x'^* \text{ and } x^* \vee x^{**} \approx 1.$$

4   The Variety of Almost Gautama Heyting Algebras ($\mathbb{AGH}$)

Observe that the implication connective is missing in algebras in $\mathbb{AG}$. Let us, therefore, consider the language $\mathbf{L} = \langle \vee, \wedge, \rightarrow, ', 0, 1 \rangle$. We will now define a new variety of algebras, namely the variety $\mathbb{AGH}$ of Almost

Gautama Heyting algebras of type **L** and show that it is term-equivalent to the variety $\mathbb{AG}$ of Almost Gautama algebras. This fact will play a crucial role later in "logicizing" the variety $\mathbb{AG}$.

Actually, $\mathbb{AGH}$ turns out, to our surprise, to coincide with the variety $\mathbb{RDQDS}t\mathbb{H}_1$, already introduced in [60], which is a subvariety of $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras. The variety $\mathbb{DHMSH}$ and its many subvarieties have been investigated in a series of papers, some of which are: [4, 32, 45, 48, 46, 58, 59, 60, 61, 62, 63, 15, 16, 64, 17]. We, therefore, need to recall some preliminaries.

Semi-Heyting algebras were introduced in [47]; but the first results about them were published in [57]. For further results on semi-Heyting algebras, see, for example, [1, 2, 3, 14, 13].

Definition 4.1. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra if $\mathbf{A}$ satisfies the following conditions:

   (i) $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice,

  (ii) $x \wedge (x \rightarrow y) \approx x \wedge y$,

 (iii) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$,

 (iv) $x \rightarrow x \approx 1$.

The variety of semi-Heyting algebras is denoted by $\mathbb{SH}$.
$\mathbf{A} \in \mathbb{SH}$ is a Heyting algebra if it satisfies:
   (H) $(x \wedge y) \rightarrow x \approx 1$.
The variety of Heyting algebras is denoted by $\mathbb{H}$.

We can now define the crucial notion of a dually hemimorphic semi-Heyting algebra fundamental to the rest of this paper.

Definition 4.2. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ is a dually hemimorphic semi-Heyting algebra (see [58]). if $\mathbf{A}$ satisfies the following conditions:

(E1) $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra,

(E2) $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is a dually hemimorphic algebra (see Definition 2.3).

The variety of dually hemimorphic semi-Heyting algebras will be denoted by $\mathbb{DHMSH}$.
$\mathbf{A} \in \mathbb{DHMSH}$ is a dually hemimorphic Heyting algebra if it satisfies:
   (H) $(x \wedge y) \rightarrow x \approx 1$.
$\mathbb{DHMH}$ denotes the variety of dually hemimorphic Heyting algebras. $\mathbf{A} \in \mathbb{DHMSH}$ is a dually quasi-De Morgan semi-Heyting algebra if its reduct $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is in $\mathbb{DQD}$.

The varieties of dually quasi-De Morgan semi-Heyting algebras and of dually quasi-De Morgan Heyting algebras are, respectively, denoted by $\mathbb{DQDSH}$ and $\mathbb{DQDH}$.
$\mathbf{A} \in \mathbb{DHMSH}$ *is regular* if **L** satisfies:

(R1) $x \wedge x^+ \leq y \vee y^*$, where $x^* := x \rightarrow 0$ and $x^+ := x'^{*'}$.

The variety of regular dually hemimorphic [quasi-De Morgan] Heyting algebras is denoted by $\mathbb{RDHMH}$ [$\mathbb{RDQDH}$].

$\mathbf{A} \in \mathbb{RDQDH}$ is a *regular dually quasi-De Morgan Stone Heyting algebra* if $\mathbf{A}$ satisfies:

(St) $x^* \vee x^{**} \approx 1$.

Let $\mathbb{RDQDS}t\mathbb{H}$ denote the variety of regular dually quasi De Morgan Stone Heyting algebras.

Remark 4.3. The reader is cautioned here not to confuse the notion of regularity given in the above definition with the one given in [58].

The varieties $\mathbb{DHMSH}$, $\mathbb{DHMH}$, $\mathbb{DQDH}$, $\mathbb{RDQDH}$, $\mathbb{RDQDStH}$ and many of their subvarieties are examined, in [58, 59, 60, 61]. The logics associated with those subvarieties of the variety $\mathbb{DHMSH}$ are investigated in [16].

The notion of "level n" has played an important role in the classification of subvarieties of $\mathbb{DHMSH}$ in [58], although this name was not explicitly used there. We only need the definition of "level 1" here.

Definition 4.4. An algebra $\mathbf{A} \in \mathbb{DHMSH}$ is of level 1 if it satisfies the identity:
$$x \wedge x'^* \approx x \wedge x'^* \wedge x'^{*'*} \quad \text{(Level 1)}.$$

Let $\mathbb{DHMSH}_1$ denote the subvariety of $\mathbb{DHMSH}$ of level 1. For a subvariety $\mathbb{V}$ of $\mathbb{DHMSH}$, we let $\mathbb{V}_1 := \mathbb{V} \cap \mathbb{DHMSH}_1$. Thus, $\mathbb{RDQDStH}_1$ denotes the subvariety of $\mathbb{RDQDStH}$ of level 1.

We are ready to define the variety of Almost Gautama Heyting algebras.

Definition 4.5. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ is an Almost Gautama Heyting algebra if $\mathbf{A} \in \mathbb{DHMSH}$ and satisfies the following additional axioms:

(1) $(x \wedge y) \rightarrow x \approx 1$ (H),

(2) $x^* \vee x^{**} \approx 1$ (St), where $x^* := x \rightarrow 0$,

(3) $(x \vee y)'' \approx x'' \vee y''$,

(4) $x'' \leq x$,

(5) $x \wedge x'^{*'} \leq y \vee y^*$ (R1),

(6) $x^{*''} \approx x^*$ (*)$_w$,

(7) $(x \wedge x'^*)'^* \approx x \wedge x'^*$ (L1).

The variety of Almost Gautama Heyting algebras will be denoted by $\mathbb{AGH}$.

Remark 4.6. It is clear that $\mathbb{AGH} \subseteq \mathbb{DQDH}$. Hence, it follows from Lemma 2.4 (5) of [59] that the identity (Level 1) is equivalent to the following identity in $\mathbb{GH}$:
$$\text{(L1)} \qquad (x \wedge x'^*)'^* \approx x \wedge x'^*.$$

Proposition 4.7. $\mathbb{AGH} \subseteq \mathbb{RDQDStH}_1 \subset \mathbb{DQDH}_1 \subset \mathbb{DHMSH}_1$.

Proof. Axioms (1) –(5) of $\mathbb{AGH}$ imply that $\mathbb{AGH} \subseteq \mathbb{RDQDStH}$. Also, the variety $\mathbb{AGH}$ is of level 1 by definition and Remark 4.6, whence $\mathbb{AGH} \subseteq \mathbb{RDQDStH}_1$. The rest of the inclusions are also immediate from the relevant definitions. $\square$

Remark 4.8.

(1) Let $\mathbf{3_{dsth}} := \langle \mathbf{3}, \vee, \wedge, \rightarrow, {}^+, 0, 1\}$ be the algebra, where $\mathbf{3}$ is the 3-element chain, $0 < a < 1$ (viewed as a bounded distributive lattice), the operation ${}^+$ is defined as: $0^+ = 1$, $a^+ = 1$ and $1^+ = 0$, and $\rightarrow$ is defined as follows:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Clearly, $\mathbf{3_{dsth}}$ is an algebra in $\mathbb{AGH}$.

(2) Let $\mathbf{3_{klh}} := \langle \mathbf{3}, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ be the algebra, where $\mathbf{3}$ is the 3-element chain, $0 < a < 1$ (viewed as a bounded distributive lattice), the operation $'$ is defined as: $0' = 1$, $a' = a$ and $1' = 0$, and $\rightarrow$ is defined as in (1). Note that $\mathbf{3_{klh}}$ is also an algebra in $\mathbb{AGH}$.

(3) Let $\mathbf{4_{dmh}} := \langle \mathbf{4}, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ be the De Morgan Heyting (Boolean) algebra, where $\mathbf{4}$, is the 4-element Boolean lattice shown below and the operation $'$ is defined as: $0' = 1$, $a' = a$, $b' = b$ and $1' = 0$; and $\rightarrow$ is the Boolean implication:
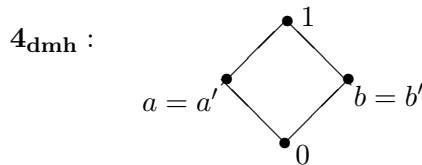

Figure 4

It is easy to see that $\mathbf{4_{dmh}} \in \mathbb{AGH}$.

We, now, wish to give an explicit description of the subdirectly irreducible algebras in $\mathbb{AGH}$. To achieve this goal, we need some definitions and results from [59].

The following lemma is a special case of Lemma 4.8 of [59], when the underlying semi-Heyting algebra is actually a Heyting algebra.

Lemma 4.9. *Let* $\mathbf{A} \in \mathbb{RDQDS}t\mathbb{H}_1$ *satisfy the simplicity condition:*
(SC) For every $x \in L$, if $x \neq 1$, then $x \wedge x'^* = 0$.
Then $\mathbf{A}$ is of height at most 2.

Corollary 4.10. *Let* $\mathbf{A} \in \mathbb{AGH}$*. If* $\mathbf{A}$ *satisfies* (SC)*, then* $\mathbf{A}$ *is of height at most 2.*

Proof. We know $\mathbb{AGH} \subseteq \mathbb{RDQDS}t\mathbb{H}_1$ by Proposition 4.7. Now apply Lemma 4.9.   □

The following lemma is a special case of Corollary 4.1 of [59].

Lemma 4.11. *Let* $\mathbf{A} \in \mathbb{RDQDS}t\mathbb{H}_1$ *with* $|A| \geq 2$*. Then TFAE:*

(1) $\mathbf{A}$ *is simple,*

(2) $\mathbf{A}$ *is subdirectly irreducible,*

(3) *For every* $x \in \mathbf{A}$*, if* $x \neq 1$*, then* $x \wedge x'^* = 0$*.*

We are now ready to give a concrete description of the subdirectly irreducible algebras in $\mathbb{AGH}$.

Theorem 4.12. *Let* $\mathbf{L} \in \mathbb{AGH}$ *with* $|L| \geq 2$*. Then TFAE:*

(1) **L** *is simple,*

(2) **L** *is subdirectly irreducible,*

(3) *For every $x \in L$, if $x \neq 1$, then $x \wedge x'^* = 0$,*

(4) $\mathbf{L} \in \{\mathbf{2}, \ \mathbf{3_{dsth}}, \ \mathbf{3_{klh}}, \mathbf{4_{dmh}}, \}$, *up to isomorphism.*

Proof. $(1) \Rightarrow (2)$ is well-known, while $(2) \Rightarrow (3)$ by Lemma 4.11. Suppose (3) holds. Then **A** is of height at most 2 by Corollary 4.10. Then it is easy to see that the algebras of height at most 2 in $\mathbb{AGH}$ are, up to isomorphism, precisely $\mathbf{2}$, $\mathbf{3_{dsth}}$, $\mathbf{3_{klh}}$, $\mathbf{4_{dmh}}$, and $\mathbf{2} \times \mathbf{2}$. It is also clear that the algebra $\mathbf{2} \times \mathbf{2}$ does not satisfy the hypothesis (3), implying that (4) holds. Thus (3) implies (4), while it is routine to verify that (4) implies (1), proving the theorem.   □

Corollary 4.13.

   (i) *The smallest non-trivial subvariety of $\mathbb{AGH}$ is $\mathbb{BA}$.*

  (ii) *The lattice of nontrivial subvarieties of $\mathbb{AGH}$ has exactly 3 atoms: $\mathbb{V}(\mathbf{3_{dsth}})$, $\mathbb{V}(\mathbf{3_{klh}})$, and $\mathbb{V}(\mathbf{4_{dmh}})$.*

 (iii) *The lattice of nontrivial subvarieties of $\mathbb{AGH}$ is isomorphic to 8-element Boolean algebra.*

   Let $\mathbb{GH}$ denote the variety $\mathbb{V}(\mathbf{3_{dsth}}), \mathbf{3_{klh}})$. We will call its elements Gautama Heyting algebras.

Corollary 4.14. *Let $\mathbf{A} := \langle A, \vee, \wedge, \rightarrow^{\mathbf{A}}, \ ', 0, 1 \rangle \in \mathbb{AGH}$. Define $\rightarrow_k$ on A by*

$$x \rightarrow_k y := (x^* \vee y^{**})^{**} \wedge [(x \vee x^*)'^{*'} \vee x^* \vee y \vee y^*],$$

where $x^* := x \rightarrow^{\mathbf{A}} 0$. Then, $\rightarrow^{\mathbf{A}} = \rightarrow_k$.

Proof. It suffices to show that the equality holds on the (non-trivial) subdirectly irreducible algebras in $\mathbb{AGH}$, which, in view of Theorem 4.12, are $\mathbf{2}, \mathbf{3_{dsth}}$, and $\mathbf{3_{klh}}$, and $\mathbf{4_{dmh}}$, up to isomorphism. Now it is routine to verify the equality of $\rightarrow^{\mathbf{A}}$ and $\rightarrow_k$ on these four algebras.   □

Corollary 4.15. $\mathbb{AGH} = \mathbb{RDQDS}t\mathbb{H}_1$.

Proof. It is clear from Theorem 4.12 and Theorem 4.9 of [59] that both the varieties have the same subdirectly irreducible algebras.   □

5   Applications

5.1   Term-Equivalence between Almost Gautama algebras and $\mathbb{AGH}$-algebras

The following theorem will play a crucial role in describing the logic associated with the variety of Almost Gautama algebras.

Theorem 5.1. *The varieties $\mathbb{AG}$ and $\mathbb{AGH}$ are term-equivalent. More explicitly,*

  (a) *For $\mathbf{A} := \langle A, \vee, \wedge, ^*, ', 0, 1 \rangle \in \mathbb{AG}$, let $\mathbf{A}_{agh} := \langle A, \vee, \wedge, \rightarrow_k, ', 0, 1 \rangle$, where $\rightarrow_k$ is defined by:*

$$x \rightarrow_k y := (x^* \vee y^{**})^{**} \wedge [(x \vee x^*)'^{*'} \vee x^* \vee y \vee y^*].$$

   Then $\mathbf{A}_{agh} \in \mathbb{AGH}$.

(b) *For* $\mathbf{A} := \langle A, \vee, \wedge, \rightarrow^{\mathbf{A}}, ', 0, 1 \rangle \in \mathbb{AGH}$, *let* $\mathbf{A}_{ag} := \langle A, \vee, \wedge, \circ, ', 0, 1 \rangle$, *where* $\circ$ *is defined by* $x^{\circ} := x \rightarrow^{\mathbf{A}}$ 0. *Then* $\mathbf{A}_{ag} \in \mathbb{AG}$.

(c) *If* $\mathbf{A} \in \mathbb{AG}$, *then* $(\mathbf{A}_{agh})_{ag} = \mathbf{A}$.

(d) *If* $\mathbf{A} \in \mathbb{AGH}$, *then* $(\mathbf{A}_{ag})_{agh} = \mathbf{A}$.

Proof. (a): Observe that it suffices to verify that (a) holds for subdirectly irreducible members of $\mathbb{AG}$. So, let $\mathbf{A}$ be a nontrivial subdirectly irreducible algebra in $\mathbb{AG}$. Then $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}, \mathbf{4_{dmbl}}\}$ by Theorem 3.12. It is obvious that $\mathbf{A}_{agh} \in \{\mathbf{2}, \mathbf{3_{dsth}}, \mathbf{3_{klh}}, \mathbf{4_{dmh}}\}$. It is now routine to verify that $\mathbf{A}_{agh} \in \mathbb{AGH}$, whence (a) is proved.

(b): The proof of (b) is similar to that of (a), in view of Theorem 4.12.

(c): Let $\mathbf{A} := \langle A, \vee, \wedge, {}^{*}, ', 0, 1 \rangle \in \mathbb{AG}$. Then $\mathbf{A}_{agh} \in \mathbb{AGH}$ by (a). Now, let $\mathbf{A_1} := (\mathbf{A}_{agh})_{gh} := \langle A, \vee, \wedge, \circ, ', 0, 1 \rangle$, where $x^{\circ} := x \rightarrow_k 0$. It is clear that $x \rightarrow_k 0 = x^{*}$. Then it follows that $x^{\circ} = x^{*}$, implying $\mathbf{A_1} = \mathbf{A}$.

(d): Let $\mathbf{A} := \langle A, \vee, \wedge, \rightarrow^{\mathbf{A}}, ', 0, 1 \rangle \in \mathbb{AGH}$. Then $\mathbf{A}_{ag} \in \mathbb{AG}$ by (b). Now, let $\mathbf{A_1} := (\mathbf{A}_{gh})_{agh} := \langle A, \vee, \wedge, \rightarrow_k, 0, 1 \rangle$, Observe that $\rightarrow_k = \rightarrow^{\mathbf{A}}$ by Corollary 4.14. Hence, $\mathbf{A_1} = \mathbf{A}$, completing the proof. $\quad\square$

5.2   Discriminator Subvarieties of $\mathbb{AGH}$

Recall that the notions of a discriminator term, a discriminator variety and a quasiprimal algebra were defined in Section 1. Discriminator varieties have been a popular topic with a considerable amount of research (see for example, [10, 72]).

Theorem 5.2.

(i) *The variety $\mathbb{AGH}$ is a discriminator variety with the discriminator term*

$$t(x, y, z) := [z \wedge d((x \vee y) \rightarrow (x \wedge y))] \vee [x \wedge (d((x \vee y) \rightarrow (x \wedge y)))^{*}], \text{ where } d(x) = x \wedge x'^{*}.$$

(ii) *The algebras $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$ and $\mathbf{4_{dmbl}}$ are quasiprimal.*

Proof. From Theorem 4.12 (3), it is clear that $x \neq 1 \Rightarrow d(x) = 0$ and $x = 1 \Rightarrow d(x) = 1$ on simple algebras. Hence, in view of Theorem 4.12 (4), if $\mathbf{L} \in \{\mathbf{2}, \mathbf{3_{dsth}}, \mathbf{3_{klh}}, \mathbf{4_{dmh}}, \}$, then it is easy to verify the following two conditions: (a) $x \neq y \Rightarrow t(x, y, z) = x$ and (b) $x = y \Rightarrow t(x, y, z) = z$. Hence $t(x, y, z)$ is a discriminator term and hence, $\mathbb{AGH}$ is a discriminator variety. $\quad\square$

Since $\mathbb{AG}$ and $\mathbb{AGH}$ are term-equivalent by Theorem 5.1, the following corollary is immediate.

Corollary 5.3. *The varieties $\mathbb{AG}$, $\mathbb{G}$, $\mathbb{RDBLSt}$, $\mathbb{RKLSt}$ and the remaining subvarieties of $\mathbb{AG}$ are discriminator varieties.*

The algebras in $\mathbb{AG}$ have a nice representation as mentioned in the next corollary which is a considerable improvement of Corollary 3.13.

Corollary 5.4. *If $\mathbb{V} \in \{\mathbb{AG}, \mathbb{G}, \mathbb{RDBLSt}, \mathbb{RKLSt}\}$, then every algebra in $\mathbb{V}$ is isomorphic to a Boolean Product of simple algebras in $\mathbb{V}$.*

Proof. Apply [10, Chapter IV, Theorem 9.4] and Corollary 5.3. $\quad\square$

The following corollary is a considerable improvement of Corollary 3.23.

Corollary 5.5.

(a) *The first-order theory of* $\mathbb{AGH}$ *is decidable,.*

(b) *The first-order theories of* $\mathbb{AG}$, $\mathbb{G}$, $\mathbb{RDBLSt}$, $\mathbb{RKLSt}$ *and the remaining subvarieties of* $\mathbb{AG}$ *are all decidable.*

Proof. The corollary follows from a well known result (see [11]) that a finitely generated discriminator variety of finite type has a decidable first-order theory. (b) follows from (a), in view of Theorem 5.1.   □

We close this section by mentioning two new axiomatizations for the variety $\mathbb{AGH}$. The proofs of these theorems are left to the reader.

Theorem 5.6. *The variety* $\mathbb{AGH}$ *is also defined, modulo* $\mathbb{RDQDH}_1$, *by*
$x^{*''*} = x^{**}$.

Theorem 5.7. *The variety* $\mathbb{AGH}$ *is also defined, modulo* $\mathbb{RDQDH}_1$ *by* $(x \wedge y)^{*'} = x^{*'} \wedge y^{*'}$.

6   Classical Nelson algebras, $\mathbb{RKLSt}$, $\mathbb{RKLStH}$ and 3-valued Lukasiewicz algebras

Nelson [34], Markov [26] and Vorobév [71] were the early contributors to the constructive logic with strong negation. Later, Rasiowa [39] introduced Nelson algebras (= quasi-pseudo-Boolean algebra) and used them to prove that the constructive logic with strong negation is implicative (see also [19]). Soon thereafter, Vakarelov [67] introduced the notion of classical Nelson algebras and proved that the variety of classical Nelson algebra is term equivalent to that of 3-valued Lukasiewicz algebras.

In this section we wish to prove this Vakarelov's result by (universal) algebraic means and then derive our main result of this section that the varieties of regular Kleene Stone algebras, of regular Kleene Stone Heyting algebras, of 3-valued Lukasiewicz algebras and of classical Nelson algebra with strong negation are all term-equivalent to one another. A logical consequence of this result will be presented in Section 8.1.

We will first recall the definition of Nelson algebras.

Definition 6.1. [33] A Nelson algebra is an algebra $\langle A, \vee, \wedge, \rightarrow, ', 1 \rangle$ such that the following conditions are satisfied for all x, y, z in A:

(N1) $x \wedge (x \vee y) = x$,

(N2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

(N3) $x'' = x$,

(N4) $(x \wedge y)' = x' \vee y'$,

(N5) $x \wedge x' = (x \wedge x') \wedge (y \vee y')$,

(N6) $x \rightarrow x = 1$,

(N7) $x \wedge (x \rightarrow y) = x \wedge (x' \vee y)$,

(N8) $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

The variety of Nelson algebras is denoted by $\mathbb{N}$. Let $1' := 0$.

A complete proof of the following Theorem, which was first proved in [31], is available in [69, Corollary 2.5].

Theorem 6.2. *A nontrivial algebra* $\mathbf{A} \in \mathbb{N}$ *is simple if and only if* $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_N}\}$, *where* $\mathbf{3_N}$ *is the algebra shown in Figure 5, with* $'$ *defined as:* $0' = 1$, $a' = a$, $1' = 0$.

$\to$	0	a	1
0	1	1	1
a	1	1	1
1	0	a	1
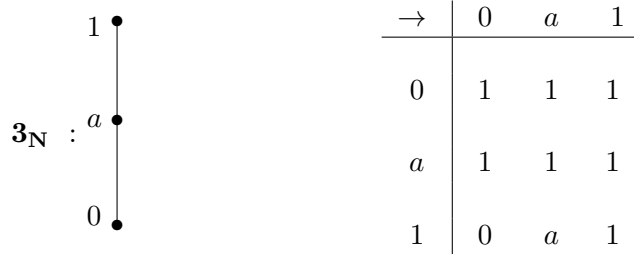
The following definition is due to [67].

Definition 6.3. A Nelson algebra is a classical Nelson algebra if it satisfies:
(C) $x \vee x^+ \approx 1$, where $x^+ := x \to 0$.

We will denote by $\mathbb{CN}$ the variety of Classical Nelson algebras. Observe that $\mathbf{2}, \mathbf{3_N} \in \mathbb{CN}$. In fact, we wish to show that $\mathbb{CN}$ is generated by $\mathbf{3_N}$.

The following theorem was proved in [69, Theorem 4.13].

Theorem 6.4. $\mathbf{A} \in \mathbb{CN}$ *is semisimple if and only if* $\mathbf{A} \models x \vee x^+ \approx 1$.

Corollary 6.5. $\mathbb{CN} = \mathbb{V}(\mathbf{3_N})$.

Proof. The corollary is immediate from Theorem 6.2 and Theorem 6.4. □

Corollary 6.6. *Let* $\mathbb{V}$ *be a subvariety of* $\mathbb{N}$. *Then the following are equivalent:*
(1) $\mathbb{V}$ *is a discriminator variety,*
(2) $\mathbb{V}$ *is semisimple,*
(3) $\mathbb{V} = \mathbb{V}(\mathbf{3_N})$.
(4) $\mathbb{V} = \mathbb{CN}$.

Corollary 6.7. *Let* $\mathbf{A}$ *be a classical Nelson algebra. For* $x \in A$, *set* $x^+ := x \to 1$. *Then the reduct* $\langle A, \vee, \wedge, ^+, 0, 1 \rangle$ *is a dually pseudocomplemented lattice.*

Proof. Observe that for $\mathbf{A} = \mathbf{3_N}$ the reduct in question is a dually pseudocomplemented lattice. Now apply Corollary 6.5. □

Let $\mathbf{3_L}$ denote the 3-element Łukasiewicz algebra.

Lemma 6.8. $\mathbf{3_L}$ *and* $\mathbf{3_N}$ *are term-equivalent.*

Proof. Given $\mathbf{3_N} = \langle \{0, a, 1\}, \vee, \wedge, \to, ', 1 \rangle \in \mathbb{CN}$, define a new operation $\rightsquigarrow$ on $\{0, a, 1\}$ by:
$x \rightsquigarrow y := (x \to y) \wedge (y' \to x')$.
Then $\rightsquigarrow$ and $\sim$, given by:

$\rightsquigarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

and

	$\sim$
0	1
a	a
1	0

are the well known operations of Łukasiewicz's three–valued algebra. Thus $\langle\{0, a, 1\}, \vee, \wedge, \rightsquigarrow, \sim, 1\rangle$ is a 3-valued Łukasiewicz algebra isomorphic to $\mathbf{3_L}$.

On the other hand, suppose $\mathbf{3}_L = \langle\{0, a, 1\}, \vee, \wedge, \rightsquigarrow, \sim, 1\rangle$ is the three-valued Łukasiewicz algebra. Then, consider the algebra $\bar{\mathbf{3}}_\mathbf{L} := \langle\{0, a, 1\}, \vee, \wedge, \rightarrow, ', 1\rangle$, where the operations $'$ and $\rightarrow$ are defined by:

$x' := x \rightsquigarrow \sim x$,

$x \rightarrow y = x \rightsquigarrow (x \rightsquigarrow y)$, for $x \in \{0, a, 1\}$.

(We could also define $\vee$ and $\wedge$ as follows: $x \vee y := (x \rightsquigarrow y) \rightsquigarrow y$, and $x \wedge y :=\sim (\sim x \vee \sim y)$.)

Then it is easy to verify that $\bar{\mathbf{3}}_\mathbf{L}$ is a classical Nelson algebra isomorphic to $\mathbf{3_N}$. The lemma follows.   □

We are now ready to prove our main theorem of this section.

Theorem 6.9. *The following varieties are term equivalent to one another:*
(a) The variety $\mathbb{RKLSt}$ of regular Kleene Stone algebras,
(b) The variety $\mathbb{RKLStH}$ of regular Kleene Stone Heyting algebras,
(c) The variety of 3-valued Lukasiewicz algebras,
(d) The variety of classical Nelson algebras with strong negation.

Proof. The equivalence of (a) and (b) follows from [64, Corollary 10]. The equivalence of (b) and (c) is proved in [15, Theorem 7.14]. It is well-known that the variety of 3-valued Lukasiewicz algebras is generated by $\mathbf{3}_L$, and we know from Theorem 6.5 that the variety of classical Nelson algebras is generated by $\mathbf{3_N}$. So, the equivalence of (c) and (d) follows from Lemma 6.8.   □

We close this section by pointing out that Nelson algebras are recently generalized to semi-Nelson algebras in [18] and to quasi-Nelson algebras in [42].

7   Logical Aspects of $\mathbb{AG}$

The rest of the paper, for the most part, is concerned with defining and investigating a propositional logic, in Hilbert-style, called $\mathcal{AG}$ (also known as $\mathcal{AGH}$) from the point of view of Abstract Algebraic Logic, with the ultimate goal of showing that the logic $\mathcal{AG}$ is algebraizable (in the sense of Blok and Pigozzi [9] with the variety $\mathbb{AG}$ of Almost Gautama algebras as its equivalent algebraic semantics. Logics corresponding to the subvarieties of $\mathbb{AG}$ are also defined and studied.

7.1   Abstract Algebraic Logic

In this subsection, we present the basic definitions and results of Abstract Algebraic Logic that will play a crucial role later.

Languages, Formulas and Logics

A language $\mathbf{L}$ is a set of finitary operations (or connectives), each with a fixed arity $n \geq 0$. In this paper, we identify $\perp$ and $\top$ with 0 and 1 respectively and thus consider the languages $\langle\vee, \wedge, \rightarrow, \sim, \perp, \top\rangle$ and $\langle\vee, \wedge, \rightarrow, ', 0, 1\rangle$ as the same. For a countably infinite set *Var* of propositional variables, the *formulas* of the language $\mathbf{L}$ are inductively defined as usual. The set of formulas in the language $\mathbf{L}$ will be denoted by $Fm_\mathbf{L}$

The set of formulas $Fm_\mathbf{L}$ can be turned into an algebra of formulas, denoted by $\mathbf{Fm_L}$, in the usual way. In what follows, Γ denotes a set of formulas and lower case Greek letters denote formulas. The homomorphisms from the formula algebra $\mathbf{Fm_L}$ into an $\mathbf{L}$-algebra (i.e, an algebra of type $\mathbf{L}$) $\mathbf{A}$ are called *interpretations* (or *valuations*) in A. The set of all such interpretations is denoted by $Hom(\mathbf{Fm_L}, \mathbf{A})$. If $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$ then the *interpretation of a formula* α under h is its image $h\alpha \in A$, while $h\Gamma$ denotes the set $\{h\phi \mid \phi \in \Gamma\}$.

Consequence Relations:

A *consequence relation* on $Fm_{\mathbf{L}}$ is a binary relation $\vdash$ between sets of formulas and formulas that satisfies the following conditions for all $\Gamma, \Delta \subseteq Fm_{\mathbf{L}}$ and $\phi \in Fm_{\mathbf{L}}$:

(i) $\phi \in \Gamma$ implies $\Gamma \vdash \phi$,

(ii) $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash \phi$,

(iii) $\Gamma \vdash \phi$ and $\Delta \vdash \beta$ for every $\beta \in \Gamma$ imply $\Delta \vdash \phi$.

A consequence relation $\vdash$ is *finitary* if $\Gamma \vdash \phi$ implies $\Gamma' \vdash \phi$ for some finite $\Gamma' \subseteq \Gamma$.

Structural Consequence Relations:

A consequence relation $\vdash$ is *structural* if

$\Gamma \vdash \phi$ implies $\sigma(\Gamma) \vdash \sigma(\phi)$ for every substitution σ ($\in Hom(\mathbf{Fm_L}, \mathbf{Fm_L})$), where $\sigma(\Gamma) := \{\sigma\alpha : \alpha \in \Gamma\}$.

Logics:

A **logic** (or **deductive system**) is a pair $\mathcal{S} := \langle \mathbf{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathbf{L}$ is a propositional language and $\vdash_{\mathcal{S}}$ is a finitary and structural consequence relation on $\mathrm{Fm}_{\mathbf{L}}$.

A *rule of inference* is a pair $\langle \Gamma, \phi \rangle$, where Γ is a finite set of formulas (the premises of the rule) and ϕ is a formula.

One way to present a logic $\mathcal{S}$ is by displaying it (syntactically) in **Hilbert-style**; that is, giving its axioms and rules of inference which induce a consequence relation $\vdash_S$ as follows:

$\Gamma \vdash_S \phi$ if there is a a **proof** (or, a **derivation**) of ϕ from Γ, where a proof is defined as a sequence of formulas $\phi_1, \ldots, \phi_n$, $n \in \mathbb{N}$, such that $\phi_n = \phi$, and for every $i \leq n$, one of the following conditions holds:

(i) $\phi_i \in \Gamma$,

(ii) there is an axiom ψ and a substitution σ such that $\phi_i = \sigma\psi$,

(iii) there is a rule $\langle \Delta, \psi \rangle$ and a substitution σ such that $\phi_i = \sigma\psi$ and $\sigma(\Delta) \subseteq \{\phi_j : j < i\}$.

Equational Consequence Relations

Let $\mathbf{L}$ denote a language. Identities in $\mathbf{L}$ are ordered pairs of $\mathbf{L}$-formulas that will be written in the form $\alpha \approx \beta$. An interpretation h in $\mathbf{A}$ satisfies an identity $\alpha \approx \beta$ if $h\alpha = h\beta$. We denote this satisfaction relation by the notation: $\mathbf{A} \models_h \alpha \approx \beta$. An algebra $\mathbf{A}$ *satisfies the equation* $\alpha \approx \beta$ if all the interpretations in $\mathbf{A}$ satisfy it; in symbols,

$$\mathbf{A} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models_h \alpha \approx \beta, \text{ for all } h \in Hom(\mathbf{Fm_L}, \mathbf{A}).$$

A class $\mathbb{K}$ of algebras *satisfies the identity* $\alpha \approx \beta$ when all the algebras in $\mathbb{K}$ satisfy it; i.e.

$$\mathbb{K} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models \alpha \approx \beta, \text{ for all } \mathbf{A} \in \mathbb{K}.$$

If $\bar{x}$ is a sequence of variables and h is an interpretation in $\mathbf{A}$, then we write $\bar{a}$ for $h(\bar{x})$. For a class $\mathbb{K}$ of $\mathbf{L}$-algebras, we define the relation $\models_{\mathbb{K}}$ that holds between a set Δ of identities and a single identity $\alpha \approx \beta$ as follows:

$$\Delta \models_{\mathbb{K}} \alpha \approx \beta \text{ if and only if}$$

for every $\mathbf{A} \in \mathbb{K}$ and every interpretation $\bar{a}$ of the variables of $\Delta \cup \{\alpha \approx \beta\}$ in $\mathbf{A}$,

if $\phi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a})$, for every $\phi \approx \psi \in \Delta$, then $\alpha^{\mathbf{A}}(\bar{a}) = \beta^{\mathbf{A}}(\bar{a})$.

In this case, we say that $\alpha \approx \beta$ is a $\mathbb{K}$-consequence of Δ. The relation $\models_K$ is called the *semantic equational consequence relation* determined by K.

Algebraic Semantics for a logic

Let $\langle \mathbf{L}, \vdash_{\mathbf{L}} \rangle$ be a finitary logic (i.e., deductive system) and $\mathbb{K}$ a class of $\mathbf{L}$-algebras. $\mathbb{K}$ is called an *algebraic semantics* for $\langle \mathbf{L}, \vdash_{\mathbf{L}} \rangle$ if $\vdash_{\mathbf{L}}$ can be interpreted in $\vdash_{\mathbb{K}}$ in the following sense:

There exists a finite set $\delta_i(p) \approx \epsilon_i(p)$, for $i \leq n$, of identities with a single variable p such that, for all

$\Gamma \cup \phi \subseteq Fm$,

$$(A) \quad \Gamma \vdash_{\mathbf{L}} \phi \Leftrightarrow \{\delta_i[\psi/p] \approx \epsilon_i[\psi/p], i \leq n, \psi \in \Gamma\} \models_K \delta_i[\phi/p] \approx \epsilon_i[\phi/p],$$

where $\delta[\psi/p]$ denotes the formula obtained by the substitution of ψ at every occurrence of p in δ. The identities $\delta_i \approx \epsilon_i$, for $i \leq n$, are called *defining identities* for $\langle L, \vdash_L \rangle$ and $\mathbb{K}$.

Equivalent Algebraic Semantics and Algebraizable Logics

Let $\mathcal{S}$ be a logic over a language $\mathbf{L}$ and $\mathbb{K}$ an algebraic semantics of S with defining equations $\delta_i(p) \approx \epsilon_i(p)$, $i \leq n$. Then, $\mathbb{K}$ is an **equivalent algebraic semantics** of $\mathbf{S}$ if there exists a finite set $\{\Delta_j(p,q) : j \leq m\}$ of formulas in two variables satisfying the condition:

For every $\phi \approx \psi$ in the language $\mathbf{L}$,

$$\phi \approx \psi \models_K \{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\}$$

and

$$\{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\} \models_K \phi \approx \psi.$$

The set $\{\Delta_j(p,q) : j \leq m\}$ is called an **equivalence system**.

A logic is **BP-algebraizable (in the sense of Blok and Pigozzi)** if it has an equivalent algebraic semantics.

Axiomatic Extensions of Algebraizable logics

A logic S' is an *axiomatic extension* of S if S' is obtained by adjoining new axioms but keeping the rules of inference the same as in S. Let $Ext(S)$ denote the lattice of axiomatic extensions of a logic S and $\mathbf{L_V}(\mathbb{K})$ denote the lattice of subvarieties of a variety $\mathbb{K}$ of algebras.

The following important theorems, due to Blok and Pigozzi, were first proved in [9].

Theorem 7.1. [9] *Let S be a BP-algebraizable logic whose equivalent algebraic semantics $\mathbb{K}$ is a variety. Then $Ext(S)$ is dually isomorphic to $\mathbf{L_V}(\mathbb{K})$.*

Theorem 7.2. [9] *Let S be a BP-algebraizable logic and S' be an axiomatic extension of S. Then $Ext(S')$ is also BP-algebraizable.*

7.2   Dually Hemimorphic Intuitionistic Logic $\mathcal{DHMH}$

The Logic $\mathcal{DHMH}$, which was first defined in [15], is slightly simplified below.

The Logic $\mathcal{DHMH}$ is defined as follows:

LANGUAGE: $\langle \vee, \wedge, \rightarrow, \sim, \bot, \top \rangle$

AXIOMS:

(a) Axioms of the Intuitionistic Logic $\mathcal{I}$ (Rasiowa-Sikorski, p.379):

(Ax1) $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$,

(Ax2) $\alpha \rightarrow (\alpha \vee \beta)$,

(Ax3) $\beta \rightarrow (\alpha \vee \beta)$,

(Ax4) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$,

(Ax5) $(\alpha \wedge \beta) \rightarrow \alpha$,

(Ax6) $(\alpha \wedge \beta) \rightarrow \beta$,

(Ax7) $(\gamma \rightarrow \alpha) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow (\alpha \wedge \beta)))$,

(Ax8) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$,

(Ax9) $((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$,

(Ax10) $((\alpha \wedge \neg\alpha) \to \beta$, where $\neg\alpha := \alpha \to \bot$,

(Ax11) $(\alpha \to (\alpha \wedge \neg\alpha)) \to \neg\alpha$.

(b) Additional axioms:

(Ax12) $\top \to \sim \bot$,

(Ax13) $\sim \top \to \bot$,

(Ax14) $\sim (\alpha \wedge \beta) \leftrightarrow (\sim \alpha \vee \sim \beta)$.

RULES OF INFERENCE:

    (MP)   From ϕ and $\phi \to \gamma$, deduce γ (Modus Ponens),

    (CP)   From $\phi \to \gamma$, deduce $\sim \gamma \to \sim \phi$ (Contraposition).

    The following lemma is crucial in proving Lemma 7.9.

Lemma 7.3.

  (i) *If* $\Gamma \vdash_{\mathcal{I}} \psi$, *then* $\Gamma \vdash_{\mathcal{DHMH}} \psi$,

  (ii) *If* $\Gamma \vdash_{DHMH} \psi$, *then* $\Gamma \vdash_{\mathcal{DHMH}} \alpha \to \psi$,

  (iii) $\Gamma \vdash_{\mathcal{DHMH}} \bot \to \alpha$.

Proof. We only prove (iii), for which it suffices to prove that $\vdash_{\mathcal{I}} \bot \to \alpha$. Then, in view of Completeness Theorem of intuitionistic logic $\mathcal{I}$, that is equivalent to proving that the identity $0 \to x \approx 1$ holds in the variety of Heyting algebras, which immediately follows from the axiom (H): $(x \wedge y) \to x \approx 1$.   □

7.2.1   The logic $\mathcal{DHMH}$ as an implicative logic

We first recall the definition of implicative logics that was introduced by Rasiowa [40] in 1974 (see also [20]).

Definition 7.4. [40] Let S be a logic in a language **L** that includes a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then S is called an implicative logic with respect to the binary connective $\to$, if the following conditions are satisfied:

(IL1) $\vdash_S \alpha \to \alpha$,

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_S \alpha \to \gamma$,

(IL3) For each operation symbol $f \in \mathbf{L}$ of arity $n \geq 1$,
$$\left\{ \begin{array}{l} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n, \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_S f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n),$$

(IL4) $\alpha, \alpha \to \beta \vdash_S \beta$,

(IL5) $\alpha \vdash_S \beta \to \alpha$.

    The following theorem is well-known.

Theorem 7.5. *The intuitionistic logic I is implicative with respect to the connective* $\to$.

Theorem 7.6. *The logic $\mathcal{DHMH}$ is implicative with respect to the connective* $\to$.

Proof. In view of Theorem 7.5, it only remains to prove (IL3) for the (unary) operation $'$ which is fulfilled by the rule **CP**.   □

    We also note here that Theorem 7.6 is a special case of [15, Theorem 3.7].

7.2.2   Algebraic Completeness of $\mathcal{DHMH}$

Definition 7.7. Rasiowa [40] Let S be an implicative logic in $\mathbf{L}$ with $\to$.
An S-algebra is an algebra $\mathbf{A}$ in the language $\mathbf{L}$ that has an element 1 with the following properties:

(LALG1) For all $\Gamma \cup \{\phi\} \subseteq Fm$ and all $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$,
        if $\Gamma \vdash_S \phi$ and $h\Gamma \subseteq \{1\}$ then $h\phi = 1$,

(LALG2) For all $a, b \in A$, if $a \to b = 1$ and $b \to a = 1$ then $a = b$.

The class of S-algebras is denoted by $Alg^*(S)$.

Since $\mathcal{DHMH}$ is an implicative logic we obtain the following result, in view of Rasiowa's Theorem [40, Theorem 7.1, pag 222].

Theorem 7.8. *The logic $\mathcal{DHMH}$ is complete with respect to the class $Alg^*(\mathcal{DHMH})$. In other words,*
    For all $\Gamma \cup \{\phi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{DHMH}} \phi$ if and only if $\Gamma \models_{Alg^*(\mathcal{DHMH})} \phi$.

The following lemma will help us improve the above theorem. Recall the definition of the variety $\mathbb{DHMH}$ given in Definition 4.2.

Lemma 7.9. $Alg^*(\mathcal{DHMH}) = \mathbb{DHMH}$.

Proof. First of all, we note that this proof is an adaptation of the proof of [15, Lemma 4.4]. First, we wish to prove that $\mathbb{DHMH} \subseteq Alg^*(\mathcal{DHMH})$.

Let $\mathbf{A} \in \mathbb{DHMH}$, $\Gamma \cup \{\phi\} \subseteq Fm$ and $h \in Hom(Fm_\mathbf{L}, \mathbf{A})$ such that $\Gamma \vdash_{Alg^*(\mathcal{DHMH})} \phi$ and $h\Gamma \subseteq \{1\}$. We need to verify that $h\phi = 1$. We will proceed by induction on the length of the proof of $\Gamma \vdash_{Alg^*(\mathcal{DHMH})} \phi$.

- Assume that ϕ is an axiom.

  If ϕ is one of the axioms (Ax1) to (Ax11) then $\vdash_\mathcal{I} \phi$. Hence, $\models_{\mathbb{DHMH}} \phi$ and so, $h(\phi) = \top$.

  If ϕ is the axiom (Ax12) then, using (E2), we have $h(\phi) = h(\top \to \sim \bot) = 1 \to 0' = 1$.

  If ϕ is the axiom (Ax13) then, using (E3), we get that $h(\phi) = h(\sim \top \to \bot) = 0 \to 0 = 1$.

- If ϕ is the axiom (Ax14) then, using (E4), we obtain that $h(\phi) = h(\sim (\alpha \wedge \beta) \to (\sim \alpha \vee \sim \beta)) = (h(\alpha) \wedge h(\beta))' \to (h(\alpha)' \vee h(\beta)') = (h(\alpha) \wedge h(\beta))' \to (h(\alpha) \wedge h(\beta))' = 1$.

- If $\phi \in \Gamma$ then $h(\phi) = \top$ by hypothesis.

- Assume now that $\Gamma \vdash_\mathcal{L} \phi$ is obtained from an application of (MP). Then there exists a formula ψ such that $\Gamma \vdash_\mathcal{L} \psi$ and $\Gamma \vdash_\mathcal{L} \psi \to \phi$. By induction, $h(\psi) = 1$ and $h(\psi \to \phi) = 1$. Then $1 = h(\psi) \to h(\phi) = 1 \to h(\phi) = h(\phi)$.

- Assume that $\Gamma \vdash_\mathcal{L} \phi$ is the result of an application of the rule (CP). Then for $\alpha, \beta \in Fm$, $\phi = \sim \beta \to \sim \alpha$ and $\Gamma \vdash_\mathcal{L} \alpha \to \beta$. By induction, $1 = h(\alpha \to \beta) = h(\alpha) \to h(\beta)$ and, consequently $h(\alpha) \leq h(\beta)$. Then, using condition (E4), $h(\beta)' \leq h(\alpha)'$. Hence $h(\beta)' \to h(\alpha)' = 1$. Therefore $h(\phi) = h(\sim \beta \to \sim \alpha) = h(\beta)' \to_H h(\alpha)' = 1$.

Hence, the induction is complete and so, we conclude that $\mathbf{A}$ satisfies (LALG1). It is easy to see that the condition (LALG2) also holds, implying $\mathbf{A} \in Alg^*(\mathcal{DHMH})$.

Next, we prove the other inclusion. Let $\mathbf{A} = \langle A, \vee, \wedge, \to, ', 0, 1 \rangle \in Alg^*(\mathcal{DHMH})$. Notice that $\langle A, \vee, \wedge, \to, 0, 1 \rangle \in Alg^*(I)$. So, $\langle A, \vee, \wedge, \to, 0, 1 \rangle \in \mathbb{H}$. Now, it only remains to show that $\mathbf{A}$ satisfies the conditions (E2) to (E4).

In view of axiom $(Ax12)$ and (LALG1), we have that $\mathbf{A} \models 1 \to 0' \approx 1$. Using (LALG1) and Lemma 7.3 (i), we get $\mathbf{A} \models 0' \to 1 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1 \approx 0'$. In view of Lemma 7.3 (ii) and $(Ax13)$, together with (LALG1), we have that $\mathbf{A} \models 0 \to 1' \approx 1$ and $\mathbf{A} \models 1' \to 0 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1' \approx 0$.

Using (LALG1), it can be shown that $\mathbf{A}$ satisfies the identity $(x' \vee y') \to (x \wedge y)' \approx 1$ and the identity $(x \wedge y)' \to (x' \vee y') \approx 1$. Then applying (LALG2), we see that the algebra satisfies (E4). Consequently $\mathbf{A} \in \mathbb{DHMH}$. This completes the proof. $\quad \square$

We are now ready to present the algebraic completeness theorem for the logic $\mathcal{DHMH}$.

Theorem 7.10. *The logic $\mathcal{DHMH}$ is complete with respect to the variety $\mathbb{DHMH}$.*

Proof. We know $Alg^*(\mathcal{DHMH}) = \mathbb{DHMH}$ by Lemma 7.9. So, the theorem follows from Theorem 7.8. $\quad \square$

7.2.3   The algebraizability of the logic $\mathcal{DHMH}$, $Ext(\mathcal{DHMH})$ and $\mathbf{L_V}(\mathbb{DHMH})$

The following theorem of Blok and Pigozzi shows that Rasiowa's implicative logics provide a class of examples of algebraizable logics and was proved in [9].

Theorem 7.11. [9, 20]
Every implicative logic L is algebraizable with respect to the class $Alg^*(L)$ and the algebraizability is witnessed by the set of defining identities $E = \{x \approx x \to x\}$ and the set of equivalence formulas $\Delta = \{p \to q, q \to p\}$.

Corollary 7.12. *The logic $\mathcal{DHMH}$ is algebraizable, and the variety $\mathbb{DHMH}$ is the equivalent algebraic semantics for $\mathcal{DHMH}$ with the set of defining identities $E = \{x \approx x \to x\}$ (equivlently, $x \approx 1$) and the set of equivalence formulas $\Delta = \{p \to q, q \to p\}$.*

The following theorem is immediate from Theorem 7.1 and Corollary 7.12.

Theorem 7.13. *The lattice $Ext(\mathcal{DHMH})$ of axiomatic extensions of $\mathcal{DHMH}$ is dually isomorphic to the lattice $\mathbf{L_V}(\mathbb{DHMH})$ of subvarieties of the variety $\mathbb{DHMH}$.*

8   The logic $\mathcal{AG}$

We will now present a new logic, $\mathcal{AG}$ (also known as $\mathcal{AGH}$) and its axiomatic extensions.

The logic $\mathcal{AG}$ is defined as follows:

LANGUAGE: $\langle \vee, \wedge, \to, \sim, \bot, \top \rangle$, where $\vee, \wedge,$ and $\to$ are binary, $\sim$ is unary, and $\bot, \top$ are constants.

Let $\leftrightarrow$ be defined by: $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$.

Define $\neg$ by $\neg \alpha := \alpha \to \bot$.

AXIOMS:

$(1), (2), \ldots, (14)$ of the logic $\mathcal{DHMH}$, plus the following axioms:

(15) $\sim\sim (\alpha \vee \beta) \leftrightarrow (\sim\sim \alpha \vee \sim\sim \beta)$,

(16) $(\alpha \vee \sim\sim \alpha) \leftrightarrow \alpha$,

(17) $(\alpha \wedge \sim \neg \sim \alpha) \vee (\beta \vee \neg\beta) \leftrightarrow (\beta \vee \neg\beta)$ (Regularity),

(18) $\neg\alpha \vee \neg\neg\alpha$ (Stone or the Weak Law of Excluded Middle),

(19) $\sim\sim\neg\alpha \leftrightarrow \neg\alpha$ (week *-regular),

(20) $\neg\sim(\alpha \wedge \neg\sim\alpha) \approx \alpha \wedge \neg\sim\alpha$ (Level 1).

RULES OF INFERENCE:

(a) **(MP)**   From ϕ and $\phi \rightarrow \gamma$, deduce γ (Modus Ponens),

(b) **(SCP)**   From $\phi \rightarrow \gamma$, deduce $\sim\gamma \rightarrow \sim\phi$ (contraposition rule).

Remark 8.1. The logic $\mathcal{AG}$ is an axiomatic extension of $\mathcal{DHMH}$.

Definition 8.2.

(a) The logic $\mathcal{G}$ is the axiomatic extension of $\mathcal{AG}$ defined by the following axiom:

(G) $\sim\neg\alpha \leftrightarrow \neg\neg\alpha$.

(b) The logic $\mathcal{RDBLSt}$ is the axiomatic extension of $\mathcal{G}$ defined by the following axiom:

(DSt) $(\sim\alpha \wedge \sim\sim\alpha) \leftrightarrow \bot$.

(c) The logic $\mathcal{RKLSt}$ is the axiomatic extension of $\mathcal{G}$ defined by the following axioms:

(1) $[(\alpha \wedge \sim\alpha) \vee (\beta \vee \sim\beta)] \leftrightarrow (\beta \vee \sim\beta)$.

(2) $\sim\sim\alpha \leftrightarrow \alpha$.

Let L be an algebraizable logic with $\mathbb{K}$ as its equivalent algebraic semantics and let $\mathbb{K}'$ be a variety term-equivalent to $\mathbb{K}$. Then $\mathbb{K}'$ can be considered as an equivalent algebraic semantics for the logic L.

Corollary 8.3. *The logic $\mathcal{AG}$ is algebraizable with the variety $\mathbb{AGH}$ as its equivalent algebraic semantics, and hence with the variety $\mathbb{AG}$ of Almost Gautama algebras as its equivalent algebraic semantics.*

Corollary 8.4. *The logic $\mathcal{G}$ is algebraizable with the variety $\mathbb{GH}$ as its equivalent algebraic semantics,, and hence with the variety $\mathbb{G}$ of Gautama algebras as its equivalent algebraic semantics.*

Since the logics $\mathcal{RDBLSt}$ and $\mathcal{RKLSt}$ are axiomatic extensions of the logic $\mathcal{G}$, we have the following corollaries.

Corollary 8.5. *The logic $\mathcal{RDBLSt}$ is algebraizable with the variety $\mathbb{RDBLSt}$ of regular double Stone algebras as its equivalent algebraic semantics.*

Corollary 8.6. *The logic $\mathcal{RKLSt}$ is algebraizable with the variety $\mathbb{RKLSt}$ of regular Kleene algebras as the equivalent algebraic semantics.*

In a similar fashion the logics corresponding to the remaining subvarieties of $\mathbb{AG}$ can be easily axiomatized by translating the known equational bases of the corresponding subvarieties of $\mathbb{AG}$.

Corollary 8.7. *The logics $\mathcal{AG}$, $\mathcal{G}$, $\mathcal{RDBLSt}$ and $\mathcal{RKLSt}$ and the other axiomatic extensions of $\mathcal{AG}$ are decidable.*

We now consider the question as to whether $\mathcal{AG}$ or any of its axiomatic extensions have the Disjunction Property.

Definition 8.8. Let **L** be a language containing a binary connective $\vee$ and a constant 1 and let $\mathcal{L}$ be a logic in **L**. We say $\mathcal{L}$ has the Disjunction Property if the following condition holds:

For any formulas α and β, $\vdash_{\mathcal{L}} (\alpha \vee \beta)$ implies either $\vdash_{\mathcal{L}} \alpha$ or $\vdash_{\mathcal{L}} \beta$.

Since the Stone axiom holds in $\mathcal{AG}$, the following corollary is immediate.

Corollary 8.9. *The logics $\mathcal{AG}$, $\mathcal{G}$, $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ and the other axiomatic extensions of $\mathcal{AG}$ do not have the Disjunction Property.*

Definition 8.10. Let $\mathcal{L}$ be an algebraizable logic. We say that $\mathcal{L}$ is a discriminator logic if its equivalent algebraic semantics is a discriminator variety. Furthermore, $\mathcal{L}$ is a primal logic if its equivalent algebraic semantics is a variety generated by a primal algebra. $\mathcal{L}$ is a quasiprimal logic if its equivalent algebraic semantics is a variety generated by a quasiprimal algebra.

The classical logic is the first well-known example of a primal logic (as the Boolean algebra **2** is a primal algebra).

Remark 8.11. It follows from Corollary 5.3 that $\mathcal{AG}$ and all its extensions are discriminator logics, while $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ are quasiprimal logics.

8.1   Classical Logic with Strong Negation

Here we will give logical applications of Corollary 6.5 and Theorem 6.9.

Vakarelov introduced the notion of classical logic with strong negation. As a consequence of a completeness theorem he obtained the equivalence of this logic with the three-valued Łukasiewicz logic. In this subsection, using Corollary 6.5 and Theorem 6.9, we will show that the classical logic with strong negation is algebraizable with $\mathbb{CN}$ as its algebraic semantics and that the logics $\mathcal{RKLS}t$, $\mathcal{RKLS}t\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent, thus strengthening Vakarelov's results.

Definition 8.12. [67] The logic, in the language $\langle \vee, \wedge, \rightarrow,', 1 \rangle$, which is obtained by adding the following axioms (C1) – (C6) (for the "strong" negation) to the axioms of classical propositional calculus (also in the lanuage $\langle \vee, \wedge, \rightarrow,', 1 \rangle$), is called the *classical logic with strong negation*:

(C1) $\alpha' \rightarrow (\alpha \rightarrow \beta)$,
(C2) $(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \beta')$,
(C3) $(\alpha \wedge \beta)' \leftrightarrow (\alpha' \vee \beta')$,
(C4) $(\alpha \vee \beta)' \leftrightarrow (\alpha' \wedge \beta')$,
(C5) $\alpha^{*'} \leftrightarrow \alpha$, where $\alpha^* := \alpha \rightarrow 0$,
(C6) $\alpha'' \leftrightarrow \alpha$.

Let $\mathcal{CN}$ denote the classical logic with strong negation.

The following theorem is a strengthened version of Vakarelov's completeness theorem for $\mathcal{CN}$ (with a different proof).

Theorem 8.13. *$\mathcal{CN}$ is BP-algebraizable with $\mathbb{CN}$ as its algebraic semantics.*

Proof. It is well-known (see, for example, [20, Page 85] that the Nelson logic with strong negation is implicative and hence is BP-algebraizable with $\mathbb{N}$ as its algebraic semantics. Hence, it is easy to see, in view of Corollary 6.5, that $\mathcal{CN}$ is BP-algebraizable with $\mathbb{CN}$ as its algebraic semantics. $\square$

The following corollary is immediate from Theorem 6.9.

Corollary 8.14. *The logics $\mathcal{RKLS}t$, $\mathcal{RKLS}t\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent.*

9   Concluding Remarks

In a forthcoming paper [17], we completely describe the subvarieties of $\mathbb{AG}$ with the Amalgamation Property and the ones without (AP).

Note that the variety $\mathbb{DHM}$ introduced in Definition 2.3 is a far-reaching–and a common– generalization of both p-algebras–more generally, semi-De Morgan algebras– and Ockham algebras. Observe also that the new variety $\mathbb{DHMP}$ that was introduced in Definition 2.4 is a sweeping generalization of the variety of Almost Gautama algebras.

We now introduce a subvariety of the variety $\mathbb{DHMP}$ whose members are called "quasi-Gautama algebras."

An algebra $\mathbf{A} = \langle A, \vee, \wedge, {}^*, {}', 0, 1\rangle$ is a *quasi-Gautama algebra* if $\mathbf{A}$ satisfies:

(1) $\langle A, \vee, \wedge, {}^*, 0, 1\rangle$ is a p-algebra,

(2) $\langle A, \vee, \wedge, {}', 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(3) $\mathbf{A}$ is regular; that is, $\mathbf{A}$ satisfies the identity: (R1) $x \wedge x'^{*'} \leq y \vee y^*$.

Let the variety of quasi-Gautama algebras be denoted by $\mathbb{QG}$. Notice that $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{QG} \subset \mathbb{RDHMP} \subset \mathbb{DHMP}$, where $\mathbb{RDHMP}$ consists of $\mathbb{DHMP}$-algbras satisfying (R1).

Similarly, we can generalize the variety $\mathbb{AGH}$ to a new variety whose members are called quasi-Gautama semi-Heyting algebras as follows:

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, {}', 0, 1\rangle$ is a quasi-Gautama semi-Heyting algebra if $\mathbf{A}$ satisfies:

(1) $\langle A, \vee, \wedge, {}^*, 0, 1\rangle$ is a semi-Heyting algebra,

(2) $\langle A, \vee, \wedge, {}', 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(3) $\mathbf{A}$ is regular; that is, $\mathbf{A}$ satisfies the identity: (R1) $x \wedge x'^{*'} \leq y \vee y^*$.

Let the variety of quasi-Gautama semi-Heyting algebras be denoted by $\mathbb{QGSH}$, and $\mathbb{QGH}$ denotes the subvariety of $\mathbb{QGSH}$ consisting of those algebras whose semi-Heyting reduct is a Heyting algebra. Note that $\mathbb{GH} \subset \mathbb{AGH} \subset \mathbb{QGH} \subset \mathbb{QGSH} \subset \mathbb{DHMSH}$.

We know that the cardinality of the lattice of subvarieties of $\mathbb{QG}$ is 2^ω since $\mathbb{QG}$ contains each of the varieties of regular double p-algebras and of regular Kleene p-algebras, each of whose lattice of subvarieties is of cardinality 2^ω (see [5, 6]). The results presented in the present paper only describe the "bottom" of the lattices of subvarieties of $\mathbb{QG}$ and of $\mathbb{QGH}$.

We conclude the paper with some open problems for further research.

OPEN PROBLEMS:

PROBLEM 1: Find a Priestley-type duality for the varieties $\mathbb{AG}$ and $\mathbb{AGH}$. (One can ask the same question for the varieties $\mathbb{QG}$, $\mathbb{QGH}$ and $\mathbb{QGSH}$.)

PROBLEM 2: There is a representation of regular double Stone algebras in terms of rough sets. Is their a similar representation for the varieties $\mathbb{AG}$ and $\mathbb{AGH}$?

PROBLEM 3: Katriňák has given a "triple" construction for regular double Stone algebras. Is their a similar construction for the variety $\mathbb{AG}$? The same question for $\mathbb{AGH}$.

PROBLEM 4: Investigate the lattice of subvarieties of $\mathbb{QG}$, and of $\mathbb{QGSH}$ of level 1 (i.e., satisfying $x \wedge x'^* \wedge x'^{*'^*} \approx x \wedge x'^*$.

PROBLEM 5: Investigate the lattice of subvarieties of the varieties $\mathbb{QG}$ and $\mathbb{QGSH}$.

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