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Completeness for Saturated L-Quasi-Uniform Limit Spaces

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Abstract. We define and study two completeness notions for saturated L-quasi-uniform limit spaces. The one, that we term Lawvere completeness, is defined using the concept of promodule and lends a lax algebraic interpretation of completeness also for saturated L-quasi-uniform limit spaces. The other, termed Cauchy completeness, is defined using saturated Cauchy pair prefilters. We show that both concepts coincide with related notions in the case of saturated L-quasi-uniform spaces and that also for saturated L-quasi-uniform limit spaces, both completeness notions are equivalent.

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1 Introduction

Generalizing an approach in [2], completeness has recently been studied from a categorical point of view for different kinds of many-valued quasi-uniform (convergence) spaces, [12, 13, 14]. This paper adds to these investigations by considering many-valued quasi-uniform limit spaces based on saturated L-prefilters. These spaces are a slight generalization of $\top$-uniform limit spaces [6, 7, 9] and of probabilistic quasi-uniform spaces [5, 14]. We define a completeness notion using adjoint promodules, thus providing a categorical framework for completeness. Also, we define completeness with the help of saturated pair L-prefilters. The main result of the paper shows that both these approaches are equivalent.

The paper is organized as follows. In the second section we collect the necessary concepts about lattices, L-subsets, saturated L-prefilters and prorelations. The third section studies saturated L-quasi-uniform limit spaces and promodules. Sections 4 and 5 are devoted to the two concepts of completeness studied in this paper. Finally, we draw some conclusions.

2 Preliminaries

In this paper, we will consider *commutative and integral quantales* $\mathsf{L} = (L, \leq, *)$. Here, $(L, \leq)$ is a complete lattice with distinct top and bottom elements $\top \neq \bot$, $(L, *)$ is a commutative semigroup with the top element of L as the unit, that is, $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, that is, $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$ for all $\alpha_i, \beta \in L$, $i \in J$, see for example [4].

The *implication* in a quantale is defined by $\alpha \to \beta = \bigvee\{\delta \in L \ : \ \delta * \alpha \leq \beta\}$ and characterized by $\delta \leq \alpha \to \beta$ if and only if $\delta * \alpha \leq \beta$.

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Typical examples of commutative and integral quantales are $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm on $[0,1]$ or *Lawvere's quantale* $\mathsf{L} = ([0,\infty], \geq, +)$. Another example is given by the *quantale of distance distribution functions* $\mathsf{L} = (\Delta^+, \leq, *)$, where Δ^+ is the set of all distance distribution functions $\varphi : [0,\infty] \longrightarrow [0,1]$ which are left-continuous in the sense that $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0,\infty]$ and $*$ is a *sup-continuous triangle function*, see [3, 11].

An *L-subset of X* is a mapping $a : X \longrightarrow L$ and we denote the set of L-subsets of X by L^X. For $A \subseteq X$ we define $\top_A \in L^X$ by $\top_A(x) = \top$ if $x \in A$ and $= \bot$ otherwise. The lattice operations are extended pointwisely from L to L^X. For a mapping $\varphi : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ we define $\varphi(a) \in L^Y$ by $\varphi(a)(y) = \bigvee_{\varphi(x)=y} a(x)$ for $y \in Y$ and $\varphi^{\leftarrow}(b) = b \circ \varphi \in L^X$.

For L-subsets $u \in L^{X\times Y}$ and $v \in L^{Y\times Z}$, we define $v \circ u \in L^{X\times Z}$ by $v \circ u(x,z) = \bigvee_{y\in Y} u(x,y) * v(y,z)$ for all $x \in X$ and $z \in Z$.

For $a, b \in L^X$ we denote the *fuzzy inclusion order* $[a,b] = \bigwedge_{x\in X}(a(x) \to b(x))$, [1]. The following properties are well-known.

Lemma 2.1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$, $u_1, u_2 \in L^{X\times Y}$, $, v_1, v_2 \in L^{Y\times Z}$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a,b] = \top$;

(ii) $a \leq a'$ *implies* $[a',b] \leq [a,b]$ *and* $b \leq b'$ *implies* $[a,b] \leq [a,b']$;

(iii) $[a,c] \wedge [b,c] = [a \vee b, c]$;

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$;

(v) $[u_1, v_1] * [u_2, v_2] \leq [u_2 \circ u_1, v_2 \circ v_1]$.

Definition 2.2. [5, 14] A subset $\mathbb{F} \subseteq L^X$ is called a *saturated* L*-prefilter* (on X) if

(SP1) $\top_X \in \mathbb{F}$;

(SP2) $a, b \in \mathbb{F}$ implies $a \wedge b \in \mathbb{F}$;

(SP3) $\bigvee_{b\in\mathbb{F}}[b,c] = \top$ implies $c \in \mathbb{F}$.

We denote the set of all saturated L-prefilters on X by $\mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ and we use the subsethood order on $\mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$.

The condition (SP3) implies $a \leq b, a \in \mathbb{F} \Longrightarrow b \in \mathbb{F}$. If additionally $\bigvee_{x\in X} a(x) = \top$ for all $a \in \mathbb{F}$, then we speak of a $\top$*-filter* [5, 14].

Example 2.3. For $x \in X$, $[x] = \{a \in L^X \ : \ a(x) = \top\}$ is a saturated L-prefilter, the *saturated point* L*-prefilter of* x. We note that $[x]$ is a $\top$-filter. More generally, for an L-set $a \in L^X$, then $[a] = \{b \in L^X \ : \ a \leq b\}$ is a saturated L-prefilter and we have, in particular, $[x] = [\top_{\{x\}}]$.

Definition 2.4. [5, 14] A subset $\mathbb{B} \subseteq L^X$ is called a *saturated* L*-prefilter base* (on X) if

(SPB) $a, b \in \mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a saturated L-prefilter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X \ : \ \bigvee_{b\in\mathbb{B}}[b,a] = \top\}$ is the saturated L-prefilter generated by $\mathbb{B}$.

For a saturated L-prefilter $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) \ : \ a \in \mathbb{F}\}$ is a saturated L-prefilter base on Y and we denote $\varphi(\mathbb{F})$ the generated saturated L-prefilter on Y, the *image of* $\mathbb{F}$ *under* φ, see e.g. [5].

A *prorelation (from X to Y)* is a set of saturated L-prefilters $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ which satisfies the axioms

(PR1) $\mathbb{F} \leq \mathbb{G}$, $\mathbb{F} \in \Phi$ implies $\mathbb{G} \in \Phi$;

(PR2) $\mathbb{F}, \mathbb{G} \in \Phi$ implies $\mathbb{F} \wedge \mathbb{G} \in \Phi$.

For $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ the set $[\mathbb{F}] = \{\mathbb{K} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y) \, : \, \mathbb{F} \leq \mathbb{K}\}$ is a prorelation.

We consider now two prorelations $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z)$ and define

$$\Psi \circ \Phi = \{\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Z) \, : \, \exists \mathbb{F} \in \Phi, \mathbb{G} \in \Psi \text{ s.t. } \mathbb{G} \circ \mathbb{F} \leq \mathbb{H}\}.$$

Here, it is defined $\mathbb{G} \circ \mathbb{F} = [\{g \circ f \, : \, g \in \mathbb{G}, f \in \mathbb{F}\}]$ with $g \circ f(x,z) = \bigvee_{y \in Y} f(x,y) * g(y,z)$ for all $x \in X, z \in Z$. It is straightforward to show that $\Psi \circ \Phi$ is a prorelation from X to Z.

We denote $\Delta_X = \{(x,x) \, : \, x \in X\} \subseteq X \times X$. Then $[\top_{\Delta_X}] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$ and hence $[[\top_{\Delta_X}]]$ is a prorelation from X to X.

Proposition 2.5. *For a prorelation $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$, we have $\Phi \circ [[\top_{\Delta_X}]] = \Phi$ and $[[\top_{\Delta_Y}]] \circ \Phi = \Phi$.*

Proof. Let $\mathbb{H} \in \Phi \circ [[\top_{\Delta_X}]]$. Then there is $\mathbb{F} \in \Phi$ such that $\mathbb{F} \circ [\top_{\Delta_X}] \leq \mathbb{H}$. For $f \in \mathbb{F}$ we have $f \circ \top_{\Delta_X}(x,y) = \bigvee_{z \in X} \top_{\Delta_X}(x,z) * f(z,y) = f(x,y)$ and hence we conclude that $g \in \mathbb{F} \circ [\top_{\Delta_X}]$ if and only if $\top = \bigvee_{f \in \mathbb{F}} [f \circ \top_{\Delta_X}, g] = \bigvee_{f \in \mathbb{F}} [f, g]$ if and only if $g \in \mathbb{F}$, as $\mathbb{F}$ is a saturated L-prefilter. Hence, $\mathbb{F} = \mathbb{F} \circ [\top_{\Delta_X}] \leq \mathbb{H}$ and we have $\mathbb{H} \in \Phi$ by (PR1). Conversely, for $\mathbb{F} \in \Phi$ we have $\mathbb{F} = \mathbb{F} \circ [\top_{\Delta_X}] \in \Phi \circ [[\top_{\Delta_X}]]$.

The second equation can be shown in a similar way. □

For $f \in L^{X \times Y}, g \in L^{Y \times Z}$ and $h \in L^{Z \times U}$ it is not difficult to show that $h \circ (g \circ f) = (h \circ g) \circ f$. From this we conclude $\mathbb{H} \circ (\mathbb{G} \circ \mathbb{F}) = (\mathbb{H} \circ \mathbb{G}) \circ \mathbb{F}$ for saturated L-prefilters $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y), \mathbb{G} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z), \mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Z \times U)$ and we obtain

Proposition 2.6. *For prorelations $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y), \Psi \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z)$ and $\Theta \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Z \times U)$ we have $(\Phi \circ \Psi) \circ \Theta = \Phi \circ (\Psi \circ \Theta)$.*

Consider now a mapping $\varphi : X \longrightarrow Y$. We define the L-relation (and denote it again by φ), $\varphi(x,y) = \top$ if $y = \varphi(x)$ and $\varphi(x,y) = \bot$ otherwise. Similarly, the opposite L-relation φ° is defined by $\varphi^\circ(y,x) = \top$ if $y = \varphi(x)$ and $\varphi^\circ(y,x) = \bot$ otherwise. Hence, $\varphi \in L^{X \times Y}$ and $\varphi^\circ \in L^{Y \times X}$ and therefore $[\varphi] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $[\varphi^\circ] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times X)$ and we obtain prorelations $[[\varphi]] \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $[[\varphi^\circ]] \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times X)$.

If $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$, then it is not difficult to show that $[\psi \circ \varphi] = [\psi] \circ [\varphi]$. From this we immediately conclude $[[\psi]] \circ [[\varphi]] = [[\psi \circ \varphi]]$.

Proposition 2.7. *Let $\varphi : X \longrightarrow Y$. Then $[[\varphi]] \circ [[\varphi^\circ]] \subseteq [[\top_{\Delta_Y}]]$ and $[[\top_{\Delta_X}]] \subseteq [[\varphi^\circ]] \circ [[\varphi]]$.*

Proof. We have, for $y, y' \in Y$, $\varphi \circ \varphi^\circ(y,y') = \bigvee_{x \in X} \varphi^\circ(y,x) * \varphi(x,y') = \top$ if $y' = \varphi(x) = y$ for some $x \in X$ and $= \bot$ otherwise. Hence $\varphi \circ \varphi^\circ \leq \top_{\Delta_Y}$ which implies $[\top_{\Delta_Y}] \leq [\varphi \circ \varphi^\circ]$ and hence $[[\varphi]] \circ [[\varphi^\circ]] = [[\varphi \circ \varphi^\circ]] \subseteq [[\top_{\Delta_Y}]]$.

Similarly, we have, for $x, x' \in X$ that $\varphi^\circ \circ \varphi(x,x') = \bigvee_{y \in Y} \varphi(x,y) * \varphi^\circ(y,x') = \top$ if $\varphi(x') = \varphi(x)$ and $= \bot$ otherwise. Hence $\top_{\Delta_X} \leq \varphi^\circ \circ \varphi$, implying $[\top_{\Delta_X}] \geq [\varphi^\circ \circ \varphi]$. From this we conclude $[[\top_{\Delta_X}]] \subseteq [[\varphi^\circ \circ \varphi]] = [[\varphi^\circ]] \circ [[\varphi]]$. □

Lemma 2.8. *Let $\varphi : X \longrightarrow Y$ and $b \in L^{X \times X}$. Then $(\varphi \times \varphi)^{\leftarrow}(b) = \varphi^\circ \circ b \circ \varphi$.*

Proof. For all $x, x' \in X$ we have $(\varphi^\circ \circ b) \circ \varphi(x,x') = \bigvee_{y \in Y} (\varphi^\circ \circ b)(y,x') * \varphi(x,y) = \bigvee_{y \in Y} \bigvee_{x : \varphi(x) = y} \varphi^\circ \circ b(y,x') = \bigvee_{x \in X} \varphi^\circ \circ b(\varphi(x),x') = \bigvee_{y \in Y} \varphi^\circ(y,x') * b(\varphi(x),y) = b(\varphi(x),\varphi(x')) = (\varphi \times \varphi)^{\leftarrow}(b)(x,x')$. □

Lemma 2.9. *Let $\varphi : X \longrightarrow Y$ and $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. Then we have, for $b \in L^{X \times Y}$, that $b \in [\varphi] \circ \mathbb{H}$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H}$.*

Proof. We have with Lemma 2.1 (v), noting $[\varphi^\circ, \varphi^\circ] = \top = [\varphi, \varphi]$, for $h \in \mathbb{H}$,

$$[\varphi \circ h, b] \leq [\varphi^\circ \circ \varphi \circ h, \varphi^\circ \circ b] \leq [h, \varphi^\circ \circ b] \leq [\varphi \circ h, \varphi \circ \varphi^\circ \circ b] \leq [\varphi \circ h, b].$$

We conclude that $b \in [\varphi] \circ \mathbb{H}$ if, and only if, $\top = \bigvee_{h\in\mathbb{H}}[\varphi \circ h, b] = \bigvee_{h\in\mathbb{H}}[h, \varphi^\circ \circ b]$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H}$. $\square$

Lemma 2.10. *Let $\varphi : X \longrightarrow Y$ and $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. Then we have, for $a \in L^{Y\times X}$, that $a \in \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $a \circ \varphi \in \mathbb{H}$.*

Proof. Similar as in the last proof, we have, for $h \in \mathbb{H}$,

$$[h \circ \varphi^\circ, a] \leq [h \circ \varphi^\circ \circ \varphi, a \circ \varphi] \leq [h, a \circ \varphi] \leq [h \circ \varphi^\circ, a \circ \varphi \circ \varphi^\circ] \leq [h \circ \varphi^\circ, a].$$

We conclude that $a \in \mathbb{G} \circ [\varphi^\circ]$ if, and only if, $\top = \bigvee_{h\in\mathbb{H}}[h \circ \varphi^\circ, a] = \bigvee_{h\in\mathbb{H}}[h, a \circ \varphi]$ if, and only if, $a \circ \varphi \in \mathbb{H}$. $\square$

Proposition 2.11. *For $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$ and $\varphi : X \longrightarrow Y$ we have $(\varphi \times \varphi)(\mathbb{H}) = [\varphi] \circ \mathbb{H} \circ [\varphi^\circ]$.*

Proof. We have $b \in [\varphi] \circ \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $(\varphi \times \varphi)^{\leftarrow}(b) = \varphi^\circ \circ b \circ \varphi \in \mathbb{H}$ if, and only if, $b \in (\varphi \times \varphi)(\mathbb{H})$. $\square$

3 Saturated L-Quasi-Uniform Limit Spaces and Promodules

Definition 3.1. Let X be a set and let $\Lambda \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. The pair (X, Λ) is called a *saturated* L*-quasi-uniform limit space* if

(SLUL1) $[\top_{\Delta_X}] \in \Lambda$;

(SLUL2) $\mathbb{H} \in \Lambda$, $\mathbb{H} \leq \mathbb{K}$ implies $\mathbb{K} \in \Lambda$;

(SLUL3) $\mathbb{H}, \mathbb{K} \in \Lambda$ implies $\mathbb{H} \wedge \mathbb{K} \in \Lambda$;

(SLUL4) $\mathbb{H}, \mathbb{K} \in \Lambda$ implies $\mathbb{H} \circ \mathbb{K} \in \Lambda$.

A mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ is called *uniformly continuous* if $(\varphi \times \varphi)(\mathbb{H}) \in \Lambda'$ whenever $\mathbb{H} \in \Lambda$.

The axioms (SLUL2) and (SLUL3) show that Λ is a prorelation from X to X that satisfies, via (SLUL1) and (SLUL4), the additional axioms

$$[[\top_{\Delta_X}]] \subseteq \Lambda \quad \text{and} \quad \Lambda \circ \Lambda \subseteq \Lambda.$$

Uniform continuity of a mapping can be characterized as follows.

Proposition 3.2. *Let (X, Λ) and (X', Λ') be saturated* L*-quasi-uniform limit spaces and $\varphi : X \longrightarrow X'$ be a mapping. The following statements are equivalent.*
(1) φ is uniformly continuous.
(2) $[[\varphi]] \circ \Lambda \subseteq \Lambda' \circ [[\varphi]]$.
(3) $\Lambda \circ [[\varphi^\circ]] \subseteq [[\varphi^\circ]] \circ \Lambda'$.

Proof. We first show that (1) implies (2). Let φ be uniformly continuous and let $\mathbb{K} \in [[\varphi]] \circ \Lambda$. Then $\mathbb{K} \geq [\varphi] \circ \mathbb{H}$ for some $\mathbb{H} \in \Lambda$ and hence $\mathbb{K} \circ [\varphi^\circ] \geq [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] = (\varphi \times \varphi)(\mathbb{H}) \in \Lambda'$. We conclude $\mathbb{K} = \mathbb{K} \circ [\top_{\Delta_X}] \geq \mathbb{K} \circ [\varphi^\circ] \circ [\varphi] \in \Lambda' \circ [[\varphi]]$ and we have $\mathbb{K} \in \Lambda' \circ [[\varphi]]$.

Now we show that (2) implies (3).Let $\mathbb{K} \in \Lambda \circ [[\varphi^\circ]]$. Then $\mathbb{K} \geq \mathbb{H} \circ [\varphi^\circ]$ for some $\mathbb{H} \in \Lambda$. Hence $[\varphi] \circ \mathbb{K} \geq [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] \in \Lambda' \circ [[\varphi]] \circ [[\varphi^\circ]] \subseteq \Lambda' \circ [[\top_{\Delta_Y}]] = \Lambda'$ and we have that $[\varphi] \circ \mathbb{K} \in \Lambda'$. We conclude $\mathbb{K} = [\top_{\Delta_X}] \circ \mathbb{K} \geq [\varphi^\circ] \circ [\varphi] \circ \mathbb{K} \in [[\varphi^\circ]] \circ \Lambda'$ and we have $\mathbb{K} \in [[\varphi^\circ]] \circ \Lambda'$.

Finally we show that (3) implies (1). Let $\mathbb{H} \in \Lambda$. Then $(\varphi \times \varphi)(\mathbb{H}) = [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] \in [[\varphi]] \circ \Lambda \circ [[\varphi^\circ]] \subseteq [[\varphi]] \circ [[\varphi^\circ]] \circ \Lambda' \subseteq [[\top_{\Delta_Y}]] \circ \Lambda' = \Lambda'$ and φ is uniformly continuous. $\square$

Example 3.3 ([13]). Let X be a set. A saturated L-prefilter $\mathcal{U} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$ is called a *saturated* L*-quasi-uniformity* if

(U0) for all $x \in X$ and $u \in \mathcal{U}$ we have $u(x,x) = \top$;

(UC) for all $u \in \mathcal{U}$ we have $\bigvee_{v \in \mathcal{U}} [v \circ v, u] = \top$.

The pair $(X,\mathcal{U})$ is the called a *saturated* L*-quasi-uniform space*. A mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ between the saturated L-quasi-uniform spaces $(X,\mathcal{U}), (X',\mathcal{U}')$ is called *uniformly continuous* if $(\varphi \times \varphi)^{\leftarrow}(v) \in \mathcal{U}$ for all $v \in \mathcal{U}'$.

We note that the conditions (U0) and (UC) are equivalent to (U0') $\mathcal{U} \leq [\top_{\Delta_X}]$ and (UC') $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$. Uniform continuity of a mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ can equivalently be expressed by $[\varphi] \circ \mathcal{U} \geq \mathcal{U}' \circ [\varphi]$.

Wang and Yue [13] call a saturated L-quasi-uniform space a fuzzy quasi-uniform space. Also, they use as order on the set of saturated L-prefilters the opposite order of the subsethood order.

For a saturated L-quasi-uniform space $(X,\mathcal{U})$ then $(X,[\mathcal{U}])$ is a saturated L-quasi-uniform limit space and a uniformly continuous mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ is also uniformly continuous as a mapping $\varphi : (X,[\mathcal{U}]) \longrightarrow (X',[\mathcal{U}'])$.

Definition 3.4. Let (X,Λ) and (X',Λ') be saturated L-quasi-uniform limit spaces. A prorelation from X to X', $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X')$, is called a *promodule* (from (X,Λ) to (X',Λ')) if $\Phi \circ \Lambda \subseteq \Phi$ and $\Lambda' \circ \Phi \subseteq \Phi$.

We note that for a promodule $\Phi = \Phi \circ [[\top_{\Delta_X}]] \subseteq \Phi \circ \Lambda$ and hence we even have $\Phi \circ \Lambda = \Phi$. Similarly we can see also that $\Lambda' \circ \Phi = \Phi$. Also, from (SLUL4) we see that Λ is a promodule from (X,Λ) to (X,Λ).

Example 3.5. Let $\varphi : (X,\Lambda) \longrightarrow (X',\Lambda')$ be uniformly continuous. Then $\varphi_* = \Lambda' \circ [[\varphi]]$ is a promodule from (X,Λ) to (X',Λ') and $\varphi^* = [[\varphi^\circ]] \circ \Lambda'$ is a promodule from from (X',Λ') to (X,Λ). It is easy to see that φ_* and φ^* are prorelations. Furthermore $\varphi_* \circ \Lambda = \Lambda' \circ [[\varphi]] \circ \Lambda \subseteq \Lambda' \circ \Lambda' \circ [[\varphi]] \subseteq \Lambda' \circ [[\varphi]] = \varphi_*$ and, similarly, $\Lambda' \circ [[\varphi_*]] = \Lambda' \circ \Lambda' \circ [[\varphi]] \subseteq \Lambda' \circ [[\varphi]] = \varphi_*$. The proof that φ^* is a promodule is similar and not shown.

Definition 3.6. Let (X,Λ) and (X',Λ') be saturated L-quasi-uniform limit spaces, let $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X,X')$ be a promodule from (X,Λ) to (X',Λ') and let $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X' \times X)$ be a promodule from from (X',Λ') to (X,Λ). Φ is called *left-adjoint* for Ψ (and Ψ is called *right-adjoint* for Φ) if $\Lambda \subseteq \Psi \circ \Phi$ and $\Phi \circ \Psi \subseteq \Lambda'$. In this case we write $\Phi \dashv \Psi$.

Example 3.7. For a uniformly continuous mapping $\varphi : (X,\Lambda) \longrightarrow (X',\Lambda')$ we have $\varphi_* \dashv \varphi^*$. In fact, we have $\Lambda = \Lambda \circ [[\top_{\Delta_X}]] = \Lambda \circ [[\varphi^\circ]] \circ [[\varphi]] \subseteq [[\varphi^\circ]] \circ \Lambda' \circ [[\varphi]] = [[\varphi^\circ]] \circ \Lambda' \circ \Lambda' \circ [[\varphi]] = \varphi^* \circ \varphi_*$ and also $\varphi_* \circ \varphi^* = \Lambda' \circ [[\varphi]] \circ [[\varphi^\circ]] \circ \Lambda' \subseteq \Lambda' \circ [[\top_{\Delta_Y}]] \circ \Lambda' = \Lambda' \circ \Lambda' = \Lambda'$.

We note that for a promodule $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X' \times X)$ its left-adjoint $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X,X')$ is unique. In fact, if we have $\Phi_1 \dashv \Psi$ and $\Phi_2 \dashv \Psi$, then $\Phi_1 = \Phi_1 \circ \Lambda \subseteq \Phi_1 \circ (\Psi \circ \Psi_2) = (\Phi_1 \circ \Psi) \circ \Phi_2 \subseteq \Lambda' \circ \Phi_2 = \Phi_2$. Similarly we see that $\Phi_2 \subseteq \Phi_1$ and hence $\Phi_1 = \Phi_2$. In the same way, also for a promodule $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X,X')$ its right-adjoint $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X' \times X)$ is unique.

The following lemma will come in handy later.

Lemma 3.8. *Let (X,Λ) and (X',Λ') be saturated* L*-quasi-uniform limit spaces, let $\Phi,\Phi' \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X,X')$ be promodules from (X,Λ) to (X',Λ') and let $\Psi,\Psi' \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X'\times X)$ be promodules from from (X',Λ') to (X,Λ). If $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$, then $\Phi' = \Phi$ and $\Psi' = \Psi$.*

Proof. We have $\Phi' = \Lambda' \circ \Phi' \supseteq (\Phi \circ \Psi) \circ \Phi' \supseteq (\Phi \circ \Psi') \circ \Phi' = \Phi \circ (\Psi' \circ \Phi') \supseteq \Phi \circ \Lambda = \Phi$. Similarly we can show $\Psi \subseteq \Psi'$. $\square$

4 Lawvere Completeness of Saturated L-Quasi-Uniform Limit Spaces

We consider a one-point set $1 = \{\bullet\}$ and the unique saturated L-quasi-uniform limit structure $\Pi = [[\top_{\{(\bullet,\bullet)\}}]]$. A mapping $\varphi : 1 \longrightarrow X$, $\varphi(\bullet) = x$ will be identified with $x \in X$ and we shall write $x : 1 \longrightarrow X$ for it. We note that $x : (1,\Pi) \longrightarrow (X,\Lambda)$ is uniformly continuous: For $\mathbb{H} \geq [\top_{\{(\bullet,\bullet)\}}]$ we find $(\varphi\times\varphi)(\mathbb{H}) \geq (\varphi\times\varphi)([\top_{\{(\bullet,\bullet)\}}]) = [\top_{\{(\varphi(\bullet),\varphi(\bullet))\}}] = [\top_{\{(x,x)\}}] \geq [\top_{\Delta_X}] \in \Lambda$ and hence $(\varphi\times\varphi)(\mathbb{H}) \in \Lambda$.

Definition 4.1. A saturated L-quasi-uniform limit space (X,Λ) is called *Lawvere complete* if for all promodules $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(1\times X)$ from $(1,\Pi)$ to (X,Λ), $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X\times 1)$ from (X,Λ) to $(1,\Pi)$ with $\Phi \dashv \Psi$ there is $x \in X$ such that $\Phi = x_*$ and $\Psi = x^*$.

In the sequel, we want to identify $X \times 1$ and $1 \times X$ with X. This leads to some adaptation in the concepts and definitions. For a mapping $x : 1 \longrightarrow X$ we note that $x(\bullet, y) = \top$ if and only if $x = x(\bullet) = y$ and $x(\bullet,y) = \bot$ otherwise. Hence, $x(\bullet,y) = \top_{\{x\}}(y)$ and we can write $x_* = \Lambda \circ [[x]]$ with the saturated point L-prefilter $[x]$. Similarly, $x^\circ(y,\bullet) = \top$ if $x = x(\bullet) = y$ and $x^\circ(y,\bullet) = \bot$ otherwise, so that also $x^* = [[x]] \circ \Lambda$.

More generally, for $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X\times 1)$ (or, similarly, for $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(1\times X)$) we identify $f \in \mathbb{F}$ with an L-subset of X (denoted again by f) via $f(x) = f(x,\bullet)$. In this sense, we define for $\mathbb{H} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X\times X)$ and $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$

$$\mathbb{H} \circ \mathbb{F} = [\{h \circ f \ : \ h \in \mathbb{H}, f \in \mathbb{F}\}]$$

with $h \circ f(x) = h \circ f(\bullet, x) = h \circ f(\bullet, x) = \bigvee_{y\in X} f(\bullet,y) * h(y,x) = \bigvee_{y\in X} f(y) * h(y,x)$ for all $x \in X$. Similarly, we define

$$\mathbb{F} \circ \mathbb{H} = [\{f \circ h \ : \ f \in \mathbb{F}, h \in \mathbb{H}\}]$$

with $f \circ h(x) = f \circ h(x,\bullet) = \bigvee_{y\in X} h(x,y) * f(y,\bullet) = \bigvee_{y\in X} h(x,y) * f(y)$.

A promodule $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(1\times X)$ from $(1,\Pi)$ to (X,Λ) then satisfies the conditions $\Phi \circ \Pi \subseteq \Phi$ and $\Lambda \circ \Phi \subseteq \Phi$. We note that the first of these conditions is always satisfied: $\Phi \circ \Pi = \Phi \circ [[\top_{\{(\bullet,\bullet)\}}]] = \Phi \circ [[\top_{\Delta_1}]] = \Phi$. Hence it is sufficient to demand the condition $\Lambda \circ \Phi \subseteq \Phi$ in this case. Identfying $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(1\times X)$ with $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$, we call a prorelation $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$ a *left-Λ-promodule* if $\Lambda \circ \Phi \subseteq \Phi$. If the saturated L-quasi-uniform limit space (X,Λ) is clear from the context, we simply speak of a *left-promodule* in this case.

Similarly, for a promodule $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X\times 1)$ from (X,Λ) to $(1,\Pi)$ we have the conditions $\Psi \circ \Lambda \subseteq \Psi$ and $\Pi \circ \Psi \subseteq \Psi$ and again the second of these conditions will be always satisfied. We therefore call a prorelation $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$ a *right-Λ-promodule* if $\Psi \circ \Lambda \subseteq \Psi$. Again, if the saturated L-quasi-uniform limit space (X,Λ) is clear from the context, we simply speak of a *right-promodule*.

For adjoint promodules, we consider prorelations $\Phi,\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$ as promodules (from $(1,\Pi)$ to (X,Λ) for Φ and from (X,Λ) to $(1,\Pi)$ for Ψ). Then, by definition, $\Phi \dashv \Psi$ if and only if $\Phi \circ \Psi \subseteq \Lambda$ and $\Pi \subseteq \Psi \circ \Phi$. The first condition, $\Phi \circ \Psi \subseteq \Lambda$, means that for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$ we have $\mathbb{F} \circ \mathbb{G} \in \Lambda$. Now we note that for $f \in \mathbb{F}$ and $g \in \mathbb{G}$ we have

$$f \circ g(x,y) = \bigvee_{z\in 1} g(x,z) * f(z,y) = f(\bullet,y) * g(x,\bullet) = f(y) * g(x) = g \otimes f(x,y)$$

and hence, $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$.

The second condition, $\Pi \subseteq \Psi \circ \Phi$, means that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\mathbb{G} \circ \mathbb{F} \leq [\top_{\{(\bullet,\bullet)\}}]$, that is, that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\top = g \circ f(\bullet,\bullet) = \bigvee_{x\in X} f(\bullet,x) * g(x,\bullet) = \bigvee_{x\in X} f(x) * g(x)$ for all $f \in \mathbb{F}, g \in \mathbb{G}$. So we arrive at the following characterization.

Proposition 4.2. *Let (X,Λ) be a saturated L-quasi-uniform limit space and let $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ be a left-promodule and $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ be a right-promodule. Then Φ is left-adjoint to Ψ, $\Phi \dashv \Psi$, if, and only if,*
(1) $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$; and
(2) there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that for all $f \in \mathbb{F}$ and all $g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x) * g(x) = \top$.

Proposition 4.3. *The saturated L-quasi-uniform limit space (X,Λ) is Lawvere complete if, and only if, for all left-promodules $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ and all right-promodules $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ with $\Phi \dashv \Psi$ there is $x \in X$ such that $\Phi = \Lambda \circ [[x]]$ and $\Psi = [[x]] \circ \Lambda$.*

In [6, 13, 14], for a saturated L-quasi-uniform space $(X,\mathcal{U})$ a prorelation is defined to be a saturated prefilter $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$. A prorelation $\mathbb{H}$ is a *left-$\mathcal{U}$-promodule* if $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H}$ and a prorelation $\mathbb{K}$ is a *right-$\mathcal{U}$-promodule* if $\mathbb{K} \leq \mathbb{K} \circ \mathcal{U}$. (Note that in [6] the composition was defined in a different order.) A left-$\mathcal{U}$-promodule $\mathbb{H}$ is *left-adjoint* to the right-$\mathcal{U}$-promodule $\mathbb{K}$, $\mathbb{H} \dashv \mathbb{K}$, if $\mathcal{U} \leq \mathbb{K} \otimes \mathbb{H}$ and $\bigvee_{x\in X} h(x) * k(x) = \top$ for all $h \in \mathbb{H}$ and all $k \in \mathbb{K}$. Then $\mathbb{H}$ is a left-$\mathcal{U}$-promodule if and only if $[\mathbb{H}]$ is a left-$[\mathcal{U}]$-promodule. In fact, if $\mathbb{H}$ is a left-$\mathcal{U}$-promodule and $\mathbb{F} \in [\mathcal{U}] \circ [\mathbb{H}]$, then $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H} \leq \mathbb{F}$ and hence, $\mathbb{F} \in [\mathbb{H}]$. Conversely, if $[\mathbb{H}]$ is a left-$[\mathcal{U}]$-promodule, then $\mathcal{U} \circ \mathbb{H} \in [\mathcal{U}] \circ [\mathbb{H}] \subseteq [\mathbb{H}]$, so that $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H}$. In a similar way, we see that $\mathbb{K}$ is a right-$\mathcal{U}$-promodule if and only if $[\mathbb{K}]$ is a right-$[\mathcal{U}]$-promodule.

Furthermore, it is not difficult to show that $\mathbb{H} \dashv \mathbb{K}$ (in $(X,\mathcal{U})$) if and only if $[\mathbb{H}] \dashv [\mathbb{K}]$ (in $(X,[\mathcal{U}])$).

A saturated L-quasi-uniform space $(X,\mathcal{U})$ is called *Lawvere complete* [13] (see also [6]) if for all left-$\mathcal{U}$-promodules $\mathbb{H}$ and all right-$\mathcal{U}$-promodules $\mathbb{K}$ with $\mathbb{H} \dashv \mathbb{K}$ there is $x \in X$ such that $\mathbb{H} = \mathcal{U}(x,\cdot) = [\{u(x,\cdot) : u \in \mathcal{U}\}]$ and $\mathbb{K} = \mathcal{U}(\cdot,x) = [\{u(\cdot,x) : u \in \mathcal{U}\}]$.

Proposition 4.4. *A saturated L-quasi-uniform space $(X,\mathcal{U})$ is Lawvere complete if, and only if, $(X,[\mathcal{U}])$ is Lawvere complete.*

Proof. Let first $(X,\mathcal{U})$ be Lawvere complete and let $\Phi \dashv \Psi$. From Proposition 4.2 we see that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\mathbb{F} \dashv \mathbb{G}$. By Lawvere completeness, there is $x \in X$ such that $\mathbb{F} = \mathcal{U}(x,\cdot)$ and $\mathbb{G} = \mathcal{U}(\cdot,x)$. For $u \in L^{X\times X}$ we have $u \circ \top_{\{x\}}(y) = \bigvee_{z\in X} \top_{\{x\}}(z) * u(z,y) = u(x,y)$ for all $y \in X$ and hence $\mathcal{U} \circ [x] = \mathcal{U}(x,\cdot)$. Similarly we can show $[x] \circ \mathcal{U} = \mathcal{U}(\cdot,x)$. We conclude $[\mathbb{F}] = [\mathcal{U}] \circ [[x]]$ and $[\mathbb{G}] = [[x]] \circ [\mathcal{U}]$. Clearly, we have $[\mathbb{F}] \dashv [\mathbb{G}]$ and $[\mathbb{F}] \subseteq \Phi$ and $[\mathbb{G}] \subseteq \Psi$. Lemma 3.8 implies $\Phi = [\mathbb{F}] = [\mathcal{U}] \circ [[x]] = x_*$ and $\Psi = [\mathbb{G}] = [[x]] \circ [\mathcal{U}] = x^*$ and hence $(X,[\mathcal{U}])$ is Lawvere complete.

For the converse, let $(X,[\mathcal{U}])$ be Lawvere complete and let $\mathbb{H} \dashv \mathbb{G}$. Then $[\mathbb{H}] \dashv [\mathbb{G}]$ and hence there is $x \in X$ such that $[\mathbb{H}] = [\mathcal{U}] \circ [[x]]$ and $[\mathbb{G}] = [[x]] \circ [\mathcal{U}]$. We conclude $\mathbb{H} \geq \mathcal{U} \circ [x] = \mathcal{U}(x,\cdot)$ and $\mathbb{K} \geq [x] \circ \mathcal{U} = \mathcal{U}(\cdot,x)$. As $\mathcal{U}(x,\cdot) \dashv \mathcal{U}(\cdot,x)$, see [6], we obtain $\mathbb{H} = \mathcal{U}(x,\cdot)$ and $\mathbb{K} = \mathcal{U}(\cdot,x)$ and $(X,\mathcal{U})$ is Lawvere complete. □

5 Cauchy Completeness of Saturated L-Quasi-Uniform Limit Spaces

Let (X,Λ) be a saturated L-quasi-uniform limit space and let $\mathbb{F},\mathbb{G} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$. The following concepts were introduced in [13].

(1) $(\mathbb{F},\mathbb{G})$ are called a *saturated pair* L*-prefilter* if for all $f \in \mathbb{F}$ and all $g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x) * g(x) = \top$.

(2) A saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called a *Cauchy pair* if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$.

(3) A saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ converges to $x \in X$, $(\mathbb{F},\mathbb{G}) \to x$, if $[x] \otimes \mathbb{F} \in \Lambda$ and $\mathbb{G} \otimes [x] \in \Lambda$.

We note that if a saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ converges to x, then $([x] \otimes \mathbb{F}) \circ (\mathbb{G} \otimes [x]) = \mathbb{G} \otimes \mathbb{F} \in \Lambda$, that is, $(\mathbb{F},\mathbb{G})$ is a Cauchy pair.

Proposition 5.1 (see also [6]). *Let (X,Λ) be a saturated* L*-quasi-uniform limit space and let $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ be saturated pair* L*-prefilters on X.*

(SCP1) $([x],[x])$ *is a Cauchy pair for all $x \in X$;*

(SCP2) If $(\mathbb{F},\mathbb{G})$ is a Cauchy pair and if $\mathbb{F}' \geq \mathbb{F}$ and $\mathbb{G}' \geq \mathbb{G}$, then $(\mathbb{F}',\mathbb{G}')$ is a Cauchy pair.

(SCP3) If $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ are Cauchy pairs and if $\bigvee_{x\in X} f(x) * g'(x) = \top$ for all $f \in \mathbb{F}$ and all $g' \in \mathbb{G}'$ and also $\bigvee_{x\in X} f'(x) * g(x) = \top$ for all $f' \in \mathbb{F}$ and all $g \in \mathbb{G}'$, then $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ is a Cauchy pair.

Proof. We show only (SCP3). Obviously, $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ is a pair L-prefilter. $\bigvee_{x\in X} f(x) * g'(x) = \top$ for all $f \in \mathbb{F}$ and all $g' \in \mathbb{G}'$, we conclude $(\mathbb{G}'\otimes\mathbb{F}')\circ(\mathbb{G}\otimes\mathbb{F}) = \mathbb{G}\otimes\mathbb{F}'$, see [7]. Similarly, we have $(\mathbb{G}\otimes\mathbb{F})\circ(\mathbb{G}'\otimes\mathbb{F}') = \mathbb{G}'\otimes\mathbb{F}$. By (SLUL2) then $\mathbb{G}\otimes\mathbb{F}' \in \Lambda$ and $\mathbb{G}'\otimes\mathbb{F} \in \Lambda$. Hence, using Proposition 3.10 [7], we obtain $\Lambda \ni (\mathbb{G}\otimes\mathbb{F})\wedge(\mathbb{G}\otimes\mathbb{F}')\wedge(\mathbb{G}'\otimes\mathbb{F})\wedge(\mathbb{G}'\otimes\mathbb{F}') = (\mathbb{G}\wedge\mathbb{G}')\otimes(\mathbb{F}\wedge\mathbb{F}')$. □

This proposition shows that a saturated L-quasi-uniform limit space has an underlying ⊤-quasi-Cauchy space. These spaces were introduced in [8].

Definition 5.2. A saturated L-quasi-uniform limit space (X,Λ) is called *Cauchy complete* if for all Cauchy pairs $(\mathbb{F},\mathbb{G})$ there is $x \in X$ such that $(\mathbb{F},\mathbb{G}) \to x$.

For a saturated L-quasi-uniform space $(X,\mathcal{U})$, a saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called a *Cauchy pair* [13] if $\mathbb{G}\otimes\mathbb{F} \geq \mathcal{U}$, that is, if $(\mathbb{F},\mathbb{G})$ is a Cauchy pair in $(X,[\mathcal{U}])$. The saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called *convergent* to $x \in X$ if $\mathbb{F} \geq \mathcal{U}(x,\cdot)$ and $\mathbb{G} \geq \mathcal{U}(\cdot,x)$. From $([x]\otimes\mathbb{F})\circ[x] = \mathbb{F}$ we obtain $[x]\otimes\mathbb{F} \geq \mathcal{U}$ if, and only if, $\mathbb{F} \geq \mathcal{U}\circ[x] = \mathcal{U}(x,\cdot)$ and similarly we have $\mathcal{G}\otimes[x] \geq \mathcal{U}$ if, and only if, $\mathcal{G} \geq [x]\circ\mathcal{U} = \mathcal{U}(\cdot,x)$. Hence we have $(\mathbb{F},\mathbb{G}) \to x$ in $(X,\mathcal{U})$ if, and only if, $(\mathbb{F},\mathbb{G}) \to x$ in $(X,[\mathcal{U}])$. From these observations we immediately obtain the following result.

Proposition 5.3. *A saturated* L*-quasi-uniform space $(X,\mathcal{U})$ is Cauchy complete if, and only if, $(X,[\mathcal{U}])$ is Cauchy complete.*

It is shown in [13, 14] that a saturated L-quasi-uniform space is Cauchy complete if, and only if, it is Lawvere complete. Hence, by Propositions 4.4 and 5.3, for a saturated L-quasi-uniform space $(X,\mathcal{U})$, the saturated L-quasi-uniform limit space $(X,[\mathcal{U}])$ is Cauchy complete if, and only if, it is Lawvere complete. This is also true for arbitrary saturated L-quasi-uniform limit spaces. We first show the following Lemma.

Lemma 5.4. *Let (X,Λ) be a saturated* L*-quasi-uniform limit space, $x \in X$ and $\mathbb{F},\mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$. Then*
(1) $[x]\otimes\mathbb{F} \in \Lambda$ if, and only if, $\mathbb{F} \in \Lambda\circ[[x]]$.
(2) $\mathbb{G}\otimes[x] \in \Lambda$ if, and only if, $\mathbb{G} \in [[x]]\circ\Lambda$.

Proof. (1) Let first $[x]\otimes\mathbb{F} \in \Lambda$. Then $\mathbb{F} = ([x]\otimes\mathbb{F})\circ[x] \in \Lambda\circ[[x]]$. (We have $(\top_{\{x\}}\otimes f)\circ\top_{\{x\}}(y) = \bigvee_{z\in X}\top_{\{x\}}(z) * (\top_{\{x\}}\otimes f)(z,y) = \top_{\{x\}}\otimes f(x,y) = f(y)$.)

Let now $\mathbb{F} \in \Lambda\circ[[x]]$. Then there is $\mathbb{L} \in \Lambda$ such that $\mathbb{L}\circ[x] \leq \mathbb{F}$. We conclude $\mathbb{L} \leq [x]\otimes(\mathbb{L}\circ[x]) \leq [x]\otimes\mathbb{F}$ and hence $[x]\otimes\mathbb{F} \in \Lambda$. (We have $\top_{\{x\}}\otimes(l\circ\top_{\{x\}})(s,t) = \top_{\{x\}}(s) * \bigvee_{y\in X}\top_{\{x\}}(y) * l(y,t) = \top_{\{x\}}(s) * l(x,t) \leq l(s,t)$.)

(2) can be shown in a similar way. □

Theorem 5.5. *A saturated* L*-quasi-uniform limit space (X,Λ) is Cauchy complete if, and only if, it is Lawvere complete.*

Proof. Let first (X,Λ) be Lawvere complete and let $(\mathbb{F},\mathbb{G})$ be a Cauchy pair. We define $\Phi = \Lambda\circ[\mathbb{F}]$ and $\Psi = [\mathbb{G}]\circ\Lambda$. It is not difficult to see that Φ,Ψ are prorelations. As $\Lambda\circ\Phi = \Lambda\circ\Lambda\circ[\mathbb{F}] \subseteq \Lambda\circ\mathbb{F} = \Phi$, Φ is a left-promodule. Similarly, $\Psi\circ\Lambda = [\mathbb{G}]\circ\Lambda\circ\Lambda \subseteq [\mathbb{G}]\circ\Lambda = \Psi$, that is, Ψ is a right promodule. We show $\Phi \dashv \Psi$. Let $\mathbb{H} \in \Phi$ and $\mathbb{K} \in \Psi$. Then there are $\mathbb{L}_1,\mathbb{L}_2 \in \Lambda$ such that $\mathbb{H} \geq \mathbb{L}_1\circ\mathbb{F}$ and $\mathbb{K} \geq \mathbb{G}\circ\mathbb{L}_2$. A straightforward calculation shows that for $l_2, l_2 \in L^{X\times X}$ and $f, g \in L^X$ we have $l_1 \circ (g \otimes f) \circ l_2 = (g \circ l_2) \otimes (l_1 \circ f)$. Hence $\mathbb{K} \otimes \mathbb{H} \geq (\mathbb{G} \circ \mathbb{L}_2) \otimes (\mathbb{L}_1 \circ \mathbb{F}) = \mathbb{L}_1 \circ (\mathbb{G} \otimes \mathbb{F}) \circ \mathbb{L}_2 \in \Lambda$ by (SLUL4). Furthermore, we have $\mathbb{F} = [\top_{\Delta_X}] \circ \mathbb{F} \in \Phi$ and $\mathbb{G} = \mathbb{G} \circ [\top_{\Delta_X}] \in \Psi$ and therefore $\Phi \dashv \Psi$. As (X, Λ) is Lawvere complete, there is $x \in X$ such that $\Phi = x_*$ and $\Psi = x^*$, that is, $\Lambda \circ [\mathbb{F}] = \Lambda \circ [[x]]$ and $[\mathbb{G}] \circ \Lambda = [[x]] \circ \Lambda$. As $\mathbb{F} = [\top_{\Delta_X}] \circ \mathbb{F} \in \Lambda \circ [\mathbb{F}] = \Lambda \circ [[x]]$ we conclude with Lemma 5.4 that $[x] \otimes \mathbb{F} \in \Lambda$. In a similar way we see that $\mathbb{G} \otimes [x] \in \Lambda$ and hence $(\mathbb{F}, \mathbb{G}) \to x$ and (X, Λ) is Cauchy complete.

Let now (X, Λ) be a Cauchy complete. Let $\Phi \dashv \Psi$. From Proposition 4.2 we see that there is a Cauchy pair $(\mathbb{F}, \mathbb{G})$ with $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$. By Cauchy completeness there is $x \in X$ such that $[x] \otimes \mathbb{F} \in \Lambda$ and $\mathbb{G} \otimes [x] \in \Lambda$, that is, $\mathbb{F} \in \Lambda \circ [[x]]$ and $\mathbb{G} \in [[x]] \circ \Lambda$. We define $\overline{\Phi} = \Lambda \circ [\mathbb{F}]$ and $\overline{\Psi} = [\mathbb{G}] \circ \Lambda$. Then, as in the first part of the proof, $\overline{\Phi} \dashv \overline{\Psi}$. We have $\overline{\Phi} = \Lambda \circ [\mathbb{F}] \subseteq \Lambda \circ \Phi \subseteq \Phi$. In a similar way we conclude $\overline{\Psi} \subseteq \Psi$ and hence, by Lemma 3.8, $\Phi = \Lambda \circ [\mathbb{F}]$. From $\mathbb{F} \in \Lambda \circ [[x]]$ we conclude $[\mathbb{F}] \subseteq \Lambda \circ [[x]]$ and hence $\Phi = \Lambda \circ [\mathbb{F}] \subseteq \Lambda \circ \Lambda \circ [[x]] \subseteq \Lambda \circ [[x]] = x_*$.

Let $\overline{\mathbb{F}} \in x_* = \Lambda \circ [[x]]$. Then there is $\mathbb{L} \in \Lambda$ such that $\mathbb{L} \circ [x] \leq \overline{\mathbb{F}}$. We note that for $f \in \mathbb{F}, g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x) * g(x) = \top$ and therefore $(g \otimes \top_{\{x\}}) \circ f = \top_{\{x\}}$. Hence we have $[x] = (\mathbb{G} \otimes [x]) \circ \mathbb{F} \in \Lambda \circ \Phi = \Phi$. It follows that $\overline{\mathbb{F}} \geq \mathbb{L} \circ [x] \in \Lambda \circ \Phi = \Phi$ and we have $\overline{\mathbb{F}} \in \Phi$, that is $x_* \subseteq \Phi$. Similar arguments show that $\Psi = x^*$ and (X, Λ) is Lawvere complete. □

6 Conclusion

We studied two completeness notions for saturated L-quasi-uniform limit spaces. The one is based on the concept of adjoint promodules and generalizes an approach of Clementino and Hofmann [2]. The other uses the concept of the Cauchy pair and generalizes a classical approach due to Lindgren and Fletcher [10]. We show that both approaches are equivalent.

An open problem is the construction of a completion based on either of the two completeness notions. This will still deserve more work.

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