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Fuzzy (Soft) Quasi-Interior Ideals of Semirings

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Abstract. In this paper, as a further generalization of fuzzy ideals, we introduce the notion of a fuzzy (soft) quasi-interior ideals of semirings and characterize regular semiring in terms of fuzzy (soft) quasi-interior ideals of semirings. We prove that (μ, A) is a fuzzy soft left quasi-interior ideal over a regular semiring M, if and only if (μ, A) is a fuzzy soft quasi-ideal over a semiring M, and study some of the properties.

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1 Introduction

The notion of ideals introduced by Dedekind for the theory of algebraic numbers was generalized by E. Noether for associative rings. The one and two-sided ideals presented by her are still central concepts in ring theory. We know that the notion of a one-sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are the generalization of left and right ideals, whereas the bi-ideals are the generalization of quasi ideals. The notion of bi-ideals in semigroups was introduced by Lajos [8]. Iseki introduced the concept of quasi ideal for semiring [4, 5, 6]. M. Henriksen studied ideals in semirings [3]. As a further generalization of ideals, Steinfeld first introduced the notion of quasi ideals for semigroups and then for rings. We know that the notion of the bi-ideal in semirings is a special case of the (m, n) ideal introduced by S. Lajos. The concept of bi-ideals was first introduced by R. A. Good and D. R. Hughes for a semigroup[2]. Lajos and Szasz introduced the concept of bi-ideals for rings[9].

Many real-world problems are complicated due to various uncertainties. In addressing them, classical methods may not be the best option. To overcome such, several theories like randomness, rough set, and fuzzy set were introduced. L. A. Zadeh developed the fuzzy set theory in 1965 [18]. Many papers on fuzzy sets appeared, showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. N. Kuroki studied fuzzy interior ideals in semigroups [7].

Molodtsov introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties, only partially resolving the problem because objects in a universal set often do not precisely satisfy the parameters associated to each of the elements in the set[11]. Soft set theory has wide applications in fields like game theory, operations research, data analysis, decision making, probability theory, and measurement

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theory. Acar et. al. gave the basic concept of soft rings. Feng et al. studied soft semirings using the soft set theory[1]. Then Maji et. al.[10] extended soft set theory to fuzzy soft set theory. Fuzzy soft set theory is a unification of fuzzy set theory and soft set theory. Aktas and Cagman defined the soft set and soft groups. Fuzzy soft set theory has wide applications in medical diagnosis.

M. Murali Krishna Rao introduced the notion of (quasi-interior, bi-interior, bi-quasi, tri, and tri-quasi interior) ideals as a generalization of (quasi, bi and interior) ideals of a semiring, semigroup, $\Gamma-$semiring, $\Gamma-$semigroup and studied their properties[12, 13, 14, 15, 17]. M. Murali Krishna Rao studied fuzzy bi-interior ideals of $\Gamma-$semiring [16].

This paper aims to introduce the notion of fuzzy quasi-interior ideal and fuzzy soft quasi-interior ideal of a semiring. We prove that every fuzzy soft left quasi-interior ideal over a regular semiring if and only if it is a fuzzy soft quasi ideal over a semiring. Regular semiring is characterized in terms of fuzzy(soft) quasi-interior ideals of a semiring. We study, M is regular if and only if $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a, a \in A$, for any fuzzy left quasi-interior ideals of fuzzy soft quasi-interior ideals (μ, A) over a semiring M.

2 Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. [13] A set M together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Example 2.2. Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring.

Definition 2.3. [13] A non-empty subset A of a semiring M is called:

(i) a *subsemiring* of M, if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$,

(ii) a *quasi ideal* of M, if A is a subsemiring of M and $AM \cap MA \subseteq A$,

(iii) a *bi-ideal* of M, if A is a subsemiring of M and $AMA \subseteq A$,

(iv) an *interior ideal* of M, if A is a subsemiring of M and $MAM \subseteq A$,

(v) a $left$ $(right)$ *ideal* of M, if A is a subsemiring of M and $MA \subseteq A(AM \subseteq A)$,

(vi) an *ideal*, if A is a subsemiring of $M, AM \subseteq A$ and $MA \subseteq A$,

(vii) a $left(right)$ *bi-quasi ideal* of M, if A is a subsemiring of M and $MA \cap AMA(AM \cap AMA) \subseteq A$,

(ix) a *bi- quasi ideal* of M, if A is a left bi- quasi ideal and a right bi- quasi ideal of M.

(x) a left(right) quasi-interior ideal of M, if A is a subsemiring of M and $MAMA(AMAM) \subseteq A$.

Definition 2.4. [13] An element a of a semiring M is called a regular element if there exists an element b of M such that $a = aba$.

Definition 2.5. [13] A semiring M is called a regular semiring if every element of M is a regular element.

Definition 2.6. [16] Let A be a non-empty subset of M. The *characteristic function* of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.7. [16] A function $f : R \to M$, where R and M are semirings. Then f is called a semiring homomorphism, if $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Definition 2.8. [16] Let U be an initial Universe set and E be the set of parameters. Let $P(U)$ denotes the power set of U. A pair (μ, E) is called a soft set over U where μ is a mapping given by $\mu : E \to P(U)$.

Definition 2.9. [16] Let U be an initial Universe set and E be the set of parameters, $A \subseteq E$. A pair (μ, A) is called fuzzy soft set over U where μ is a mapping given by $\mu : A \to I^U$ where I^U denotes the collection of all fuzzy subsets of U. $\mu(a)$, $a \in A$, be a fuzzy subset and is denoted by μ_a.

Definition 2.10. [16] Let $(\mu, A), (\lambda, B)$ be fuzzy soft sets over U then (μ, A) is said to be a fuzzy soft subset of (λ, B), denoted by $(\mu, A) \subseteq (\lambda, B)$ if $A \subseteq B$ and $\mu_a \subseteq \lambda_a$ (μ_a, λ_a are fuzzy subsets) for all $a \in A$.

Definition 2.11. [16] Let (μ, A), (λ, B) be fuzzy soft sets. The *intersection* of (μ, A) and (λ, B), denoted by $(\mu, A) \cap (\lambda, B) = (\gamma, C)$, where $C = A \cup B$, is defined as:

$$\gamma_c = \begin{cases} \mu_c, & \text{if } c \in A \setminus B; \\ \lambda_c, & \text{if } c \in B \setminus A; \\ \mu_c \cap \lambda_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.12. [16] Let (μ, A), (λ, B) be fuzzy soft sets. The *union* of (μ, A) and (λ, B), denoted by $(\mu, A) \cup (\lambda, B) = (\gamma, C)$, where $C = A \cup B$, is defined as:

$$\gamma_c = \begin{cases} \mu_c, & \text{if } c \in A \setminus B; \\ \lambda_c, & \text{if } c \in B \setminus A; \\ \mu_c \cup \lambda_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.13. [16] Let M be a semiring, E be a parameter set and $A \subseteq E$. Let $\mu : A \to [0,1]^M$ be a mapping, where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (μ, A) is called a *fuzzy soft left (right) ideal over* M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0,1]$ is a fuzzy left(right) ideal of M, i.e., for all $x, y \in M$,

(i) $\mu_a(x+y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xy) \geq \mu_a(y)(\mu_a(x))$.

(μ, A) is called a *fuzzy soft ideal* over M, if

(i) $\mu_a(x+y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xy) \geq \max\{\mu_a(x), \mu_a(y)\}$.

Definition 2.14. [16] Let M be a semiring, E be a parameter set, let $A \subseteq E$ and let $\mu : A \to [0,1]^M$ be a mapping. Then (μ, A) is called a *fuzzy soft quasi ideal* over M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0,1]$ is a fuzzy quasi ideal of M, i.e. for all $x, y \in M$,

(i) $\mu_a(x+y) \geq \min(\mu_a(x), \mu_a(y))$, (ii) $\mu_a \circ \chi_M \wedge \chi_M \circ \mu_a \subseteq \mu_a$.

(μ, A) is called a *fuzzy soft interior ideal* over M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0,1]$ is a fuzzy interior ideal of M, i.e., for all $x, y \in M$, (i) $\mu_a(x+y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xyz) \geq \max\{\mu_a(y)\}$.

3 Fuzzy quasi interior ideals

In this section, we introduce the notion of fuzzy (right, left) quasi interior ideal and study the properties of fuzzy (right, left) quasi interior ideals of semirings.

Definition 3.1. A fuzzy subset μ of a semiring M, is called a fuzzy left (right) quasi interior ideal if μ satisfies the following conditions

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$.

(ii) $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu\big)$.

A fuzzy subset μ of a semiring M, is called a fuzzy quasi interior ideal if it is both left fuzzy quasi interior ideal and right fuzzy quasi interior ideal of M.

Theorem 3.2. *Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Then I is a right quasi interior ideal of a semiring M if and only if χ_I is a fuzzy right quasi interior ideal of a semiring M.*

Proof. Let I be a non-empty subset of the semiring M and χ_I be the characteristic function of I. Suppose I is a right quasi interior ideal of the semiring M. Obviously, χ_I is a fuzzy subsemiring of M. We have $IMIM \subseteq I$. Then $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M = \chi_{IMIM} \subseteq \chi_I$. Therefore χ_I is a fuzzy right quasi interior ideal of the semiring M.
Conversely, suppose that χ_I is a fuzzy right quasi interior ideal of M. Then I is a subsemiring of M. We have $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \subseteq \chi_I$, implies that $\chi_{IMIM} \subseteq \chi_I$. Therefore $IMIM \subseteq I$. Hence I is a right quasi interior ideal of the semiring M. □

Theorem 3.3. *Let I be a right quasi interior ideal of a semiring M and μ be a fuzzy subset of M, is defined by* $\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1, & \text{otherwise.} \end{cases}$
for all $x \in M$, $\alpha_0, \alpha_1 \in [0,1]$ such that $\alpha_0 > \alpha_1$. Then μ is a fuzzy right quasi interior ideal of M and $\mu_{\alpha_0} = I$.

Theorem 3.4. *Every fuzzy (right, left) ideal of a semiring M is a fuzzy right quasi-interior ideal of M.*

Proof. Let μ be a fuzzy right ideal of the semiring M and $x \in M$.

$$\begin{aligned}
\mu \circ \chi_M(x) &= \sup_{x=ab} \min\{\mu(a), \chi_M(b)\}\ a, b \in M. \\
&= \sup_{x=ab} \mu(a) \\
&\leq \sup_{x=ab} \mu(ab) \\
&= \mu(x).
\end{aligned}$$

Therefore $\mu \circ \chi_M(x) \leq \mu(x)$. Then

$$\begin{aligned}
\mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=uvs} \min\{\mu \circ \chi_M(uv), \mu \circ \chi_M(s)\} \\
&\leq \sup_{x=uvs} \min\{\mu(uv), \mu(s)\} \\
&= \mu(x).
\end{aligned}$$

Hence μ is a fuzzy right quasi-interior ideal of M.

Let μ be a fuzzy left ideal of the semiring M and $x \in M$.

$$\begin{aligned}\chi_M \circ \mu(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu(b)\}\ a, b \in M.\\ &= \sup_{x=ab} \mu(b)\\ &\le \sup_{x=ab} \mu(ab)\\ &= \mu(x).\end{aligned}$$

Therefore $\chi_M \circ \mu(x) \le \mu(x)$. Then

$$\begin{aligned}\chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=uvs} \min\{\chi_M \circ \mu(u), \chi_M \circ \mu(vs)\}\\ &\le \sup_{x=uvs} \min\{\mu(u), \mu(vs)\}\\ &= \mu(x).\end{aligned}$$

Hence μ is a fuzzy left quasi-interior ideal of M. $\square$

Example 3.5. Let $M = \{a, b, c, d\}$. The binary operation is defined by the following tables

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	a

then $(M, +, \cdot)$ is a semiring.
I) Let $J = \{a, d\}$, then J is a subsemiring.
J is not a(ideal, left, right, bi, quasi, interior) ideal.
J is a right quasi-interior ideal.
1) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.
2) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in J, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.

II) Let $J_1 = \{a, c\}$, then J_1 is a subsemiring.
J_1 is not a(ideal, left, right, bi, quasi, interior) ideal.
J_1 is a left quasi-interior ideal.
1) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 1 & \text{if } x \in J_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy left quasi-interior ideal of M and μ is not a fuzzy left ideal.
2) Define $\mu = M \to [0, 1]$

$\mu(x) = \begin{cases} 0.6 & \text{if } x \in J_1, \\ 0.3, & \text{otherwise.} \end{cases}$

Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.

Theorem 3.6. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy left quasi interior ideal of a semiring M if and only if the level subset μ_t of μ is a left quasi interior ideal of a semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left quasi interior ideal of the semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$. Then $\mu(a) \geq t, \mu(b) \geq t,$ *so* $\mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t$, therefore $a + b \in \mu_t$ *and* $\mu(ab) \geq \min\{\mu(a), \mu(b)\} \geq t$, hence $ab \in \mu_t$.
Let $x \in M\mu_t M\mu_t$. Then $x = badc$, where $b, d \in M, a, c \in \mu_t$, thus
$\chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t$, so $\mu(x) \geq \chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t$.
Therefore $x \in \mu_t$. Hence μ_t is a left quasi interior ideal of M.
Conversely, suppose that μ_t is a left quasi interior ideal of the semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$, so $x + y \in \mu_{t_2}$ and $xy \in \mu_{t_2}$, then $\mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(x + y) \geq t_2 = \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(xy) \geq t_2 = \min\{\mu(x), \mu(y)\}$. We have $M\mu_l M\mu_l \subseteq \mu_l$, for all $l \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $M\mu_t M\mu_t \subseteq \mu_t$. Therefore $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy left quasi interior ideal of M. □

Corollary 3.7. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy (right) quasi interior ideal of a semiring if and only if the level subset μ_t of μ is a (right) quasi-interior ideal of a semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Theorem 3.8. *If μ and λ are fuzzy left quasi interior ideals of a semiring M, then $\mu \cap \lambda$ is a fuzzy left quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy left quasi interior ideals of M and $x, y \in M$.

$$\begin{aligned} \mu \cap \lambda(x + y) &= \min\{\mu(x + y), \lambda(x + y)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y).\} \end{aligned}$$

$$\begin{aligned} \mu \cap \lambda(xy) &= \min\{\mu(xy), \lambda(xy)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \end{aligned}$$

Then $\mu \cap \lambda$ is a fuzzy subsemiring. And

$$\begin{aligned} \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cap \lambda(b)\} \\ &= \sup_{x=ab} \min\Big\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\Big\} \\ &= \sup_{x=ab} \min\Big\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\Big\} \\ &= \min\Big\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\Big\} \\ &= \min\{\chi_M \circ \mu(x). \chi_M \circ \lambda(x)\} \\ &= \chi_M \circ \mu \cap \chi_M \circ \lambda(x). \end{aligned}$$

Therefore $\chi_M \circ \mu \cap \chi_M \circ \lambda = \chi_M \circ \mu \cap \lambda$.

$$
\begin{aligned}
&(\chi_M \circ \mu \cap \lambda) \circ (\chi_M \circ \mu \cap \lambda)(x)\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \chi_M \circ \lambda(a), \chi_M \circ \mu \cap \lambda(bc)\}\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(bc)\}\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}\}, \min\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\}\\
&= \sup_{x=abc} \min\Big\{ \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\}\\
&= \min\Big\{ \sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \sup_{x=abc} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\}\\
&= \min\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\}\\
&= \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}
$$

Then $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda$
Therefore $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda$.
Hence $\mu \cap \lambda$ is the fuzzy left quasi-interior ideal of M. □

Corollary 3.9. *If μ and λ are fuzzy (right) quasi interior ideals of a semiring M, then $\mu \cap \lambda$ is a fuzzy (right) quasi interior ideal of a semiring M.*

Theorem 3.10. *If μ and λ are fuzzy left quasi interior ideals of a semiring M, then $\mu \cup \lambda$ is a fuzzy left quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy left quasi interior ideals of M and $x, y \in M$.

$$
\begin{aligned}
\mu \cup \lambda(x+y) &= \max\{\mu(x+y), \lambda(x+y)\}\\
&\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\
&= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\
&= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}
\end{aligned}
$$

$$
\begin{aligned}
\mu \cup \lambda(xy) &= \max\{\mu(xy), \lambda(xy)\}\\
&\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\
&= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\
&= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}
\end{aligned}
$$

Then $\mu \cup \lambda$ is a fuzzy subsemiring. And

$$
\begin{aligned}
\chi_M \circ \mu \cup \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cup \lambda(b)\}\\
&= \sup_{x=ab} \min\{\chi_M(a), \max\{\mu(b), \lambda(b)\}\}\\
&= \sup_{x=ab} \max\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\
&= \max\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\}\\
&= \max\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\}\\
&= \chi_M \circ \mu \cup \chi_M \circ \lambda(x).
\end{aligned}
$$

Therefore $\chi_M \circ \mu \cup \chi_M \circ \lambda = \chi_M \circ \mu \cup \lambda$.

$$\begin{aligned}
&(\chi_M \circ \mu \cup \lambda) \circ (\chi_M \circ \mu \cup \lambda)(x) \\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cup \chi_M \circ \lambda(a), \chi_M \circ \mu \cup \chi_M \circ \lambda(bc)\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}, \max\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}, \max\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \max\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \max\Big\{\sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \sup_{x=abc} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \max\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

Then $\chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda$
Therefore $\chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cup \lambda$.
Hence $\mu \cup \lambda$ is a fuzzy left quasi-interior ideal of M. □

Corollary 3.11. *If μ and λ are fuzzy (right) quasi-interior ideals of a semiring M, then $\mu \cup \lambda$ is a fuzzy (right) quasi-interior ideal of a semiring M.*

Theorem 3.12. *Let M be a semiring. Then M is a regular if and only if $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi interior ideal μ of a semiring M.*

Proof. Let μ be a fuzzy left quasi interior ideal of the regular semiring M and $x, y \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.

$$\begin{aligned}
\chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=xyx} \Big\{\min\{\chi_M \circ \mu(x), \chi_M \circ \mu(yx)\}\Big\} \\
&\geq \sup_{x=xyx} \Big\{\min\{\mu(x), \mu(x)\}\Big\} \\
&= \mu(x).
\end{aligned}$$

Therefore $\mu \subseteq \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\chi_M \circ \mu \circ \chi_M \circ \mu = \mu$.
Conversely suppose that $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi interior ideal μ of the semiring M. Let B be a left quasi interior ideal of the semiring M.
By Theorem 3.2, χ_B is a fuzzy left quasi interior ideal of the semiring M.
Then $\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{MBMB}$. Therefore $B = MBMB$.
Hence M is the regular semiring . □

4 Fuzzy Soft (left, right) Quasi Interior Ideals

In this section, we introduce the notion of fuzzy soft right(left) quasi interior ideal, fuzzy soft quasi interior ideal of a semiring and study their properties.

Definition 4.1. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let μ be a mapping given by $\mu : A \rightarrow [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (μ, A) is called a fuzzy soft left (right) quasi interior ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset satisfies the following conditions

(i) $\mu_a(x+y) \geq \min\{\mu_a(x), \mu_a(y)\}$ for all $x, y \in M$.

(ii) $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a \big(\mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \mu_a\big)$.

A fuzzy soft set(μ, A) over a semiring M, is called a fuzzy soft quasi interior ideal if it is both fuzzy soft left quasi interior ideal and fuzzy soft right quasi interior ideal over M.

Example 4.2. Let $M = \{0, a, b, c\}$, define the binary operations " $+$ " and " $\cdot$ " on M, with the following tables

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	b	b	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then $(M, +, \cdot)$ is a semiring.
Let $E = \{e_1, e_2, e_3\}$. Choose the fuzzy set (F, E) over M.

Define

	0	a	b	c
f_{e_1}	0.7	0.4	0.6	0
f_{e_2}	0.8	0.5	0.7	0
f_{e_3}	0.9	0.6	0.8	0

$\{f_{e_i}\}, i = 1, 2, 3$ is a fuzzy right quasi interior ideal of M, and $\{f_{e_i}\}$ is not a fuzzy right ideal of M. Therefore (F, E) is not a fuzzy soft right ideal and (F, E) is a fuzzy soft right quasi interior ideal over M.

Theorem 4.3. *Let M be a semiring, E be a parameter set and $A \subseteq E$. If (μ, A) is a fuzzy soft right ideal over M, then (μ, A) is a fuzzy soft right quasi interior ideal over M.*

Proof. Let μ_a be a fuzzy soft right ideal of the semiring M and $x \in M$.

$$\begin{aligned}
\mu_a \circ \chi_M(x) &= \sup_{x=ab} \min\{\mu_a(a), \chi_M(b)\} \ a, b \in M. \\
&= \sup_{x=ab} \mu_a(a) \\
&\leq \sup_{x=ab} \mu_a(ab) \\
&= \mu_a(x).
\end{aligned}$$

Therefore $\mu_a \circ \chi_M(x) \leq \mu_a(x)$. Now

$$\begin{aligned}
\mu_a \circ \chi_M \circ \mu_a \circ \chi_M(x) &= \sup_{x=uvs} \min\{\mu_a \circ \chi_M(uv), \mu_a \circ \chi_M(s)\} \\
&\leq \sup_{x=uvs} \min\{\mu_a(uv), \mu_a(s)\} \\
&= \mu_a(x).
\end{aligned}$$

Thus μ_a is fuzzy right quasi-interior ideal of M. Hence (μ, A) is a fuzzy soft right quasi-interior ideal over M. $\square$

Corollary 4.4. *Every fuzzy soft (left) ideal of a semiring M is a fuzzy soft(left) quasi interior ideal over M.*

Theorem 4.5. *Let M be a semiring, $A \subseteq E$ and (η, A) be a non-empty fuzzy soft over M. Then (η, A) is a fuzzy soft left quasi interior ideal over M, if and only if the level subset $(\eta_a)_k$ of (η, A) is a left quasi interior ideal of M, $a \in A$, for every $k \in [0, 1]$, where $(\eta_a)_k \neq \phi$.*

Proof. The proof of the following theorem is similar to Theorem 3.6, so we omit the proof. □

Theorem 4.6. *Let M be a semiring, E be a parameter set and $A \subseteq E$, $B \subseteq E$. If (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over M, then $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M.*

Proof. Let (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals of a semiring M. By Definition 2.11, we have that $(\mu, A) \cap (\lambda, B) = (\gamma, C)$ where $C = A \cup B$.
Case (i): If $c \in A \setminus B$, then $\gamma_c = \mu_c$. Thus γ_c is a fuzzy left quasi interior ideal of M, since (μ, A) is a fuzzy soft left quasi interior ideal over M.
Case (ii): If $c \in B \setminus A$, then $\gamma_c = \lambda_c$. Therefore γ_c is a fuzzy left quasi interior ideal of M, since (λ, B) is a fuzzy soft left quasi interior ideal over M.
Case (iii): If $c \in A \cap B$, and $x, y \in M$, then $\gamma_c = \mu_c \cap \lambda_c$ and
Therefore By Theorem 3.8, γ_c is a fuzzy left quasi interior ideal of M. Hence $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M. □

Corollary 4.7. *If (μ, A) and (λ, B) are fuzzy soft(right) quasi interior ideals over semiring M, then $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft(right) quasi-interior ideal over M.*

Theorem 4.8. *Let M be a semiring, E be a parameter set and $A \subseteq E$, $B \subseteq E$. If (μ, A) and (λ, B) are fuzzy soft left quasi-interior ideals of M, then $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M.*

Proof. Let (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over the semiring M. By Definition 2.12, we have that $(\mu, A) \cup (\lambda, B) = (\gamma, C)$ where $C = A \cup B$.
Case (i): If $c \in A \setminus B$, then $\gamma_c = \mu_c$. Thus γ_c is a fuzzy left quasi-interior ideal of M, since (μ, A) is a fuzzy soft left quasi-interior ideal over M.
Case (ii): If $c \in B \setminus A$, then $\gamma_c = \lambda_c$. Therefore γ_c is a fuzzy left quasi-interior ideal of M, since $(\lambda, B)$is a fuzzy soft left quasi-interior ideal over M.
Case (iii): If $c \in A \cup B$, and $x, y \in M$, then $\gamma_c = \mu_c \cup \lambda_c$.
Therefore By Theorem 3.10, γ_c is a fuzzy left quasi-interior ideal of M. Hence $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft left quasi-interior ideal over M. □

Corollary 4.9. *If (μ, A) and (λ, B) are fuzzy soft(right) quasi-interior ideals over semiring M, then $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft(right) quasi-interior ideal over M.*

Theorem 4.10. *Let M be a semiring, E be a parameter set and $A \subseteq E$. Then (μ, A) is a fuzzy soft left quasi-interior ideal over a regular semiring M if and only if (μ, A) is a fuzzy soft quasi ideal over a semiring M.*

Proof. Let (μ, A) be a fuzzy soft left quasi interior ideal over the regular semiring M and $x \in M$. Then for each $a \in A$, $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$.
Suppose $\chi_M \circ \mu_a(x) > \mu_a(x)$ and $\mu_a \circ \chi_M(x) > \mu_a(x)$.

Since M is regular, there exists $y \in M$, such that $x = xyx$.

$$\begin{aligned}
\mu_a \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu_a(x), \chi_M(yx)\} \\
&= \sup_{x=xyx} \min\{\mu_a(x), 1\} \\
&= \sup_{x=xyx} \mu_a(x) \\
&> \mu_a(x). \textit{ And} \\
\mu_a \circ \chi_M \circ \mu_a \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu_a \circ \chi_M(x), \mu_a \circ \chi_M(yx)\} \\
&> \sup_{x=xyx} \min\{\mu_a(x), \mu_a(yx)\} \\
&= \mu_a(x)
\end{aligned}$$

Which is a contradiction. Hence (μ, A) is a fuzzy soft quasi ideal of M.
Let (μ, A) be a soft quasi interior ideal over the semiring M, and $a \in A$.
Then $\mu_a \circ \chi_M \wedge \chi_M \circ \mu_a \subseteq \mu_a$.
$\mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \mu_a \circ \chi_M$, and $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \chi_M \circ \mu_a$.
Therefore $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \wedge \mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \chi_M \circ \mu_a \wedge \mu_a \circ \chi_M \subseteq \mu_a$.
Hence (μ, A) is the fuzzy soft quasi interior ideal over M. □

Corollary 4.11. *Let M be a regular semiring, E be a parameter set and $A \subseteq E$. Then (μ, A) is a fuzzy soft(right) quasi interior ideal over a semiring M if and only if (μ, A) is a fuzzy soft quasi ideal over a semiring M.*

Theorem 4.12. *Let M be a semiring, E be a parameter set and $A \subseteq E$. Then M is a regular if and only if $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a, a \in A$, for any fuzzy left quasi interior ideal of fuzzy soft quasi interior ideal (μ, A) over a semiring M.*

Proof. Let (μ, A) be a fuzzy soft left quasi interior ideal over the regular semiring M and $x, y \in M$. Then to each $a \in A$, $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$.

$$\begin{aligned}
\chi_M \circ \mu_a \circ \chi_M \circ \mu_a(x) &= \sup_{x=xyx} \{\min\{\chi_M \circ \mu_a(x), \chi_M \circ \mu_a(yx)\}\} \\
&\geq \sup_{x=xyx} \{\min\{\mu_a(x), \mu_a(x)\}\} \\
&= \mu_a(x).
\end{aligned}$$

Therefore $\mu_a \subseteq \chi_M \circ \mu_a \circ \chi_M \circ \mu_a$. Hence $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a = \mu_a$.
Conversely suppose that $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a$, for any fuzzy soft quasi interior ideal (μ, A) over the semiring M and $a \in A$. Let B be a quasi interior ideal of the semiring M. By Theorem 3.2, χ_B be a fuzzy quasi interior ideal of the semiring M. Then $\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{MBMB}$. Thus $B = MBMB$.
Hence M is the regular semiring . □

Theorem 4.13. *Let M be a semiring, E be a parameter set and $A \subseteq E, B \subseteq E$. Then M is a regular if and only if $\mu_b \cap \gamma_a \subseteq \mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a$, for every fuzzy soft left quasi interior ideal (γ, A) and every fuzzy soft ideal (μ, B) over a semiring M, $a \in A, b \in B$.*

Proof. Let M be a regular semiring and $x \in M$. Then there exist $y \in M$ such that $x = xyx$, for each $a \in A, b \in B$, γ_a is a fuzzy left quasi interior ideal, μ_b is a fuzzy ideal of the semiring M. Then

$$\begin{aligned}
\mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a(x) &= \sup_{x=xyx} \Big\{ \min\{\mu_b \circ \gamma_a(xy),\ \mu_b \circ \gamma_a(x)\} \Big\} \\
&= \min \Big\{ \sup_{xy=xyxy} \{\min\{\mu_b(x), \gamma_a(yxy)\}, \sup_{xy=xyxy} \{\min\{\mu_b(x), \gamma_a(yxy)\}\} \Big\} \\
&\geq \min \Big\{ \min\{\mu_b(x), \gamma_a(x)\}, min\{\mu_b(x), \gamma_a(x)\} \Big\} \\
&= \min\{\mu_b(x), \gamma_a(x)\} = \mu_b \cap \gamma_a(x).
\end{aligned}$$

Hence $\mu_b \cap \gamma_a \subseteq \mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a$.
Conversely, suppose that the condition holds. Let (μ, B) be a fuzzy soft left quasi interior ideal of the semiring M. Then to each $b \in B$, $\mu_b \cap \chi_M \subseteq \chi_M \circ \mu_b \circ \chi_M \circ \mu_b, \mu_b \subseteq \chi_M \circ \mu_b \circ \chi_M \circ \mu_b$. Hence M is the regular semiring. $\square$

5 Conclusion

In this paper, we discussed the algebraic properties of fuzzy right(left) quasi interior ideal and fuzzy soft quasi interior ideal of a semiring. Regular semiring is characterized in terms of fuzzy quasi interior ideals and fuzzy soft quasi-interior ideals. We proved, that if M is a semiring, E be a parameter set and $A \subseteq E,\ B \subseteq E$ and if (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over M, then $(\mu, A) \cap (\lambda, B), ((\mu, A) \cup (\lambda, B))$ are fuzzy soft left quasi interior ideal over M. One can extend this work by studying the other algebraic structures.

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