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Weighted Graphs and Fuzzy Graphs

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Abstract. It has been shown in [3] that in the two-dimensional case, the lattices of truth values considered are pairwise isomorphic, and so are the corresponding families of fuzzy sets. Therefore, each result for one of these types of fuzzy sets can be directly rewritten for each (isomorphic) type of fuzzy sets. In this paper, we show that there is a strong connection between weighted graphs and fuzzy graphs. We accomplish this by using lattice isomorphisms. Consequently, under certain conditions, results for one area can be carried over immediately to the other. Many situations in fuzzy graph theory do not depend on the weights of the vertices. The situation of providing weights for the vertices of a weighted graph is also considered. We also consider lattice homomorphisms with an illustration involving nonstandard analysis. In particular, we consider a nonstandard weighted graph, i.e., a graph where the weights of the edges are from a nonstandard interval.

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1 Introduction

In [3], it is shown that many well-known generalizations of the concept of fuzzy sets, [8], with two-dimensional lattices of truth values are pairwise isomorphic and so are the corresponding families of fuzzy sets. Therefore, each result for one of these types of fuzzy sets can be directly rewritten for each isomorphic type of fuzzy set. In this paper, we wish to show that this also holds for weighted graphs and fuzzy graphs in certain circumstances. Many situations in fuzzy graph theory do not depend on the weights of the vertices. The situation of providing weights for the vertices of a weighted graph is also considered. We also develop some beginning results for lattice homomorphisms and illustrate the results using the nonstandard interval $[0,1]^*$ in nonstandard analysis and the standard unit interval $[0,1]$.

We begin by reviewing some results from [3]. We let $\mathbb{N}$ denote the set of positive integers.

For two partially ordered sets $(L_1, \leq_1)$ and $(L_2, \leq_2)$, a function $\varphi : L_1 \to L_2$ is called an **order homomorphism** if it preserves monotonicity, i.e., if $x \leq_1 y$ implies $\varphi(x) \leq_2 \varphi(y)$. If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are two lattices, then a function $\varphi : L_1 \to L_2$ is called a **lattice homomorphism** if it preserves finite meets and joins, i.e., if for all $x, y \in L_1$, $\varphi(x \wedge_1 y) = \varphi(x) \wedge_2 \varphi(y)$ and $\varphi(x \vee_1 y) = \varphi(x) \vee_2 \varphi(y)$. Each lattice homomorphism is an order homomorphism, but not conversely. A lattice homomorphism $\varphi : L_1 \to L_2$ is called an **embedding** (or **monomorphism**) if it is injective, an **epimorphism** if it is surjective, and an **isomorphism** if it is bijective, i.e., if it is both an embedding and an epimorphism.

Suppose that $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are isomorphic lattices and that $\varphi : L_1 \to L_2$ is a lattice isomorphism of L_1 onto L_2. Let the bottom and top elements of $(L_1, \leq_1)$ be denoted by 0_1 and 1_1, respectively. Let A_1 :

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$L_1 \times L_1 \to L_1$ be an associative, commutative order homomorphism and define the function $A_2 : L_2 \times L_2 \to L_2$ by

$$A_2(x,y) = \varphi(A_1((\varphi^{-1}(x), \varphi^{-1}(y))).$$

If A_1 is a t-norm, then A_2 is a t-norm. If A_1 is a t-conorm, then A_2 is a t-conorm, [[3], p. 5].

Note that $\varphi^{-1}(A_2(x,y)) = A_1((\varphi^{-1}(x), \varphi^{-1}(y))$ and so $\varphi^{-1}(A_2(\varphi(w), \varphi(z))) = A_1(w,z)$, where $w = \varphi^{-1}(x)$ and $z = \varphi^{-1}(y)$.

Definition 1.1. *Let c be a function of L into L. Here we are considering a lattice $(L, \vee, \wedge, 0, 1)$. Consider the following conditions:*

(1) *$c(0) = 1$ and $c(1) = 0$ (boundary conditions).*

(2) *For all $a, b \in L$, if $a \leq b$, then $c(a) \geq c(b)$ (monotonicity).*

(3) *c is continuous.*

(4) *c is involutive, i.e., $c(c(a)) = a$, for all $a \in L$.*

If c satisfies conditions (1) and (2), we say that c is a ***complement*** *on L.*

Suppose that φ is a lattice isomorphism of L_1 onto L_2. Let c_1 be a complement on L_1. Define $c_2(x) = \varphi(c_1(\varphi^{-1}(x)))$ for all $x \in L_2$. Then $c_2(1_2) = \varphi(c_1(\varphi^{-1}(1_2))) = \varphi(c_1(1_1)) = \varphi(0_1) = 0_2$. Also, $c_2(0_2) = \varphi(c_1(\varphi^{-1}(0_2))) = \varphi(c_1(0_1)) = \varphi(1_1) = 1_2$.

Let $x, y \in L_2$ be such that $x \leq_2 y$. Then $\varphi^{-1}(x) \leq_1 \varphi^{-1}(y)$. Thus $c_2(x) = \varphi(c_1(\varphi^{-1}(x))) \geq_2 \varphi(c_1(\varphi^{-1}(y)) = c_2(y)$.

Let c_2 be a complement of L_2. Define $c_1(x) = \varphi^{-1}(c_2(\varphi(x)))$ for all $x \in L_1$. Then $c_1(1_1) = \varphi^{-1}(c_2(\varphi(1_1))) = \varphi^{-1}(c_2(1_2)) = \varphi^{-1}(0_2) = 0_1$. Also, $c_1(0_1) = \varphi^{-1}(c_2(\varphi(0_1))) = \varphi^{-1}(c_2(0_2)) = \varphi^{-1}(1_2) = 1_1$.

Let $x, y \in L_1$ be such that $x \leq_1 y$. Then $\varphi(x) \leq_2 \varphi(y)$. Thus $c_1(x) = \varphi^{-1}(c_2(\varphi(x))) \geq_2 \varphi^{-1}(c_2(\varphi(y))) = c_1(y)$.

We have just shown the following result.

Theorem 1.2. *c_1 is a complement if and only if c_2 is a complement.*

Note that if $c_2(x) = \varphi(c_1(\varphi^{-1}(x)))$, then $\varphi^{-1}(c_2(x)) = c_1(\varphi^{-1}(x))$ and so $\varphi^{-1}(c_2(\varphi(y))) = c_1(\varphi^{-1}(\varphi(y))) = c_1(y)$, where $y = \varphi^{-1}(x)$.

Theorem 1.3. *c_1 is involutive if and only if c_2 is involutive.*

Proof. Suppose c_1 is involutive. Let $x \in L_2$. Then

$$\begin{aligned} c_2(c_2(x)) &= c_2(\varphi(c_1(\varphi^{-1}(x)))) = \varphi(c_1(\varphi^{-1}(\varphi(c_1(\varphi^{-1}(x)))))) \\ &= \varphi(c_1(c_1(\varphi^{-1}(x)))) = \varphi(\varphi^{-1}(x)) = x. \end{aligned}$$

The converse is now immediate. □

2 Weighted Graphs

We next apply the results in the previous section to weighted graphs.

Let (V, E, w) be a weighted graph, where V is a finite set of vertices and w is a function of the set of edges E into the positive real numbers. We assume no loops and at most one edge between two vertices. Let $m \geq \vee\{w(e) | e \in E\}$. We hold m fixed throughout. We also assume $m \geq 1$. For all $e \in E$, define

$\mu : E \to [0,1]$ by $\mu(e) = \frac{1}{m}w(e)$. Then $G = (V, E, \sigma, \mu)$ is a fuzzy graph if for all $v \in V, \sigma(v) = 1$. Consider (V, E, μ) as a weighted graph. We note below that the lattices associated with (V, E, w) and (V, E, μ) are isomorphic. (Clearly there one-to-one correspondence between their point sets which preserves adjacency, namely the identity map of V onto V.)

Define $\varphi : [0, m] \to [0, 1]$ by for all $x \in [0, m], \varphi(x) = \frac{1}{m}x$. Then φ is a lattice isomorphism of L_1 onto L_2, where $L_1 = [0, m]$ and $L_2 = [0, 1]$ and $\vee$ and $\wedge$ are the usual operations of maximum and minimum, respectively. Note that for all $x, y \in L_1, \varphi(x \wedge y) = \frac{x \wedge y}{m} = \frac{x}{m} \wedge \frac{y}{m} = \varphi(x) \wedge \varphi(y)$ and a similar result holds for $\vee$. Also, φ is continuous and preserves $<$.

We next consider Definition 1.1 for $[0, m]$.

Theorem 2.1. (See [4]) *Let $m = 1$. If $c : [0, 1] \to [0, 1]$ satisfies (2) and (4) of Definition 1.1, then c satisfies (1) and (3). Also, c is bijective.*

Let c satisfy (1) and (2) of Definition 1.1. Define $\widehat{c} : [0, m] \to [0, m]$ by for all $a \in [0, m], \widehat{c}(a) = \varphi^{-1}(c((\varphi(a))$. Then $\widehat{c}(0) = \varphi^{-1}(c(0)) = \varphi^{-1}(1) = m$ and $\widehat{c}(m) = \varphi^{-1}(c(1)) = \varphi^{-1}(0) = 0$.

Let $C = \{c |\ c$ satisfies (1) and (2) for $m = 1\}$ and $\widehat{C} = \{\widehat{c} |\ \widehat{c}$ satisfies (1) and (2) for $m > 1\}$. Define the function f of C into $\widehat{C}$ by for all $c \in C, f(c) = \widehat{c}$, where for all $a \in [0, m]$,

$$\widehat{c}(a) = mc(\frac{a}{m}).$$

We show that f is a one-to-one function of C onto $\widehat{C}$. We have that $\widehat{c}(0) = mc(0) = m$ and $\widehat{c}(m) = m(c(1)) = 0$. Since m is fixed, that $\widehat{c}$ satisfies (2) holds since c satisfies (2). Hence f maps C into $\widehat{C}$. Let $\widehat{c} \in \widehat{C}$. Define $c : [0, 1] \to [0, 1]$ by for all $a \in [0, 1], c(a) = \frac{1}{m}\widehat{c}(ma)$. Then $c(0) = \frac{1}{m}\widehat{c}(0) = 1$ and $c(1) = \frac{1}{m}\widehat{c}(m) = 0$. Thus f maps C onto $\widehat{C}$. Now

$$\begin{aligned} c_1 &= c_2 \Leftrightarrow \forall a \in [0, m], c_1(\frac{a}{m}) = c_2(\frac{a}{m}) \\ &\Leftrightarrow \forall a \in [0, m], mc_1(\frac{a}{m}) = mc_2(\frac{a}{m}) \\ &\Leftrightarrow \forall a \in [0, m], \widehat{c_1}(a) = \widehat{c_2}(a) \\ &\Leftrightarrow \widehat{c_1} = \widehat{c_2} \\ &\Leftrightarrow f(c_1) = f(c_2). \end{aligned}$$

Theorem 2.2. *c is involutive if and only if $\widehat{c}$ is involutive, where $f(c) = \widehat{c}$.*

Proof. Suppose c is involutive. Let $a \in [0, m]$. Then

$$\widehat{c}(\widehat{c}(a)) = \widehat{c}(m(c(\frac{a}{m})) = m(c(\frac{mc(\frac{a}{m})}{m}) = m(c(c(\frac{a}{m})) = m(\frac{a}{m}) = a.$$

Thus $\widehat{c}$ is involutive.

Conversely, suppose $\widehat{c}$ is involutive. Let $a \in [0, m]$. Then $c(c(\frac{a}{m})) = c(\frac{\widehat{c}(a)}{m}) = \frac{\widehat{c}(\widehat{c}(a))}{m} = \frac{a}{m}$. Hence c is involutive. □

Definition 2.3. *Let $\pi : [0, m] \times [0, m] \to [0, m]$. Then π is called a t-**norm** on $[0, m]$ if the following conditions hold for all $a, b, d \in [0, m]$:*

(1) *$\pi(a, m) = a$ (boundary condition).*

(2) *$b \le d$ implies $\pi(a, b) \le \pi(a, d)$ (monotonicity).*

(3) $\pi(a,b) = \pi(b,a)$ *(commutativity).*

(4) $\pi(a,\pi(b,d)) = \pi(\pi(a,b),d)$ *(associativity).*

Let i be a t-norm on $[0,1]$. Define $\pi : [0,m] \times [0,m] \to [0,m]$ by for all $a,b \in [0,m], \pi(a,b) = mi(\frac{a}{m},\frac{b}{m})$. Let $a \in [0,m]$. Then $\pi(a,m) = mi(\frac{a}{m},\frac{m}{m}) = m(\frac{a}{m}) = a$. Note that $\pi(a,b) = \varphi(i(\varphi^{-1}(a),\varphi^{-1}(b)))$,where $\varphi(x) = \frac{x}{m}$ and $x \in [0,m]$. Note also that $\varphi^{-1}(y) = my$, where $y \in [0,1]$. Check: $\varphi(i(\varphi^{-1}(y_1),\varphi^{-1}(y_2))) = \varphi(i(my_1,my_2)) = \frac{1}{m}i(my_1,my_2)$.

Let π be a t-norm on $[0,m]$. Define $i : [0,1] \times [0,1] \to [0,1]$ by for all $a,b \in [0,1], i(a,b) = \frac{1}{m}\pi(ma,mb)$. Let $a \in [0,1]$. Then $i(a,1) = \frac{1}{m}\pi(ma,m) = \frac{1}{m}ma = a$.

Suppose for example that i is the t-norm product on $[0,1]$. Let $a,b \in [0,m]$. Then $\pi(a,b) = mi(\frac{a}{m},\frac{b}{m}) = m\frac{a}{m}\frac{b}{m} = \frac{ab}{m}$.

Definition 2.4. *Let $\rho : [0,m] \times [0,m] \to [0,m]$. Then π is called a t-**conorm** on $[0,m]$ if the following conditions hold for all $a,b,d \in [0,m]$:*

(1) $\rho(a,0) = a$ *(boundary condition).*

(2) $b \leq d$ *implies* $\rho(a,b) \leq \rho(a,d)$ *(monotonicity).*

(3) $\rho(a,b) = \rho(b,a)$ *(commutativity).*

(4) $\rho(a,\rho(b,d)) = \rho(\rho(a,b),d)$ *(associativity).*

Let u be a t-conorm on $[0,1]$. Define $\rho : [0,m] \times [0,m] \to [0,m]$ by $a,b \in [0,m], \rho(a,b) = mu(\frac{a}{m},\frac{b}{m})$. Let $a \in [0,m]$. Then $\rho(a,0) = mu(\frac{a}{m},\frac{0}{m}) = m(\frac{a}{m}) = a$.

Let ρ be a t-conorm on $[0,m]$. Define $u : [0,1] \times [0,1] \to [0,1]$ by for all $a,b \in [0,1], u(a,b) = \frac{1}{m}\rho(ma,mb)$. Let $a \in [0,1]$. Then $u(a,0) = \frac{1}{m}\rho(ma,0) = \frac{1}{m}ma = a$.

Suppose for example that u is the t-conorm algebraic sum on $[0,1]$. Let $a,b \in [0,m]$. Then $\rho(a,b) = mu(\frac{a}{m},\frac{b}{m}) = m(\frac{a}{m}+\frac{b}{m}-\frac{a}{m}\frac{b}{m}) = a+b-\frac{ab}{m}$.

Recall that $\varphi : [0,m] \to [0,1]$, where for all $a \in [0,m], \varphi(a) = \frac{a}{m}$ is an isomorphism. Also if $e_1,e_2 \in E$, then $w(e_1) \leq w(e_2)$ if and only if $\mu(e_1) \leq \mu(e_2)$.

Define $w \circ w$ by for all $x,y \in V, (w \circ w)(x,y) = \vee\{w(xz) \wedge w(zy)|z \in V\}$. Let $w^2 = w \circ w$. Suppose n is a positive integer and that w^n has been defined. Define w^{n+1} to be $w^n \circ w$. Define w^∞ by $w^\infty(x,y) = \vee\{w^n(x,y)|\ n = 1,2,...\}$.

Define $\mu : E \to [0,1]$, by $e \in E, \mu(e) = \frac{1}{m}w(e)$. Then

$$\begin{aligned}(\mu \circ \mu)(x,y) &= \vee\{\mu(xz) \wedge \mu(zy)|z \in V\}\\ &= \vee\{\frac{1}{m}w(xz) \wedge \frac{1}{m}w(zy)|z \in V\}\\ &= \frac{1}{m} \vee \{w(xz) \wedge w(zy)|z \in V\}\\ &= \frac{1}{m}(w \circ w)(x,y).\end{aligned}$$

It follows by induction that $\mu^n(x,y) = \frac{1}{m}w^n(x,y)$ for all positive integers n. Thus $\mu^\infty(x,y) = \frac{1}{m}w^\infty(x,y)$.

Let $xy \in E$. Let $G' = (V,E\backslash\{xy\},w')$ be the weighted subgraph of $G = (V,E,w)$ obtained by deleting the edge xy from E and defining w' on $E\backslash\{xy\}$ by $w'(uv) = w(uv)$ for all $uv \in E\backslash\{xy\}$. Then xy is called a **bridge** in G if $\omega'^\infty(uv) < w^\infty(uv)$ for some $uv \in E\backslash\{xy\}$. Clearly, xy is a bridge in G if and only if xy is a bridge in the fuzzy graph (V,E,σ,μ), where $\mu(uv) = \frac{1}{m}w(uv)$ for all $uv \in E$.

It is now easy to see that the proof of the following result can be copied from the proof of Theorem 9.1, [[7], p. 90].

Theorem 2.5. *Let (V, E, w) be a weighted graph. Then the following statements are equivalent.*

(1) *xy is a bridge;*

(2) *$w'^{\infty}(xy) < w(xy)$;*

(3) *xy is not the weakest edge of any cycle.*

We next consider placing weights on the vertices of weighted graphs. Let (V, E, w) be a weighted graph. Let $m \geq \vee\{w(e)|e \in E\}$. Hold m fixed. Define $\tau : V \to [0, m]$ by for all $x \in V, \tau(x) = \vee\{w(xy)|y \in V\}$. Then for all $uv \in E, w(uv) \leq \tau(u) \wedge \tau(v)$. That is, (V, E, w, τ) is a weighted graph with a weight on the vertices. Define $\sigma : V \to [0, 1]$ by for all $x \in V, \sigma(x) = \frac{1}{m}\tau(x)$. Define $\mu : E \to [0, 1]$, by for all $xy \in E, \mu(xy) = \frac{1}{m}w(xy)$. Since $w(x) \leq \tau(x) \wedge \tau(y)$, it follows that $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$. That is, (V, E, σ, μ) is a fuzzy graph.

Let (V, E, σ, μ) be a fuzzy graph. Define $\rho : V \to [0, m]$, by $\forall v \in V, \rho(v) = m\sigma(v)$. Define w as before, i.e., $w(e) = m\mu(e)$. Then (V, E, ρ, ω) is a weighted graph with a weight on the vertices. Clearly, for all $x, y \in V, w(xy) \leq \rho(x) \wedge \rho(y)$.

3 Homomorphisms

In this section, we consider lattice homomorphisms of one lattice onto another. Our goal is to illustrate our results using the nonstandard interval $[0, 1]^*$ in nonstandard analysis. Consequently, we first review some basic properties of nonstandard analysis. We follow the approach in [1]. We do not provide a formal construction. A formal construction can be found in [5, 6].

Let F be a field and $<$ a relation on F. Suppose $<$ satisfies the following properties:

(1) $\forall x, y \in F$ such that $x \neq y$, either $x < y$ or $y < x$;

(2) $\forall x, y, z \in F, x < y$ and $y < z$ implies $x < z$;

(3) $\forall x, y, z \in F, x < y$ implies $x + z < y + z$;

(4) $\forall x, y, z \in F, x < y$ and $0 < z$ implies $xz < yz$.

Then $<$ is called an order on F and $(F, <)$ is called an **ordered field**.

Let $\mathbb{R}$ denote the field of real numbers. Let $\mathbb{R}^*$ denote a nonstandard universe, [6], with the following properties:

(1) $(\mathbb{R}, +, \bullet, 0, 1, <)$ is an ordered subfield of $(\mathbb{R}^*, +, \bullet, 0, 1, <)$;

(2) $\mathbb{R}^*$ has a positive infinitesimal element ε, that is $\varepsilon \in \mathbb{R}^*$ is such that $\varepsilon > 0$ and $\varepsilon < r$ for all positive real numbers r.

(3) For all $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a natural extension $f^* : (\mathbb{R}^*)^n \to \mathbb{R}^*$. The natural extensions of the field operations $+, \bullet : \mathbb{R}^2 \to \mathbb{R}$ coincide with the operations in $\mathbb{R}^*$. Similarly, for every $A \subseteq \mathbb{R}^n$, then is a subset $A^* \subseteq (\mathbb{R}^*)^n$ such that $A^* \cap \mathbb{R}^n = A$.

(4) $\mathbb{R}^*$ equipped with the above assignments of extensions of functions and subsets behaves logically like $\mathbb{R}$.

Definition 3.1. $\mathbb{R}^*$ *is called the* ***ordered field of hyperreals****.*

Now ε has an additive inverse $-\varepsilon$. Clearly, $-\varepsilon$ is a negative infinitesimal. For every positive real number $r, \varepsilon^{-1} > r$. Thus ε^{-1} is a positive infinite element and $-\varepsilon^{-1}$ is a negative infinite element.

Definition 3.2.
- *Let $\mathbb{R}_{fin} = \{x \in \mathbb{R}^* |\ |x| \leq n$ for some $n \in \mathbb{N}\}$. $\mathbb{R}_{fin}$ is called the set of* ***finite hyperreals****.*
- *Let $\mathbb{R}_{\text{inf}} = \mathbb{R}^* \backslash \mathbb{R}_{fin}$. $\mathbb{R}_{\text{inf}}$ is called the set of* ***infinite hyperreals****.*
- *Let $M = \{x \in \mathbb{R}^* |\ |x| \leq \frac{1}{n}$ for all $n \in \mathbb{N}, n > 0\}$. M is called the set of* ***infinitesimal hyperreals****.*

We see that $M \subseteq \mathbb{R}_{fin}, \mathbb{R} \subseteq \mathbb{R}_{fin}$, and $M \cap \mathbb{R} = \{0\}$. If $\delta \in M \backslash \{0\}$, then $\delta^{-1} \notin \mathbb{R}_{fin}$.

Proposition 3.3. *$\mathbb{R}_{fin}$ is a subring of $\mathbb{R}^*$ and M is and ideal of $\mathbb{R}_{fin}$.*

Definition 3.4. *Define the relation $\approx$ on R^* by for all $x, y \in R^*, x \approx y$ if and only if $x - y \in M$. If $x \approx y$, we say that x and y are* ***infinitely close****.*

It follows that not only is $\approx$ an equivalence relation on $\mathbb{R}^*$, but also a congruence relation.

Theorem 3.5. *(Existence of Standard Parts) Let $r \in \mathbb{R}_{fin}$. Then there exists a unique $s \in \mathbb{R}$ such that $r \approx s$. We call s the* ***standard part*** *of r and write $st(r) = s$.*

Corollary 3.6. *$\mathbb{R}_{fin} = \mathbb{R} + M$.*

Corollary 3.7. *Define $st : \mathbb{R}_{fin} \to \mathbb{R}$ by for all $r \in \mathbb{R}, st(r) = s$, where s is the standard part of r. Then st is a homomorphism of $\mathbb{R}_{fin}$ onto $\mathbb{R}$ such that $Ker(st) = M$.*

Let h be a homomorphism of a lattice $(L_1, \leq_1)$ onto a lattice $(L_2, \leq_2)$. If the lattice $(L_1, \leq_1)$ is bounded with bottom element 0_1 and top element 1_1, then $(L_2, \leq_2)$ is bounded, and the bottom and top elements of $(L_2, \leq_2)$ is $0_2 = h(0_1)$ and $1_2 = h(1_1)$, respectively.

Let $(L_1, \vee_1, \wedge_1, 0_1, 1_1)$ and $(L_2, \vee_2, \wedge_2, 0_2, 1_2)$ be lattices. (Define $<_i$ on L_i by for all $x, y \in L_i, x <_i y$ if and only if $x = x \wedge_i y$ and $x \neq y, i = 1, 2$).

We assume in the following that h is a homomorphism of L_1 onto L_2.

Define the relation $\approx$ on L_1 by for all $x, y \in L_1, x \approx y$ if and only if $h(x) = h(y)$. Then clearly $\approx$ is an equivalence relation on L_1. Now $\approx$ is also a congruence relation on L_1 since $h(x \vee_1 y) = h(x) \vee_2 h(y), h(x \wedge_1 y) = h(x) \wedge_2 h(y)$. (Suppose $x \approx y$ and $x' \approx y'$. Then $h(x) = h(y)$ and $h(x') = h(y')$. Thus $h(x \vee_1 x') = h(x) \vee_2 h(x') = h(y) \vee_2 h(y') = h(y \vee_1 y')$. Hence $x \vee_1 x' \approx y \vee_1 y'$. A similar result for $\wedge_1$and $\wedge_2$.

Let $x, y \in L_1$. Suppose $x <_1 y$. Then $x = x \wedge_1 y$ and $x \neq y$ and so $h(x) = h(x \wedge_1 y) = h(x) \wedge_2 h(y)$. It is not necessarily the case that $h(x) \neq h(y)$.

Now $x_1 \approx x_2$ and $y_1 \approx y_2 \Leftrightarrow h(x_1) = h(x_2)$ and $h(y_1) = h(y_2)$. Suppose $x_1 \approx x_2$ and $y_1 \approx y_2$. Then $h(x_1 \vee_1 y_1) = h(x_1) \vee_2 h(y_1) = h(x_2) \vee_2 h(y_2) = h(x_2 \vee_1 y_2)$. Thus $x_1 \vee_1 y_1 \approx x_2 \vee_1 y_2$. A similar result holds for $\wedge$. Hence $\approx$ is a congruence relation.

For all $x \in L_1$. Let $[x]$ denote the equivalence class of x with respect to $\approx$. Let $L_1^* = \{[x] | x \in L_1\}$. Define $\vee$ and $\wedge$ on L_1^* as follows: For all $[x], [y] \in L_1^*, [x] \vee [y] = [x \vee_1 y]$ and $[x] \wedge [y] = [x \wedge_1 y]$. Define $f : L_1 \to L_1^*$ by for all $x \in L_1, f(x) = [x]$. Then f is a function of L_1 onto L_1^* such that $f(x \vee_1 y) = [x \vee_1 y] = [x] \vee [y] = f(x) \vee f(y)$ and similarly for $\wedge$. Define $\leq$ on L_1^* by for all $[x], [y] \in L_1^*, [x] \leq [y]$ if and only if $h(x) \leq_2 h(y)$. Then $[x] = [y]$ if and only if $h(x) = h(y)$. Suppose $[x] < [y]$. Then $h(x) \leq h(y)$, but $h(x) \neq h(y)$, else $[x] = [y]$. We have L_1^* is a lattice and f is a homomorphism of L_1 onto L_1^*. Define $g : L_1^* \to L_2$ by for all $[x] \in L_1^*, g([x]) = h(x)$. Then $[x] = [y] \Leftrightarrow x \approx y \Leftrightarrow h(x) = h(y)$. Thus g is a one-to-one function of L_1^* into L_2. In fact, g maps L_1^* onto L_2 since h is onto L_2. Now $g \circ f(x) = g(f(x)) = g([x]) = h(x)$ for all $x \in L_1$. Thus $g \circ f = h$. Clearly, $[x] \leq [y]$ if and only if $g(x) \leq_2 g(y)$. Hence g is a lattice isomorphism of L_1^* onto L_2. Let L_1'be a sublattice of L_1 such that $h|_{L_1'}$ is a an isomorphism of L_1' onto L_2. Then for all $y \in L_1$, there exists unique $x \in L_1'$ such that $x \approx y$: Let $x \in L_1'$ be such that $h(x) = h(y)$. (x exists since h maps L_1' onto L_2). Then x is unique since h is one-to-one on L_1'

Assume $L_2 \subseteq L_1$. Let c_1 be a complement on L_1. Define $c_2 : L_2 \to L_2$ by $c_2(y) = c_1(y)$, where $y \in L_2$. Assume $c_1(y) \in L_2$. Then, assuming $0_1 = 0_2$ and $1_1 = 1_2$, $c_2(0_2) = c_1(0_2) = c_1(0_1) = 1_1 = 1_2$.

Let c_2 be a complement of L_2. Define $c_1 : L_1 \to L_1$ by for all $x \in L_1, c_1(x) = c_2(h(x))$. Then, assuming $1_1 = 1_2$, we have that

$$c_1(0_1) = c_2(h(0_1)) = c_2(0_2) = 1_2 = 1_1.$$

These assumptions are the case for st, i.e., $L_1 = [0,1]^*$ and $L_2 = [0,1]$. In fact, st is the identity map on L_1.

Definition 3.8. *Let c be a complement on L. We say that c is* ***infinitesimally involutive*** *if for all $x \in L, c(c(x) \approx x$.*

Proposition 3.9. *Let L_2 be a sublattice of L_1 and h a lattice homomorphism of L_1 onto L_2. Let c_2 be a complement on L_2. Define $c_1 : L_1 \to L_1$ by for all $x \in L_1, c_1(x) = c_2(h(x))$. The c_1 is a complement on L_1. If c_2 is involutive and h is the identity restricted to L_2, then c_1 is infinitesimally involutive.*

Proof. Let $x \in L_1$. Then since $c_2(h(x)) \in L_2$,

$$\begin{aligned} c_1(c_1(x)) &= c_1(c_2(h(x)) \\ &= c_2(h(c_2(h(x))) \\ &= c_2(c_2(h(x))) \\ &= h(x) \\ &\approx x. \end{aligned}$$

□

Example 3.10. Let $L_1 = [0,1]^*$ and $L_2 = [0,1]$. Let $h = st$. Let $c_2(x) = 1 - x$ for all $x \in L_2$. Let $x \in L_1$. Then there exists $a \in [0,1]$ and $m \in M$ such that $x = a + m$. Now $c_1(c_1(x)) = c_1(c_2(h(x))) = c_1(c_2(a)) = c_1(1-a) = c_2(1-a) = a \approx x$, where $c_1(x) = c_2(h(x))$.

Let i_1 be a t-norm on L_1. Define $i_2 : L_2 \times L_2 \to L_2$ by $\forall (x_2, y_2) \in L_2 \times L_2, i_2(x_2, y_2) = i_1(x_2, y_2)$. Assume $i_1(x_2, y_2) \in L_2$ and that $L_2 \subseteq L_1$. Then $i_2(y_2, 1_2) = i_1(y_2, 1_2) = i_1(y_2, 1_1) = y_2$. Assume $0_1 = 0_2$ and $1_1 = 1_2$. Now $i_2(0_2, y_2) = i_1(0_2, y_2) = i_1(0_1, y_2) = 0_1 = 0_2$.

Let i_2 be a t-norm on L_2. Define $i_1 : L_1 \times L_1 \to L_1$ by $\forall (x_1, y_1) \in L_1 \times L_1, i_1(x_1, y_1) = i_2(h(x_1), h(y_1))$. Then $i_1(x_1, 1_1) = i_2(h(x_1), h(1_1)) = i_2(h(x_1), 1_2) = h(x_1) \neq x_1$. Assume $h(h(x_1)) = h(x_1)$. Then $h(x_1) \approx x_1$. Now, $i_1(0_1, y_1) = i_2(h(0_1), h(y_1)) = i_2(0_2, h(y_1)) = 0_2$.

Definition 3.11. *$i : L \times L \to L$. We say that i satisfies the boundary conditions* ***infinitesimally*** *if $i(x, 1) \approx x$ and $i(0, x) \approx 0$. We say that i is an* ***infinitesimal t-norm*** *if it satisfies the definition of a t-norm except for the boundary conditions which it satisfies infintesimally.*

An infintesimal t-conorm is defined similarly.

Proposition 3.12. *Suppose that L_2 is a sublattice of L_1 and that h is a homomorphism L_1 onto L_2. Let i_2 be a t-norm on L_2. Define $i_1 : L_1 \times L_1 \to L_1$ by for all $(x, y) \in L_1 \times L_1, i_1(x, y) = i_2(h(x), h(y))$. If h is the identity on L_2 and h preserves $\leq$, then i_1 is an infinitesimal t-norm on L_1.*

Proof. Let $x, y \in L_1$. Then $i_1(x, y) = i_2(h(x), h(y)) = i_2(h(y), h(x)) = i_1(y, x)$. Let $x, y, z \in L_1$. Then

$$\begin{aligned} i_1(x, i_1(y, z)) &= i_2(h(x), h(i_1(y, z)) = i_2(h(x), h(i_2(h(y), h(z))) \\ &= i_2(h(x), i_2(h(y), h(z))) = i_2(i_2(h(x), h(y)), h(z)) \\ &= i_2(h(i_2(h(x), h(y)), h(z)) = i_1(i_2(h(x), h(y)), z) \\ &= i_1(i_1(x, y), z)) \end{aligned}$$

since $L_2 \subseteq L_1$ and h is the identity on L_2. Let $x \in L_1$. Then $i_1(x, 1_1) = i_2(h(x), h(1_1)) = i_2(h(x), 1_2) = h(x)$. Now $h(x) \in L_2$ and $h(h(x) = h(x)$. Hence $h(x) \approx x$. Thus $i_1(x, 1_1) \approx x$. Suppose $x \leq_1 y$. Then $h(x) \leq_2 h(y)$.Thus $i_1(z, x) = i_2(h(z), h(x)) \leq_2 i_2(h(z), h(y)) = i_1(z, y)$. □

Example 3.13. Let $L_1 = [0,1]^*, L_2 = [0,1]$, and $h = st$. Let $x, y \in L_1$. Then there exists, $a, b \in \mathbb{R}$ and $m, m' \in M$ such that $x = a + m$ and $b = b + m'$. Now $x \leq y$ if and only if $a + m \leq b + m'$ if and only if ($a = b$ and $m \leq m'$ or $a < b$). Hence $st(x) \leq st(y)$ if $x \leq y$.

Proposition 3.14. *Suppose that L_2 is a sublattice of L_1 and that h is a homomorphism L_1 onto L_2. Let i_1 be a t-norm L_1 such that i_1 maps $L_2 \times L_2$ into L_2. Define $i_2 : L_2 \times L_2 \to L_2$ by for all $x, y \in L_2, i_2(x, y) = i_1(x, y)$. If $1_1 = 1_2$,then i_2 is a t-norm on L_2.*

Proof. Let $x \in L_2$. Then $i_2(x, 1_2) = i_1(x, 1_2) = i_1(x, 1_1) = x$. Let $x, y, z \in L_2$. Suppose $x \leq_2 y$. Then $x \leq_1 y$. Thus $i_2(z, x) = i_1(z, x) \leq_1 i_1(z, y) = i_2(z, y)$. Hence $i_2(z, x) \leq_2 i_2(z, y)$. The commutative and associative properties hold for i_2 since they hold for i_1. □

Proposition 3.15. *Suppose that L_2 is a sublattice of L_1 and that h is a homomorphism L_1 onto L_2. Let u_2 be a t-conorm on L_2. Define $u_1 : L_1 \times L_1 \to L_1$ by for all $(x, y) \in L_1 \times L_1, u_1(x, y) = u_2(h(x), h(y))$. If h is the identity on L_2 and h preserves $\leq$, then u_1 is an infinitesimal t-conorm on L_1.*

Proposition 3.16. *Suppose that L_2 is a sublattice of L_1 and that h is a homomorphism L_1 onto L_2. Let u_1 be a t-conorm L_1 such that u_1 maps $L_2 \times L_2$ into L_2. Define $u_2 : L_2 \times L_2 \to L_2$ by for all $x, y \in L_2, u_2(x, y) = u_1(x, y)$. If $0_1 = 0_2$,then u_2 is a t-conorm on L_2.*

4 Nonstandard Fuzzy Graphs

We next consider a nonstandard weighted graph $G = (V, E, w)$, where $w : E \to [0, m]^*$ and where $[0, m]^* = \{x \in \mathbb{R}^* | 0 \leq x \leq m\}$. Define $g : [0, m]^* \to [0, 1]^*$, by for all $a \in [0, m]^*, g(a) = \frac{a}{m}$. Then g is a one-to-one function of $[0, m]^*$ onto $[0, 1]^*$. In fact, g is an isomorphism of $([0, m]^*, \vee, \wedge, 0, m)$ onto $([0, 1]^*, \vee, \wedge, 0, 1)$: $g(a \wedge b) = \frac{a \wedge b}{m} = \frac{a}{m} \wedge \frac{b}{m} = g(a) \wedge g(b)$ and $g(a \vee b) = \frac{a \wedge \vee b}{m} = \frac{a}{m} \vee \frac{b}{m} = g(a) \vee g(b)$. Also, g is continuous and preserves $<$.

Recall that for all $x \in [0, m]^*$, there exist $r \in \mathbb{R}$ and $y \in M$ such that $x = r + y$ and that r is called the standard part of x, written $st(x) = r$. As a matter of fact, $r \in [0, m]$. The isomorphism g also preserves $\approx$: Suppose $a, b \in [0, m]^*$ are such that $a \approx b$. Then $a - b \in M$ and so $a = b + x$ for some $x \in M$. Now $g(a) = \frac{a}{m} = \frac{b+x}{m} = g(b + x)$. Now $g(b) = \frac{b}{x}$ and $g(b + x) - g(b) = \frac{x}{m} \in M$. Thus $g(b + x) \approx g(b)$ and so $g(a) \approx g(b)$.

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