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Łukasiewicz Anti Fuzzy Set and Its Application in BE-algebras

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Abstract. The idea of Łukasiewicz t-conorm is used to construct the concept of Łukasiewicz anti fuzzy sets based on a given anti fuzzy set, and it is applied to BE-algebras. The notion of Łukasiewicz anti fuzzy BE-ideal is introduced, and its properties are investigated. The conditions under which Łukasiewicz anti fuzzy set will be Łukasiewicz anti fuzzy BE-ideal are explored, and the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal are discussed. Three types of subsets so called $\lessdot$-subset, Υ-subset, and anti subset are constructed, and the conditions under which they can be BE-ideals are explored.

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1 Introduction

BCK-algebra and BCI-algebra, introduced by Y. Imai, K. Iski and S. Tanaka in 1966, are algebraic structures of universal algebra which describe fragments of propositional calculus related to implications known as BCK and BCI-logic. After that, various generalizations were attempted, and BCC-algebras, BCH-algebras, BH-algebras, d-algebras etc. appeared. In 2007, H. S. Kim and Y. H. Kim [7] introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. Since then, the fuzzy set theory in BE-algebras has been studied (see [2, 5, 8]). S. S. Ahn and K. S. So [3] introduced the notion of ideals in BE-algebras, and S. Abdullah et al. [1] studied anti fuzzy ideals in BE-algebras. In mathematics, a triangular norm (briefly, t-norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. The Łukasiewicz t-norm is a nice example of t-norm. A t-conorm is dual to a t-norm under the order-reversing operation that assigns $1x$ to x on $[0, 1]$, and the Łukasiewicz t-conorm is dual to the Łukasiewicz t-norm. It is the standard semantics for strong disjunction in Łukasiewicz fuzzy logic.

In this paper, we establish the concept of the Łukasiewicz anti-fuzzy set using the idea of the Łukasiewicz t-conorm and anti-fuzzy set, and apply it to BE-algebra. We introduce the notion of Łukasiewicz anti fuzzy BE-ideal and investigate its properties. We explore the conditions under which Łukasiewicz anti fuzzy set will be Łukasiewicz anti fuzzy BE-ideal. We discuss the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal. We look for conditions under which the $\lessdot$-subset, Υ-subset, and anti subset can be BE-ideal.

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2 Preliminaries

This section lists the known default content that will be used later.

A *BE-algebra* (see [7]) is defined to be a set X together with a binary operation "$*$" and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,

(BE2) $(\forall a \in X)\ (a * 1 = 1)$,

(BE3) $(\forall a \in X)\ (1 * a = a)$,

(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

In the following, the BE-algebra is expressed as $(X, 1)_*$.

A relation "$\leq$" in $(X, 1)_*$ is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow \ a * b = 1). \tag{1}$$

In $(X, 1)_*$, the following conditions are valid.

$$(\forall a, b \in X)\left(a * (b * a) = 1\right). \tag{2}$$

$$(\forall a, b \in X)\left(a * ((a * b) * b) = 1\right). \tag{3}$$

A BE-algebra $(X, 1)_*$ is said to be *transitive* (see [3]) if it satisfies:

$$(\forall a, b, c \in X)\left(b * c \leq (a * b) * (a * c)\right). \tag{4}$$

A BE-algebra $(X, 1)_*$ is said to be *self-distributive* (see [7]) if it satisfies:

$$(\forall a, b, c \in X)\left(a * (b * c) = (a * b) * (a * c)\right). \tag{5}$$

Note that if a BE-algebra $(X, 1)_*$ is self-distributive, then it is transitive, but the converse is not valid (see [3]).

A subset K of X is called a *BE-ideal* of $(X, 1)_*$ (see [3]) if it satisfies:

$$(\forall a, b \in X)\left(b \in K \ \Rightarrow \ a * b \in K\right), \tag{6}$$

$$(\forall a, b, c \in X)\left(b, c \in K \ \Rightarrow \ (b * (c * a)) * a \in K\right). \tag{7}$$

Lemma 2.1 ([6])**.** *A subset K of X is a BE-ideal of $(X, 1)_*$ if and only if it satisfies:*

$$1 \in K, \tag{8}$$

$$(\forall x, y, z \in X)(x * (y * z) \in K,\ y \in K \ \Rightarrow \ x * z \in K). \tag{9}$$

Given two fuzzy sets f and g in a set X, their union $f \cup g$ and intersection $f \cap g$ are defined as follows:

$$f \cup g : X \to [0, 1],\ b \mapsto \max\{f(b), g(b)\},$$
$$f \cap g : X \to [0, 1],\ b \mapsto \min\{f(b), g(b)\}.$$

A fuzzy set g in X is called an *anti fuzzy BE-ideal* of $(X, 1)_*$ (see [1]) if it satisfies:

$$(\forall a, b \in X)\left(g(a * b) \leq g(b)\right), \tag{10}$$

$$(\forall a, b, c \in X)\left(g((b * (c * a)) * a) \leq \max\{g(b), g(c)\}\right). \tag{11}$$

3 Łukasiewicz anti fuzzy sets

A fuzzy set g in a set X of the form

$$g(b) := \begin{cases} s \in [0,1) & \text{if } b = a, \\ 1 & \text{if } b \neq a, \end{cases} \tag{12}$$

is called an *anti fuzzy point* with support a and value s, and is denoted by $\frac{a}{s}$. A fuzzy set g in a set X is said to be *non-unit* if there exists $a \in X$ such that $g(a) \neq 1$.

For a fuzzy set g in a set X, we say that an anti fuzzy point $\frac{a}{s}$ is said to

(i) *beside* in g, denoted by $\frac{a}{s} \lessdot g$, (see [4]) if $g(a) \leq s$.

(ii) *be non-quasi coincident* with g, denoted by $\frac{a}{s} \,\Upsilon\, g$, (see [4]) if $g(a) + s < 1$.

If $\frac{a}{s} \lessdot g$ or $\frac{a}{s} \,\Upsilon\, g$ (resp., $\frac{a}{s} \lessdot g$ and $\frac{a}{s} \,\Upsilon\, g$), we say that $\frac{a}{s} \lessdot\vee\Upsilon\, g$ (resp., $\frac{a}{s} \lessdot\wedge\Upsilon\, g$). Given $\beta \in \{\lessdot, \Upsilon\}$, to indicate $\frac{a}{s}\,\overline{\beta}\, g$ means that $\frac{a}{s}\,\beta\, g$ is not established.

Based on the Łukasiewicz t-conorm, we define Łukasiewicz anti fuzzy set.

Definition 3.1. Let ε be an element of the unit interval $[0,1]$ and let g be a fuzzy set in a set X. A function

$$Ł_g^\varepsilon : X \to [0,1],\ x \mapsto \min\{1, g(x) + \varepsilon\}$$

is called a *Łukasiewicz anti fuzzy set* of g in X.

Let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set of a fuzzy set g in X. If $\varepsilon = 0$, then $Ł_g^\varepsilon(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x)\} = g(x)$ for all $x \in X$. This shows that if $\varepsilon = 0$, then the Łukasiewicz anti fuzzy set of a fuzzy set g in X is the classisical fuzzy set g itself in X. If $\varepsilon = 1$, then $Ł_g^\varepsilon(x) = \min\{1, g(x)+\varepsilon\} = \min\{1, g(x)+1\} = 1$ for all $x \in X$, that is, if $\varepsilon = 1$, then the Łukasiewicz anti fuzzy set is the constant function with value 1. Therefore, in handling the Łukasiewicz anti fuzzy set, the value of ε can always be considered to be in $(0,1)$.

Let g be a fuzzy set in a set X and $\varepsilon \in (0,1)$. If $g(x) + \varepsilon \geq 1$ for all $x \in X$, then the Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ of g in X is the constant function with value 1, that is, $Ł_g^\varepsilon(x) = 1$ for all $x \in X$. Therefore, in order for the Łukasiewicz anti fuzzy set to have a meaningful shape, a fuzzy set g in X and $\varepsilon \in (0,1)$ shall be set to satisfy condition "$g(x) + \varepsilon < 1$ for some $x \in X$".

Proposition 3.2. *If g is a fuzzy set in a set X and $\varepsilon \in (0,1)$, then its Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ satisfies:*

$$(\forall x, y \in X)(g(x) \geq g(y) \;\Rightarrow\; Ł_g^\varepsilon(x) \geq Ł_g^\varepsilon(y)), \tag{13}$$

$$(\forall x \in X)\left(\tfrac{x}{\varepsilon}\,\Upsilon\, g \;\Rightarrow\; Ł_g^\varepsilon(x) = g(x) + \varepsilon\right). \tag{14}$$

$$(\forall x \in X)(\forall \varepsilon, \gamma \in (0,1))(\varepsilon \geq \gamma \;\Rightarrow\; Ł_g^\varepsilon(x) \geq Ł_g^\gamma(x)). \tag{15}$$

Proof. Straightforward. □

Proposition 3.3. *If f and g are fuzzy sets in a set X, then*

$$(\forall \varepsilon \in (0,1))\left(Ł_{f\cap g}^\varepsilon = Ł_f^\varepsilon \cap Ł_g^\varepsilon,\ Ł_{f\cup g}^\varepsilon = Ł_f^\varepsilon \cup Ł_g^\varepsilon\right). \tag{16}$$

Proof. For every $y \in X$, we have

$$\begin{aligned} Ł_{f\cap g}^\varepsilon(y) &= \min\{1, (f \cap g)(y) + \varepsilon\} = \min\{1, \min\{f(y), g(y)\} + \varepsilon\} \\ &= \min\{1, \min\{f(y) + \varepsilon, g(y) + \varepsilon\}\} \\ &= \min\{\min\{1, f(y) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\ &= \min\{Ł_f^\varepsilon(y), Ł_g^\varepsilon(y)\} = (Ł_f^\varepsilon \cap Ł_g^\varepsilon)(y), \end{aligned}$$

and

$$\begin{aligned}
Ł_{f\cup g}^{\varepsilon}(y) &= \min\{1, (f\cup g)(y)+\varepsilon\} = \min\{1, \max\{f(y), g(y)\}+\varepsilon\} \\
&= \min\{1, \max\{f(y)+\varepsilon, g(y)+\varepsilon\}\} \\
&= \max\{\min\{1, f(y)+\varepsilon\}, \min\{1, g(y)+\varepsilon\}\} \\
&= \max\{Ł_f^{\varepsilon}(y), Ł_g^{\varepsilon}(y)\} = (Ł_f^{\varepsilon}\cup Ł_g^{\varepsilon})(y).
\end{aligned}$$

Hence (16) is valid. □

Given a Łukasiewicz anti fuzzy set $Ł_g^{\varepsilon}$ of a fuzzy set g in X and $s\in[0,1)$, consider the sets:

$$(Ł_g^{\varepsilon}, s)_{\lessdot} := \{y\in X \mid \tfrac{y}{s}\lessdot Ł_g^{\varepsilon}\} \text{ and } (Ł_g^{\varepsilon}, s)_{\Upsilon} := \{y\in X \mid \tfrac{y}{s}\,\Upsilon\, Ł_g^{\varepsilon}\}$$

which are called the *$\lessdot$-subset* and *Υ-subset* of $Ł_g^{\varepsilon}$ in X. Also, we consider the following set

$$Anti(Ł_g^{\varepsilon}) := \{y\in X \mid Ł_g^{\varepsilon}(y) < 1\}$$

and it is called the *anti subset* of $Ł_g^{\varepsilon}$ in X. It is observed that

$$Anti(Ł_g^{\varepsilon}) = \{y\in X \mid g(y)+\varepsilon < 1\}.$$

It is clear that if $s<\varepsilon$, then $(Ł_g^{\varepsilon}, s)_{\lessdot} = \emptyset$, and if $s+\varepsilon\geq 1$, then $(Ł_g^{\varepsilon}, s)_q = \emptyset$.

Example 3.4. Consider a set $X := \{x\in\mathbb{N} \mid x\leq 10\}$ and define a fuzzy set g in X as follows:

$$g: X\to[0,1],\ x\mapsto\begin{cases} 0.5 & \text{if } x=5,\\ 0.3 & \text{if } x\in\{1,2\},\\ 0.6 & \text{if } x\in\{3,4\},\\ 0.8 & \text{if } x\in\{5,6,7\},\\ 0.1 & \text{if } x\in\{8,9\},\\ 1.0 & \text{if } x=10.\end{cases}$$

If we take $\varepsilon := 0.28$ and $s := 0.59$, then $(Ł_g^{\varepsilon}, s)_{\lessdot} = \{1,2,8,9\}$, $(Ł_g^{\varepsilon}, \Upsilon)_{\lessdot} = \{8,9\}$, and $Anti(Ł_g^{\varepsilon}) = \{1,2,3,4,8,9\}$.

4 Łukasiewicz anti fuzzy BE-ideals

In this section, let g and ε be a fuzzy set in X and an element of $(0,1)$, respectively, unless otherwise specified.

Definition 4.1. A Łukasiewicz anti fuzzy set $Ł_g^{\varepsilon}$ in X is called a *Łukasiewicz anti fuzzy BE-ideal* of $(X,1)_*$ if it satisfies:

$$(\forall x,y\in X)(\forall s\in[0,1))\left(\tfrac{y}{s}\lessdot Ł_g^{\varepsilon} \;\Rightarrow\; \tfrac{x*y}{s}\lessdot Ł_g^{\varepsilon}\right), \tag{17}$$

$$(\forall x,y,z\in X)(\forall s_a,s_b\in[0,1))\left(\tfrac{x}{s_a}\lessdot Ł_g^{\varepsilon},\ \tfrac{y}{s_b}\lessdot Ł_g^{\varepsilon} \;\Rightarrow\; \tfrac{(x*(y*z))*z}{\max\{s_a,s_b\}}\lessdot Ł_g^{\varepsilon}\right). \tag{18}$$

Example 4.2. Let $X=\{1,a,b,c,d,0\}$ and $*$ be given by the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X,1)_*$ is a BE-algebra (see [7]). Let g be a fuzzy set in X defined as follows:

$$g : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{1,a,b\}, \\ 0.86 & \text{if } x = c, \\ 0.67 & \text{if } x = d, \\ 0.79 & \text{if } x = 0. \end{cases}$$

Given $\varepsilon := 0.35$, the Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ of g in X is given as follows:

$$Ł_g^\varepsilon : X \to [0,1],\ y \mapsto \begin{cases} 0.78 & \text{if } y \in \{1,a,b\}, \\ 1.00 & \text{if } y \in \{c,d,0\}. \end{cases}$$

It is routine to verify that $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$.

Theorem 4.3. *A Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ if and only if it satisfies:*

$$(\forall x,y \in X)\left(Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y)\right). \tag{19}$$

$$(\forall x,y,z \in X)\left(Ł_g^\varepsilon((x*(y*z))*z) \le \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\}\right). \tag{20}$$

Proof. Assume that $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$. Let $x,y \in X$. Since $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$, we have $\frac{x*y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$ by (17), and so $Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y)$. Note that $\frac{x}{Ł_g^\varepsilon(x)} \lessdot Ł_g^\varepsilon$ and $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$ for all $x,y \in X$. It follows from (18) that $\frac{(x*(y*z))*z}{\max\{Ł_g^\varepsilon(x),Ł_g^\varepsilon(y)\}} \lessdot Ł_g^\varepsilon$, that is, $Ł_g^\varepsilon((x*(y*z))*z) \le \max\left\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\right\}$ for all $x,y,z \in X$.

Conversely, let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set satisfying (19) and (20). If $\frac{y}{s} \lessdot Ł_g^\varepsilon$ for all $y \in X$ and $s \in [0,1)$, then $Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y) \le s$ for all $x \in X$ by (19). Hence $\frac{x*y}{s} \lessdot Ł_g^\varepsilon$. Let $x,y,z \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$ and $\frac{y}{s_b} \lessdot Ł_g^\varepsilon$. Then $Ł_g^\varepsilon(x) \le s_a$ and $Ł_g^\varepsilon(y) \le s_b$. It follows from (20) that

$$Ł_g^\varepsilon((x*(y*z))*z) \le \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \le \max\{s_a, s_b\}.$$

Hence $\frac{(x*(y*z))*z}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon$, and therefore $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$. □

Proposition 4.4. *Every Łukasiewicz anti fuzzy BE-ideal $Ł_g^\varepsilon$ of $(X,1)_*$ satisfies:*

$$(\forall x \in X)(\forall s \in [0,1))\left(\tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{1}{s} \lessdot Ł_g^\varepsilon\right). \tag{21}$$

$$(\forall x,y \in X)(\forall s \in [0,1))\left(\tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{(x*y)*y}{s} \lessdot Ł_g^\varepsilon\right). \tag{22}$$

$$(\forall x,y \in X)(\forall s \in [0,1))\left(x \le y,\ \tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{y}{s} \lessdot Ł_g^\varepsilon\right). \tag{23}$$

$$(\forall x,y \in X)(\forall s_a,s_b \in [0,1))\left(\tfrac{x*y}{s_b} \lessdot Ł_g^\varepsilon,\ \tfrac{x}{s_a} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{y}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon\right). \tag{24}$$

$$(\forall x,y,z \in X)(\forall s_a,s_b \in [0,1))\left(\tfrac{x*(y*z)}{s_a} \lessdot Ł_g^\varepsilon,\ \tfrac{y}{s_b} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{x*z}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon\right). \tag{25}$$

Proof. The combination of (BE1) and (17) induces the condition (21). Let $x \in X$ and $s \in [0,1)$ be such that $\frac{x}{s} \lessdot Ł_g^\varepsilon$. Then $\frac{(x*y)*y}{s} = \frac{(x*(1*y))*y}{s} = \frac{(x*(1*y))*y}{\max\{s,s\}} \lessdot Ł_g^\varepsilon$ by (BE3), (18) and (21). The combination of (BE3), (1) and (22) induces (23). Let $x,y \in X$ and $s_a,s_b \in [0,1)$ be such that $\frac{x*y}{s_b} \lessdot Ł_g^\varepsilon$ and $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$. Then

$$\frac{y}{\max\{s_a,s_b\}} = \frac{1*y}{\max\{s_a,s_b\}} = \frac{((x*y)*(x*y))*y}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon$$

by (BE1), (BE3) and (18), which proves (24). The condition (25) is derived from the combination of (BE4) and (24). □

Lemma 4.5. *If a Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X satisfies (21) and (25), then it satisfies the conditions (22) and (23).*

Proof. Let $x, y \in X$ and $s \in [0, 1)$ be such that $x \leq y$ and $\frac{x}{s} \lessdot Ł_g^\varepsilon$. Then $x * y = 1$ and $\frac{1*(x*y)}{s} = \frac{1*1}{s} = \frac{1}{s} \lessdot Ł_g^\varepsilon$ by (BE1) and (21). It follows from (BE3) and (25) that $\frac{y}{s} = \frac{1*y}{s} \lessdot Ł_g^\varepsilon$. Hence (23) is valid. Since $x * ((x * y) * y) = (x * y) * (x * y) = 1$, i.e., $x \leq (x * y) * y$, for all $x, y \in X$, it follows from (23) that $\frac{(x*y)*y}{s} \lessdot Ł_g^\varepsilon$ which proves (22). □

Theorem 4.6. *Let $(X, 1)_*$ be a transitive BE-algebra. If a Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X satisfies conditions (21) and (25), then it is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$.*

Proof. Assume that $Ł_g^\varepsilon$ satisfies conditions (21) and (25). Since $(X, 1)_*$ is transitive, we have

$$(\forall x, y, z \in X)\, (((y * z) * z) * ((x * (y * z)) * (x * z)) = 1)\,. \tag{26}$$

Let $y \in X$ and $s \in [0, 1)$ be such that $\frac{y}{s} \lessdot Ł_g^\varepsilon$. Then $\frac{x*(y*y)}{s} = \frac{1}{s} \lessdot Ł_g^\varepsilon$ by (BE1), (BE2) and (21). It follows from (25) that $\frac{x*y}{s} \lessdot Ł_g^\varepsilon$. Let $x, y, z \in X$ and $s_a, s_b \in [0, 1)$ be such that $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$ and $\frac{y}{s_b} \lessdot Ł_g^\varepsilon$. Then $\frac{(y*z)*z}{s_b} \lessdot Ł_g^\varepsilon$ by Lemma 4.5, and so $\frac{(x*(y*z))*(x*z)}{s_b} \lessdot Ł_g^\varepsilon$ by the combination of Lemma 4.5 and (26). It follows from (25) that $\frac{(x*(y*z))*z}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon$. Therefore $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$. □

Since every self-distributive BE-algebra is transitive, we have the following corollary.

Corollary 4.7. *Let $(X, 1)_*$ be a self-distributive BE-algebra. Then every Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$ if and only if it satisfies conditions (21) and (25).*

Theorem 4.8. *If g is an anti fuzzy BE-ideal of $(X, 1)_*$, then $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$.*

Proof. Let $x, y, z \in X$. Then $Ł_g^\varepsilon(x * y) = \min\{1, g(x * y) + \varepsilon\} \leq \min\{1, g(y) + \varepsilon\} = Ł_g^\varepsilon(y)$ and

$$\begin{aligned} Ł_g^\varepsilon((x * (y * z)) * z) &= \min\{1, g((x * (y * z)) * z) + \varepsilon\} \\ &\leq \min\{1, \max\{g(x), g(y)\} + \varepsilon\} \\ &= \min\{1, \max\{g(x) + \varepsilon, g(y) + \varepsilon\}\} \\ &= \max\{\min\{1, g(x) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\ &= \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \end{aligned}$$

Hence $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$ by Theorem 4.3. □

In Example 4.2, $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$. But g is not an anti fuzzy BE-ideal of $(X, 1)_*$ since $g(b * 0) = g(c) = 0.86 \nleq 0.79 = g(0)$. Therefore, the converse of Theorem 4.8 may not be true. In the sense of Theorem 4.8, we can say that Łukasiewicz anti fuzzy BE-ideal is a generalization of anti fuzzy BE-ideal.

We explore the conditions under which $\lessdot$-subset and Υ-subset of the Łukasiewicz anti fuzzy set can be BE-ideal.

Theorem 4.9. *Let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set in X. Then $\lessdot$-subset $(Ł_g^\varepsilon, s)_\lessdot$ of $Ł_g^\varepsilon$ with value $s \in [0, 0.5)$ is a BE-ideal of $(X, 1)_*$ if and only if $Ł_g^\varepsilon$ satisfies:*

$$(\forall x \in X)\, \left(Ł_g^\varepsilon(x) \geq \min\{Ł_g^\varepsilon(1), 0.5\}\right), \tag{27}$$

$$(\forall x, y, z \in X)\, \left(\min\{Ł_g^\varepsilon(x * z), 0.5\} \leq \max\{Ł_g^\varepsilon(x * (y * z)), Ł_g^\varepsilon(y)\}\right). \tag{28}$$

Proof. Assume that $(\text{Ł}_g^\varepsilon, s)_\lessdot$ is a BE-ideal of $(X,1)_*$ for $s \in [0, 0.5)$. If $\text{Ł}_g^\varepsilon(a) < \min\{\text{Ł}_g^\varepsilon(1), 0.5\}$ for some $a \in X$, then $\text{Ł}_g^\varepsilon(a) \in [0, 0.5)$ and $\text{Ł}_g^\varepsilon(a) < \text{Ł}_g^\varepsilon(1)$. Hence $\frac{a}{\text{Ł}_g^\varepsilon(a)} \lessdot \text{Ł}_g^\varepsilon$, and so $a \in (\text{Ł}_g^\varepsilon, \text{Ł}_g^\varepsilon(a))_\lessdot$, but $1 \notin (\text{Ł}_g^\varepsilon, \text{Ł}_g^\varepsilon(a))_\lessdot$. This is a contradiction, and thus $\text{Ł}_g^\varepsilon(x) \geq \min\{\text{Ł}_g^\varepsilon(1), 0.5\}$ for all $x \in X$. If the condition (28) is not valid, then there exist $a, b, c \in X$ such that $\min\{\text{Ł}_g^\varepsilon(a * c), 0.5\} > \max\{\text{Ł}_g^\varepsilon(a * (b * c)), \text{Ł}_g^\varepsilon(b)\}$. If we take $s := \max\{\text{Ł}_g^\varepsilon(a * (b * c)), \text{Ł}_g^\varepsilon(b)\}$, then $s \in [0, 0.5)$ and $\frac{a*(b*c)}{s} \lessdot \text{Ł}_g^\varepsilon$ and $\frac{b}{s} \lessdot \text{Ł}_g^\varepsilon$, but $\frac{a*c}{s} \overline{\lessdot}\, \text{Ł}_g^\varepsilon$, that is, $a * (b * c) \in (\text{Ł}_g^\varepsilon, s)_\lessdot$ and $b \in (\text{Ł}_g^\varepsilon, s)_\lessdot$, but $a * c \notin (\text{Ł}_g^\varepsilon, s)_\lessdot$. This is a contradiction, and thus (28) is valid.

Conversely, suppose that Ł_g^ε satisfies (27) and (28), and let $s \in [0, 0.5)$. For every $x \in (\text{Ł}_g^\varepsilon, s)_\lessdot$, we have $\min\{\text{Ł}_g^\varepsilon(1), 0.5\} \leq \text{Ł}_g^\varepsilon(x) \leq s < 0.5$ by (27). Hence $1 \in (\text{Ł}_g^\varepsilon, s)_\lessdot$. Let $x, y, z \in X$ be such that $x * (y * z) \in (\text{Ł}_g^\varepsilon, s)_\lessdot$ and $y \in (\text{Ł}_g^\varepsilon, s)_\lessdot$. Then $\text{Ł}_g^\varepsilon(x * (y * z)) \leq s$ and $\text{Ł}_g^\varepsilon(y) \leq s$, which imply from (28) that

$$\min\{\text{Ł}_g^\varepsilon(x * z), 0.5\} \leq \max\{\text{Ł}_g^\varepsilon(x * (y * z)), \text{Ł}_g^\varepsilon(y)\} \leq s < 0.5.$$

Hence $\frac{x*z}{s} \lessdot \text{Ł}_g^\varepsilon$, that is, $x * z \in (\text{Ł}_g^\varepsilon, s)_\lessdot$. Therefore $(\text{Ł}_g^\varepsilon, s)_\lessdot$ is a BE-ideal of $(X,1)_*$ for $s \in [0, 0.5)$ by Lemma 2.1. □

Theorem 4.10. *The Υ-subset of the Łukasiewicz anti fuzzy BE-ideal is a BE-ideal.*

Proof. Let Ł_g^ε be a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ and let $s \in [0,1)$. If $1 \notin (\text{Ł}_g^\varepsilon, s)_\Upsilon$, then $\frac{1}{s} \overline{\Upsilon}\, \text{Ł}_g^\varepsilon$, i.e., $\text{Ł}_g^\varepsilon(1) + s \geq 1$. Since $\frac{x}{\text{Ł}_g^\varepsilon(x)} \lessdot \text{Ł}_g^\varepsilon$ for all $x \in X$, we get $\frac{1}{\text{Ł}_g^\varepsilon(x)} \lessdot \text{Ł}_g^\varepsilon$ for all $x \in X$ by (21). Hence $\text{Ł}_g^\varepsilon(1) \leq \text{Ł}_g^\varepsilon(x)$ for $x \in (\text{Ł}_g^\varepsilon, s)_\Upsilon$, and so $1 - s \leq \text{Ł}_g^\varepsilon(1) \leq \text{Ł}_g^\varepsilon(x)$. This shows that $\frac{x}{s} \overline{\Upsilon}\, \text{Ł}_g^\varepsilon$, that is, $x \notin (\text{Ł}_g^\varepsilon, s)_\Upsilon$, a contradiction. Thus $1 \in (\text{Ł}_g^\varepsilon, s)_\Upsilon$. Let $x, y, z \in X$ be such that $x * (y * z) \in (\text{Ł}_g^\varepsilon, s)_\Upsilon$ and $y \in (\text{Ł}_g^\varepsilon, s)_\Upsilon$. Then $\frac{x*(y*z)}{s} \Upsilon \text{Ł}_g^\varepsilon$ and $\frac{y}{s} \Upsilon \text{Ł}_g^\varepsilon$, that is, $\text{Ł}_g^\varepsilon(x * (y * z)) < 1 - s$ and $\text{Ł}_g^\varepsilon(y) < 1 - s$. It follows from (25) that $\text{Ł}_g^\varepsilon(x * z) \leq \max\left\{\text{Ł}_g^\varepsilon(x * (y * z)), \text{Ł}_g^\varepsilon(y)\right\} < 1 - s$. Hence $\frac{x*z}{s} \Upsilon \text{Ł}_g^\varepsilon$, and so $x * z \in (\text{Ł}_g^\varepsilon, s)_\Upsilon$. Therefore $(\text{Ł}_g^\varepsilon, s)_\Upsilon$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Corollary 4.11. *If g is an anti fuzzy BE-ideal of $(X,1)_*$, then the Υ-subset of Ł_g^ε is a BE-ideal of $(X,1)_*$.*

Theorem 4.12. *For the Łukasiewicz anti fuzzy set Ł_g^ε in X, if the Υ-subset of Ł_g^ε is a BE-ideal of $(X,1)_*$, then the following arguments are satisfied.*

$$1 \in (\text{Ł}_g^\varepsilon, s)_\lessdot, \tag{29}$$

$$\frac{x}{s_a} \Upsilon \text{Ł}_g^\varepsilon,\ \frac{y}{s_b} \Upsilon \text{Ł}_g^\varepsilon \ \Rightarrow\ (x * (y * z)) * z \in (\text{Ł}_g^\varepsilon, \min\{s_a, s_b\})_\lessdot \tag{30}$$

for all $x, y, z \in X$ and $s, s_a, s_b \in [0.5, 1)$.

Proof. Assume that the Υ-subset of Ł_g^ε is a BE-ideal of $(X,1)_*$. If $1 \notin (\text{Ł}_g^\varepsilon, s)_\lessdot$ for some $s \in [0.5, 1)$, then $\frac{1}{s} \overline{\lessdot}\, \text{Ł}_g^\varepsilon$. Hence $\text{Ł}_g^\varepsilon(1) > s \geq 1 - s$ since $s \in [0.5, 1)$, and so $\frac{1}{s} \overline{\Upsilon}\, \text{Ł}_g^\varepsilon$, i.e., $1 \notin (\text{Ł}_g^\varepsilon, s)_\Upsilon$. This is a conradiction, and thus $1 \in (\text{Ł}_g^\varepsilon, s)_\lessdot$. Let $x, y, z \in X$ and $s_a, s_b \in [0.5, 1)$ be such that $\frac{x}{s_a} \Upsilon \text{Ł}_g^\varepsilon$ and $\frac{y}{s_b} \Upsilon \text{Ł}_g^\varepsilon$. Then $x \in (\text{Ł}_g^\varepsilon, s_a)_\Upsilon \subseteq (\text{Ł}_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$ and $y \in (\text{Ł}_g^\varepsilon, s_b)_\Upsilon \subseteq (\text{Ł}_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$, from which $(x * (y * z)) * z \in (\text{Ł}_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$ is derived. Hence

$$\text{Ł}_g^\varepsilon((x * (y * z)) * z) < 1 - \min\{s_a, s_b\} \leq \min\{s_a, s_b\},$$

that is, $\frac{(x*(y*z))*z}{\min\{s_a,s_b\}} \lessdot \text{Ł}_g^\varepsilon$. Therefore $(x * (y * z)) * z \in (\text{Ł}_g^\varepsilon, \min\{s_a, s_b\})_\lessdot$. □

Theorem 4.13. *If g is an anti fuzzy BE-ideal of $(X,1)_*$, then the non-empty anti subset of Ł_g^ε is a BE-ideal of $(X,1)_*$.*

Proof. If g is an anti fuzzy BE-ideal of $(X,1)_*$, then $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ (see Theorem 4.8). It is clear that $1 \in Anti(Ł_g^\varepsilon)$. Let $x,y,z \in X$ be such that $x*(y*z) \in Anti(Ł_g^\varepsilon)$ and $y \in Anti(Ł_g^\varepsilon)$. Then $Ł_g^\varepsilon(x*(y*z)) < 1$ and $Ł_g^\varepsilon(y) < 1$. Since $\frac{x*(y*z)}{Ł_g^\varepsilon(x*(y*z))} \lessdot Ł_g^\varepsilon$ and $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$, we have $\frac{x*z}{\max\{Ł_g^\varepsilon(x*(y*z)),Ł_g^\varepsilon(y)\}} \lessdot Ł_g^\varepsilon$ by (25). It follows that

$$Ł_g^\varepsilon(x*z) \leq \max\left\{Ł_g^\varepsilon(x*(y*z)), Ł_g^\varepsilon(y)\right\} < 1.$$

Hence $x*z \in Anti(Ł_g^\varepsilon)$, and therefore $Anti(Ł_g^\varepsilon)$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Theorem 4.14. *If a Lukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X satisfies (21) and*

$$(\forall x,y,z \in X)(\forall s_a, s_b \in [0,1)) \left(\begin{array}{c} \frac{x*(y*z)}{s_a} \lessdot Ł_g^\varepsilon,\ \frac{y}{s_b} \lessdot Ł_g^\varepsilon \\ \Rightarrow \frac{x*z}{\min\{s_a,s_b\}} \Upsilon\, Ł_g^\varepsilon \end{array} \right). \tag{31}$$

then the non-empty anti subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$.

Proof. Let $Anti(Ł_g^\varepsilon)$ be a non-empty anti subset of $Ł_g^\varepsilon$. Then there exists $x \in Anti(Ł_g^\varepsilon)$, and so $s := Ł_g^\varepsilon(x) < 1$, i.e., $\frac{x}{s} \lessdot Ł_g^\varepsilon$ for $s < 1$. Hence $\frac{1}{s} \lessdot Ł_g^\varepsilon$ by (21), and thus $Ł_g^\varepsilon(1) \leq s < 1$. Thus $1 \in Anti(Ł_g^\varepsilon)$. Let $x,y,z \in X$ be such that $x*(y*z) \in Anti(Ł_g^\varepsilon)$ and $y \in Anti(Ł_g^\varepsilon)$. Then $g(x*(y*z)) + \varepsilon < 1$ and $g(y)+\varepsilon < 1$. Since $\frac{x*(y*z)}{Ł_g^\varepsilon(x*(y*z))} \lessdot Ł_g^\varepsilon$ and $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$, it follows from (31) that $\frac{x*z}{\min\{Ł_g^\varepsilon(x*(y*z)),\, Ł_g^\varepsilon(y)\}} \Upsilon\, Ł_g^\varepsilon$. If $x*z \notin Anti(Ł_g^\varepsilon)$, then $Ł_g^\varepsilon(x*z) = 1$, and so

$$\begin{aligned} & Ł_g^\varepsilon(x*z) + \min\left\{Ł_g^\varepsilon(x*(y*z)),\, Ł_g^\varepsilon(y)\right\} = 1 + \min\left\{Ł_g^\varepsilon(x*(y*z)),\, Ł_g^\varepsilon(y)\right\} \\ & = 1 + \min\left\{\min\{1, g(x*(y*z))+\varepsilon\},\, \min\{1, g(y)+\varepsilon\}\right\} \\ & = 1 + \min\left\{g(x*(y*z))+\varepsilon,\, g(y)+\varepsilon\right\} \\ & = 1 + \min\left\{g(x*(y*z)),\, g(y)\right\} + \varepsilon \\ & \geq 1 + \varepsilon > 1. \end{aligned}$$

Hence $\frac{x*z}{\min\{Ł_g^\varepsilon(x*(y*z)),\, Ł_g^\varepsilon(y)\}} \overline{\Upsilon}\, Ł_g^\varepsilon$, a contradiction. Thus $x*z \in Anti(Ł_g^\varepsilon)$, and therefore $Anti(Ł_g^\varepsilon)$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Theorem 4.15. *Let $Ł_g^\varepsilon$ be a Lukasiewicz anti fuzzy set in X that satisfies $\frac{1}{\varepsilon} \Upsilon\, g$ and the condition (30) for all $x,y,z \in X$ and $s_a, s_b \in [0,1)$. Then the anti subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$.*

Proof. Let $Anti(Ł_g^\varepsilon)$ be the anti subset of $Ł_g^\varepsilon$. If $\frac{1}{\varepsilon} \Upsilon\, g$, then $g(1)+\varepsilon < 1$ and so $Ł_g^\varepsilon(1) = \min\{1, g(1)+\varepsilon\} = g(1)+\varepsilon < 1$. Hence $1 \in Anti(Ł_g^\varepsilon)$. Let $x,y,z \in X$ be such that $x,y \in Anti(Ł_g^\varepsilon)$. Then $Ł_g^\varepsilon(x) < 1$ and $Ł_g^\varepsilon(y) < 1$, which imply that $\frac{x}{0} \Upsilon\, Ł_g^\varepsilon$ and $\frac{y}{0} \Upsilon\, Ł_g^\varepsilon$. It follows from (30) that $(x*(y*z))*z \in (Ł_g^\varepsilon, \min\{0,0\})_\lessdot = (Ł_g^\varepsilon, 0)_\lessdot$. Hence $Ł_g^\varepsilon((x*(y*z))*z) = 0 < 1$, and so $(x*(y*z))*z \in Anti(Ł_g^\varepsilon)$. Therefore $Anti(Ł_g^\varepsilon)$ is a BE-ideal of $(X,1)_*$. □

Theorem 4.16. *Let $Ł_g^\varepsilon$ be a Lukasiewicz anti fuzzy set in X that satisfies $\frac{1}{\varepsilon} \Upsilon\, g$ and*

$$(\forall x,y,z \in X)(\forall s_a, s_b \in [0,1)) \left(\begin{array}{l} \frac{x*(y*z)}{s_a} \Upsilon\, Ł_g^\varepsilon,\ \frac{y}{s_b} \Upsilon\, Ł_g^\varepsilon \\ \Rightarrow x*z \in (Ł_g^\varepsilon, \min\{s_a,s_b\})_\lessdot \end{array} \right). \tag{32}$$

Then the anti subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$.

Proof. Let $Anti(\text{Ł}_g^\varepsilon)$ be an anti subset of Ł_g^ε. Then $1 \in Anti(\text{Ł}_g^\varepsilon)$ in the proof of Theorem 4.15. Let $x, y, z \in X$ be such that $x * (y * z) \in Anti(\text{Ł}_g^\varepsilon)$ and $y \in Anti(\text{Ł}_g^\varepsilon)$. Then $\text{Ł}_g^\varepsilon(x * (y * z)) < 1$ and $\text{Ł}_g^\varepsilon(y) < 1$. Thus $\frac{x*(y*z)}{0} \Upsilon \text{Ł}_g^\varepsilon$ and $\frac{y}{0} \Upsilon \text{Ł}_g^\varepsilon$. Using (32) leads to $x * z \in (\text{Ł}_g^\varepsilon, \min\{0, 0\})_\lessdot = (\text{Ł}_g^\varepsilon, 0)_\lessdot$ Hence $\text{Ł}_g^\varepsilon(x * z) = 0 < 1$, and so $x * z \in Anti(\text{Ł}_g^\varepsilon)$. It follows from Lemma 2.1 that $Anti(\text{Ł}_g^\varepsilon)$ is a BE-ideal of $(X, 1)_*$. $\square$

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