Transactions on Fuzzy Sets and Systems

URL: https://tfss.journals.iau.ir/

Online ISSN: 2821-0131

Vol.1, No.2, (2022), 6-20

DOI: http://10.30495/tfss.2022.1958599.1029

Article Type: Original Research Article (Invited Paper)

Novel Characterizations of LM-Fuzzy Open Operators

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Abstract. In this paper, we present some characterizations of the LM-fuzzy interior operator, the LM-fuzzy closure operator, LM-fuzzy semiopen operator and LM-fuzzy preopen operator in an LM-fuzzy topological space. Based on them, we introcuced the notions of LM-fuzzy regularly open operators and LM-fuzzy regularly closed operators and show that these kinds of openness degrees are different from those defined by level L-topology.

AMS Subject Classification 2020: 54A40; 03E72

Keywords and Phrases: LM-fuzzy topology, LM-fuzzy closure operator, LM-fuzzy semiopen operator, LM-fuzzy preopen operator, LM-fuzzy regular open operator.

1 Introduction

As we all know, closure operator and interior operator are not only two important concepts in topology, but also have important applications in many other branches of mathematics. For example, it is a basic tool in functional analysis, algebra, lattice theory, matroid theory and convexity theory and so on. In [20], Shi generalized them to L-fuzzy topological spaces and called them L-fuzzy interior operators and L-fuzzy closure operators. L-fuzzy interior operators and L-fuzzy closure operators can be used to characterize L-fuzzy topology $\mathcal{T}$, but don't rely on the level L-topology $\mathcal{T}_{[r]}$.

The notions of semiopenness, preopenness and regular openness are very important in general topology [15]. They were extended to L-topological spaces by Azad, Singal and Prakash, respectively (see [1, 23]). The notions of semicontinuity and precontinuity were also extended to L-topological spaces by Azad and Nanda respectively (see [1, 17]). Moreover the notions of semiopenness and regular openness were extended to fuzzifying topological spaces by A.M. Zahran, F.M. Zeyada and A.K. Mousa respectively (see [25, 26]). Further in [12, 13, 14], S.J. Lee and E.P. Lee introduced the notions of fuzzy r-semiopen sets, fuzzy r-preopen sets and fuzzy r-regular open sets in $[0, 1]$-fuzzy topological space $(X, \mathcal{T})$ by means of the level $[0, 1]$-topology $\mathcal{T}_{[r]}$.

In 2011, Shi introduced the notions of LM-fuzzy semiopen operator and LM-fuzzy preopen operator in LM-fuzzy topological spaces by means of the idea of [20]. Further they were applied to many research fields by Ghareeb, Al-Omeri and Liang [3, 4, 5, 6, 7, 21].

In this paper, we shall present some characterizations of the LM-fuzzy interior operator, the LM-fuzzy closure operator, LM-fuzzy semiopen operator and LM-fuzzy preopen operator. We shall show that these kinds of openness degrees are different from those defined by level $[0, 1]$-topology.

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Received: 12 May 2022; Revised: 20 May 2022; Accepted: 21 May 2022; Published Online: 7 November 2022.

How to cite: F.-G. Shi, Novel characterizations of LM-fuzzy open operator, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 6-20.

2 Preliminaries

Throughout this paper, L and M denote completely distributive lattices with order-reversing involutions, X is a nonempty set. The set of all nonzero co-prime elements of L is denoted by $J(L)$. The set of all nonzero co-prime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in J(L)$).

We say that a is a wedge below b in M, denoted by $a \prec b$, if for every subset $D \subseteq M$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$ [2]. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M \mid a \prec b\}$ for each $b \in M$. $\{a \in M \mid a \prec b\}$ is called the greatest minimal family of b, denoted by $\beta(b)$. $\alpha(a) = \{b \in M \mid b' \prec a'\}$ is called the greatest maximal family of a.

In a completely distributive lattice M, α is an $\bigwedge$-$\bigcup$ map, β is a union-preserving map, and for each $a \in M$, $a = \bigvee \beta(a) = \bigwedge \alpha(a)$ (see [10, 27]).

For $A \in M^X$ and $a \in M$, we use the following symbols [18, 19].

$$A_{[a]} = \{x \in X \mid A(x) \not\geq a\}, \qquad A^{(a)} = \{x \in X \mid A(x) \not\leq a\},$$
$$A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}, \quad A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

Definition 2.1. [8, 9, 11, 22, 24] *A map $\mathcal{T} : L^X \to M$ is called an LM-fuzzy pretopology on X provided that it satisfies the following conditions:*

(LFT1) $\mathcal{T}(X) = \mathcal{T}(\emptyset) = \top_M$;

(LFT2) $\mathcal{T}\left(\bigvee_{i\in\Omega} A_i\right) \geqslant \bigwedge_{i\in\Omega} \mathcal{T}(A_i),\ \forall\{A_i \mid i \in \Omega\} \subseteq L^X$.

An LM-fuzzy pretopology $\mathcal{T}$ is called an LM-fuzzy topology if it satisfies the following condition again.

(LFT3) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V), \quad \forall U, V \in L^X$.

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an L-open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ is called the degree of closedness of U. The pair $(X, \mathcal{T})$ is called an LM-fuzzy topological space. When $L = M$, an LM-fuzzy topology is also called an L-fuzzy topology. When $L = M = [0,1]$, an LM-fuzzy topology is called a $[0,1]$-fuzzy topology. In particular, when $M = \{0,1\}$, an LM-fuzzy topology is called an L-topology and when $L = \{0,1\}$, an LM-fuzzy topology is called an M-fuzzifying topology.

A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be continuous with respect to LM-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if $\mathcal{T}_1(f_L^{\leftarrow}(U)) \geq \mathcal{T}_2(U)$ holds for all $U \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(U)(x) = U(f(x))$.

Analogous to Theorem 3.2 in [27], we have the following.

Theorem 2.2. [27] Let $\mathcal{T} : L^X \to M$ be a map. Then the following conditions are equivalent:

(T1) $\mathcal{T}$ is an LM-fuzzy topology on X.

(T2) $\forall a \in M$, $\mathcal{T}_{[a]}$ is an L-topology on X.

(T3) $\forall a \in M$, $\mathcal{T}^{[a]}$ is an L-topology on X.

Definition 2.3. [20, 22] An LM-fuzzy interior operator on X is a map $\mathrm{Int} : L^X \to M^{J(L^X)}$ satisfying the following conditions:

(FI1) $\mathrm{Int}(A)(x_\lambda) = \bigwedge\limits_{\mu \prec \lambda} \mathrm{Int}(A)(x_\mu), \quad \forall x_\lambda in J(L^X), \forall A \in L^X$;

(FI2) $\mathrm{Int}(X)(x_\lambda) = \top_M$ for any $x_\lambda \in J(L^X)$;

(FI3) $\mathrm{Int}(A)(x_\lambda) = \perp_M$ for any $x_\lambda \not\leq A$;

(FI4) $\mathrm{Int}(A \wedge B) = \mathrm{Int}(A) \wedge \mathrm{Int}(B)$;

(FI5) $\forall a \in M\backslash\{\top_M\}$, $(\mathrm{Int}(A))^{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))^{(a)}\right)\right)^{(a)}$.

Corollary 2.4. [20, 22] Let $\mathcal{T}$ be an LM-fuzzy topology on X and let $\mathrm{Int}^{\mathcal{T}}$ be the LM-fuzzy interior operator induced by $\mathcal{T}$. Then $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$,

$$\mathrm{Int}^{\mathcal{T}}(A)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq A} \mathcal{T}(V) \ \text{ and } \ \mathcal{T}(A) = \bigwedge_{x_\lambda \prec A} \mathrm{Int}^{\mathcal{T}}(A)(x_\lambda).$$

Definition 2.5. [20, 22] *An LM-fuzzy closure operator on X is a map* $\mathrm{Cl} : L^X \to M^{J(L^X)}$ *satisfying the following conditions:*

(FC1) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu), \forall x_\lambda \in J(L^X)$;

(FC2) $\mathrm{Cl}(\emptyset)(x_\lambda) = \perp_M$ for any $x_\lambda \in J(L^X)$;

(FC3) $\mathrm{Cl}(A)(x_\lambda) = \top_M$ for any $x_\lambda \leq A$;

(FC4) $\mathrm{Cl}(A \vee B) = \mathrm{Cl}(A) \vee \mathrm{Cl}(B)$;

(FC5) $\forall a \in M\backslash\{\perp_M\}$, $\left(\mathrm{Cl}\left(\bigvee(\mathrm{Cl}(A))_{[a]}\right)\right)_{[a]} \subseteq (\mathrm{Cl}(A))_{[a]}$.

Corollary 2.6. [20, 22] Let $\mathcal{T}$ be an LM-fuzzy topology on X and let $\mathrm{Cl}^{\mathcal{T}} : L^X \to M^{J(L^X)}$ be the LM-fuzzy closure operator induced by $\mathcal{T}$. Then $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$,

$$\mathrm{Cl}^{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \ \text{ and } \ \mathcal{T}(A) = \bigwedge_{x_\lambda \not\leq A'} \mathrm{Cl}(A')(x_\lambda)'.$$

3 The characterizations of LM-fuzzy interiors and closures

In this section, our aim is to present some characterizations of LM-fuzzy interiors and LM-fuzzy closures.

Theorem 3.1. *If a map* $\mathrm{Int} : L^X \to M^{J(L^X)}$ *satisfies the following* (FI1)–(FI4)*:*

(FI1) $\mathrm{Int}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Int}(A)(x_\mu), \ \ \forall x_\lambda \in J(L^X), \ \forall A \in L^X$;

(FI2) $\mathrm{Int}(X)(x_\lambda) = \top_M$ *for any* $x_\lambda \in J(L^X)$;

(FI3) $\mathrm{Int}(A)(x_\lambda) = \perp_M$ *for any* $x_\lambda \not\leq A$;

(FI4) $\mathrm{Int}(A \wedge B) = \mathrm{Int}(A) \wedge \mathrm{Int}(B)$,

then the following (FI5), (FI6) *and* (FI7) *are equivalent:*

(FI5) $\mathrm{Int}(A)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$;

(FI6) $\forall a \in M\backslash\{\top_M\}$, $(\mathrm{Int}(A))^{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))^{(a)}\right)\right)^{(a)}$;

(FI7) $\forall a \in M\backslash\{\bot_M\}$, $(\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$.

Proof. By means of Theorem 3.3 in [22] we know that (FI5) is equivalent to (FI6). Now we prove that (FI5) is equivalent to (FI7).

In order to prove (FI5) $\Rightarrow$ (FI7), suppose $x_\lambda \in (\mathrm{Int}(A))_{(a)}$. Then $a \prec \mathrm{Int}(A)(x_\lambda)$. By (FI5) we know that there exists $V \in L^X$ such that $x_\lambda \le V \le A$ and

$$a \prec \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \le \mathrm{Int}(V)(y_\mu) \le \mathrm{Int}(A)(y_\mu) \text{ for all } y_\mu \prec V\ .$$

This implies $y_\mu \in (\mathrm{Int}(V))_{(a)} \subseteq (\mathrm{Int}(A))_{(a)}$. Further we obtain $V \le \bigvee(\mathrm{Int}(V))_{(a)} \le \bigvee(\mathrm{Int}(A))_{(a)}$. Therefore it holds

$$a \prec \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \le \bigvee_{x_\lambda \le V \le \bigvee(\mathrm{Int}(A))_{(a)}} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) = \mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)(x_\lambda).$$

This shows $x_\lambda \in \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$. (FI7) is proved.

(FI7) $\Rightarrow$ (FI5). It is easy to check that $\mathrm{Int}\,(A)\,(x_\lambda) \ge \bigvee_{x_\lambda \le V \le A} \bigwedge_{y_\mu \prec V} \mathrm{Int}\,(V)\,(y_\mu)$ holds. We only need to show that $\mathrm{Int}\,(A)\,(x_\lambda) \le \bigvee_{x_\lambda \le V \le A} \bigwedge_{y_\mu \prec V} \mathrm{Int}\,(V)\,(y_\mu)$ is true.

Suppose that $a \prec \mathrm{Int}(A)(x_\lambda)$. Then by (FI7) we know $x_\lambda in (\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$. Let $V = \bigvee(\mathrm{Int}(A))_{(a)}$. Then $x_\lambda \le V \le A$ and $a \prec \mathrm{Int}(V)(x_\lambda)$. For all $y_\mu \prec V$, there exists $y_\gamma \in (\mathrm{Int}(A))_{(a)}$ such that $y_\mu \prec y_\gamma$. By (FI1) and (FI7) we know

$$y_\mu \in (\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)} = (\mathrm{Int}\,(V))_{(a)}\,, i.e., a \prec \mathrm{Int}(V)(y_\mu).$$

This implies $a \le \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$. Hence we have

$$a \quad \le \quad \bigvee_{x_\lambda \le V \le \bigvee(\mathrm{Int}(A))_{(a)}} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \quad \le \quad \bigvee_{x_\lambda \le V \le A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu).$$

This shows that $\mathrm{Int}(A)(x_\lambda) \le \bigvee_{x_\lambda \le V \le A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$ is true. The proof is completed. □

Theorem 3.2. *If a map* $\mathrm{Cl} : L^X \to M^{J(L^X)}$ *satisfies the following* (FC1)–(FC4)*:*

(FC1) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu), \forall x_\lambda \in J(L^X)$*;*

(FC2) $\mathrm{Cl}(\emptyset)(x_\lambda) = \bot_M$ *for any* $x_\lambda \in J(L^X)$*;*

(FC3) $\mathrm{Cl}(A)(x_\lambda) = \top_M$ *for any* $x_\lambda \le A$*;*

(FC4) $\mathrm{Cl}(A \vee B) = \mathrm{Cl}(A) \vee \mathrm{Cl}(B)$,

then the following (FC5), (FC6) *and* (FC7) *are equivalent:*

(FC5) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\le B \ge A} \bigvee_{y_\mu \not\le B} (\mathrm{Cl}(B))(y_\mu)$;

(FC6) $\forall a \in M\backslash\{\bot_M\}$, $\left(\mathrm{Cl}\left(\bigvee(\mathrm{Cl}(A))_{[a]}\right)\right)_{[a]} \subseteq (\mathrm{Cl}(A))_{[a]}$*;*

(FC7) $\forall a \in M\backslash\{\top_M\}$, $\left(\mathrm{Cl}\left(\bigvee(\mathrm{Cl}(A))^{[a]}\right)\right)^{[a]} \subseteq (\mathrm{Cl}(A))^{[a]}$.

Proof. By means of Theorem 3.1 in [22] we know that (FC5) is equivalent to (FC6). Now we prove that (FC5) is equivalent to (FC7).

(FC5) $\Rightarrow$ (FC7). Suppose that $x_\lambda \notin \mathrm{Cl}(A)^{[a]}$. Then by (FC5) we obtain the following fact.

$$a \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right) = \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) = \bigcup_{x_\lambda \not\leq B \geq A} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right).$$

Hence there exists $B \in L^X$ with $x_\lambda \not\leq B \geq A$ such that $a \in \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right)$, which implies $\forall y_\mu \not\leq B$, $a \in \alpha(\mathrm{Cl}(B)(y_\mu))$, i.e., $y_\mu \notin \mathrm{Cl}(B)^{[a]}$. Therefore it follows $\bigvee \mathrm{Cl}(B)^{[a]} \leq B$. Thus we have $x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(B)^{[a]} \geq \bigvee \mathrm{Cl}(A)^{[a]}$. Hence we obtain the following formula.

$$\begin{aligned} a &\in \bigcup_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[a]}} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) = \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[a]}} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \\ &= \alpha\left(\mathrm{Cl}\left(\bigvee \mathrm{Cl}(A)^{[a]}\right)(x_\lambda)\right). \end{aligned}$$

This implies $x_\lambda \notin \mathrm{Cl}\left(\bigvee \mathrm{Int}(A)^{[a]}\right)^{[a]}$. (FC7) is proved.

(FC7) $\Rightarrow$ (FC5). It is easy to check that $\mathrm{Cl}(A)(x_\lambda) \leq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$ holds. Now we prove $\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$.

Suppose that $a \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right)$. Then there exists $b \in L$ such that $a \in \alpha(b)$ and $b \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right)$. By (FC7) we know

$$x_\lambda \notin (\mathrm{Cl}(A))^{[b]} \supseteq \left(\mathrm{Cl}\left(\bigvee (\mathrm{Cl}(A))^{[b]}\right)\right)^{[b]}.$$

Let $D = \bigvee \mathrm{Cl}(A)^{[b]}$. Then $A \leq D$ and $b \in \alpha\left(\mathrm{Cl}\left(D\right)(x_\lambda)\right)$. In this case, we must have $x_\lambda \not\leq D$. In fact, if $x_\lambda \leq D$, then $x_\mu \prec x_\lambda \leq D$ for all $\mu \prec \lambda$, hence there exists $x_\gamma \in \mathrm{Cl}(A)^{[b]}$ such that $x_\gamma \geq x_\mu$. From $\mathrm{Cl}(A)(x_\gamma) \leq \mathrm{Cl}(A)(x_\mu)$ we know $b \notin \alpha(\mathrm{Cl}(A)(x_\mu))$ for all $\mu \prec \lambda$. This implies

$$b \notin \bigcup_{\mu \prec \lambda} \alpha\left(\mathrm{Cl}(A)(x_\mu)\right) = \alpha\left(\bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu)\right) = \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right),$$

which contradicts to $b \in \alpha\left(\mathrm{Cl}\left(D\right)(x_\lambda)\right)$. Therefore $x_\lambda \not\leq D \geq A$. For all $y_\mu \not\leq D$, by

$$(\mathrm{Cl}(A))^{[b]} \supseteq \left(\mathrm{Cl}\left(\bigvee (\mathrm{Cl}(A))^{[b]}\right)\right)^{[b]} = \left(\mathrm{Cl}\left(D\right)\right)^{[b]}$$

we know $y_\mu \not\leq \left(\mathrm{Cl}\left(D\right)\right)^{[b]}$, i.e., $b \in \alpha\left(\mathrm{Cl}\left(D\right)(y_\mu)\right)$. Further we have $b \geq \bigvee_{y_\mu \not\leq D} \mathrm{Cl}\left(D\right)(y_\mu)$. This shows

$$\begin{aligned} a \in \alpha\left(b\right) &\subseteq \alpha\left(\bigvee_{y_\mu \not\leq D} \mathrm{Cl}\left(D\right)(y_\mu)\right) \subseteq \bigcup_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[b]}} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \\ &= \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[b]}} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \subseteq \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq D} \mathrm{Cl}(B)(y_\mu)\right). \end{aligned}$$

Therefore it follows $\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$. The proof is completed. $\square$

Theorem 3.3. *Let $\mathcal{T}$ be an LM-fuzzy topology on X,* Int *be the LM-fuzzy interior operator in $(X,\mathcal{T})$ and* Cl *be the LM-fuzzy closure operator in $(X,\mathcal{T})$. Then for any $A \in L^X$ and for any $a \in M \setminus \{\perp_M\}$, it follows*

(1) $A \in \mathcal{T}_{[a]} \;\Leftrightarrow\; \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a \;\Leftrightarrow\; A \leq \bigvee (\mathrm{Int}(A))_{[a]}$.

(2) $A \in \mathcal{T}^*_{[a]} \;\Leftrightarrow\; \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a' \;\Leftrightarrow\; \bigvee (\mathrm{Cl}(A))^{(a')} \leq A$.

Proof. (1) From Corollary 2.4 we easily obtain

$$A \in \mathcal{T}_{[a]} \Leftrightarrow a \leq \mathcal{T}(A) \Leftrightarrow \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a.$$

Moreover it is obvious

$$\forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a \Rightarrow A \leq \bigvee \mathrm{Int}(A)_{[a]}.$$

Now we prove

$$A \leq \bigvee \mathrm{Int}(A)_{[a]} \Rightarrow \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a.$$

Suppose $x_\lambda \prec A$. By $A \leq \bigvee \mathrm{Int}(A)_{[a]}$, there exists $x_\mu \in \mathrm{Int}(A)_{[a]}$ such that $x_\lambda \prec x_\mu$. Hence

$$\mathrm{Int}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Int}(A)(x_\lambda) = \mathrm{Int}(A)(x_\mu) \geq a.$$

(2) From Corollary 2.6 we easily obtain

$$A \in \mathcal{T}^*_{[a]} \Leftrightarrow a \leq \mathcal{T}^*(A) \Leftrightarrow \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a'.$$

Moreover it is obvious

$$\bigvee \mathrm{Cl}(A)^{(a')} \leq A \Rightarrow \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a'.$$

Now we prove

$$\forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a' \Rightarrow \bigvee \mathrm{Cl}(A)^{(a')} \leq A.$$

Suppose that $x_\lambda \prec \bigvee \mathrm{Cl}(A)^{(a')}$. Then there exists $x_\mu \in \mathrm{Cl}(A)^{(a')}$ (that is, $\mathrm{Cl}(A)(x_\mu) \not\leq a'$) such that $x_\lambda \prec x_\mu$. Hence by

$$\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Cl}(A)(x_\lambda) = \mathrm{Cl}(A)(x_\mu) \not\leq a'$$

we obtain $\mathrm{Cl}(A)(x_\lambda) \not\leq a'$. This implies $x_\lambda \leq A$. $\bigvee \mathrm{Cl}(A)^{(a')} \leq A$ is proved. □

Theorem 3.4. *Let $\mathcal{T}$ be an LM-fuzzy topology on X,* Int *be the LM-fuzzy interior operator in $(X,\mathcal{T})$ and* Cl *be the LM-fuzzy closure operator in $(X,\mathcal{T})$. Then for any $A \in L^X$ and for any $a \in M \setminus \{\perp_M\}$, it follows*

(1) $A \in \mathcal{T}^{[a]} \;\Leftrightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]} \Leftrightarrow\; A \leq \bigvee \mathrm{Int}(A)^{[a]}$.

(2) $A \in \mathcal{T}^{*[a]} \;\Leftrightarrow\; \forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')} \Leftrightarrow \bigvee \mathrm{Cl}(A)_{(a')} \leq A$.

Proof. (1) From Corollary 2.4 we easily obtain

$$A \in \mathcal{T}^{[a]} \Leftrightarrow a \notin \alpha(\mathcal{T}(A)) \Leftrightarrow \forall x_\lambda \prec A,\ a \notin \alpha(\mathrm{Int}(A)(x_\lambda)) \Leftrightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]}.$$

Moreover it is obvious

$$\forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]} \Rightarrow A \leq \bigvee \mathrm{Int}(A)^{[a]}.$$

Now we prove

$$A \leq \bigvee \mathrm{Int}(A)^{[a]} \Rightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]}.$$

Suppose $x_\lambda \prec A$. By $A \leq \bigvee \mathrm{Int}(A)^{[a]}$, there exists $x_\mu \in \mathrm{Int}(A)^{[a]}$ such that $x_\lambda \prec x_\mu$. Hence by

$$\mathrm{Int}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Int}(A)(x_\lambda) = \mathrm{Int}(A)(x_\mu) \text{ and } a \notin \alpha(\mathrm{Int}(A)(x_\mu))$$

we know $a \notin \alpha(\mathrm{Int}(A)(x_\lambda))$, i.e., $x_\lambda \in \mathrm{Int}(A)^{[a]}$.

(2) From Corollary 2.6 we easily obtain

$$\begin{aligned} A \in \mathcal{T}^{*[a]} &\Leftrightarrow a \notin \alpha(\mathcal{T}^*(A)) \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ a \notin \alpha(\mathrm{Cl}(A)(x_\lambda)') \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ a' \notin \beta(\mathrm{Cl}(A)(x_\lambda)) \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')}. \end{aligned}$$

It is easy to check $\forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')} \Leftrightarrow \bigvee \mathrm{Cl}(A)_{(a')} \leq A$. □

4 The Characterizations of LM-fuzzy (semiclosed, preopen) preclosed operators

In 2011, Shi presented the notions of LM-fuzzy semiopen operator and LM-fuzzy preopen operator by means of LM-fuzzy topology $\mathcal{T}$. They were applied to many research fields by Ghareeb, Al-Omeri and Liang [3, 4, 5, 6, 7, 20]. Now we give their characterizations by means of LM-fuzzy interior operator and LM-fuzzy closure operator.

Definition 4.1. [6, 16] *Let $\mathcal{T}$ be an LM-fuzzy topology on X. For any $A \in L^X$, define two mappings $\mathcal{T}_s, \mathcal{T}_p : L^X \to M$ by*

$$\mathcal{T}_s(A) = \bigvee_{B \leq A} \left\{ \mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A} \bigwedge_{x_\lambda \not\leq D \geq B} \left(\mathcal{T}(D')\right)' \right\},$$

$$\mathcal{T}_p(A) = \bigwedge_{x_\lambda \prec A} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.$$

Then $\mathcal{T}_s$ is called the LM-fuzzy semiopen operator induced by $\mathcal{T}$, and $\mathcal{T}_p$ is called the LM-fuzzy preopen operator induced by $\mathcal{T}$. For all $A \in L^X$, define $\mathcal{T}_s^*(A) = \mathcal{T}_s(A')$ and $\mathcal{T}_p^*(A) = \mathcal{T}_p(A')$, then $\mathcal{T}_s^*$ and $\mathcal{T}_p^*$ are respectively called the LM-fuzzy semiclosed operator and the LM-fuzzy preclosed operator induced by $\mathcal{T}$.

The next theorem presents a characterization of the LM-fuzzy semiclosed operator.

Theorem 4.2. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A \in L^X$,*

$$\mathcal{T}_s^*(A) = \bigvee_{B \geq A} \left\{ \mathcal{T}^*(B) \wedge \bigwedge_{x_\mu \not\leq A} \left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)' \right\}. \quad (1)$$

Proof. On the one hand, we have

$$\begin{aligned}\mathcal{T}_s^*(A)=\mathcal{T}_s(A') &= \bigvee_{B\geq A}\left\{\mathcal{T}(B')\wedge\bigwedge_{x_\lambda\prec A'}\mathrm{Cl}^{\mathcal{T}}(B')(x_\lambda)\right\}\\ &\geq \bigvee_{B\geq A}\left\{\mathcal{T}(B')\wedge\bigwedge_{x_\lambda\leq A'}\mathrm{Cl}^{\mathcal{T}}(B')(x_\lambda)\right\}\\ &= \bigvee_{B\geq A}\left\{\mathcal{T}(B')\wedge\bigwedge_{x_\lambda\leq A'}\bigwedge_{\mu\not\leq\lambda'}\left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)'\right\}\\ &\geq \bigvee_{B\geq A}\left\{\mathcal{T}^*(B)\wedge\bigwedge_{x_\mu\not\leq A}\left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)'\right\}.\end{aligned}$$

On the other hand, we can prove

$$\begin{aligned}&\bigvee_{B\geq A}\left\{\mathcal{T}^*(B)\wedge\bigwedge_{x_\mu\not\leq A}\left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)'\right\}\\ = &\bigvee_{B\geq A}\left\{\mathcal{T}(B')\wedge\bigwedge_{A\not\leq(x_\lambda)'}\bigvee_{\mu\not\leq\lambda'}\mathrm{Cl}^{\mathcal{T}}(B')(x_\mu)\right\}\\ \geq &\bigvee_{B'\leq A'}\left\{\mathcal{T}(B')\wedge\bigwedge_{x_\mu\prec A'}\mathrm{Cl}^{\mathcal{T}}(B')(x_\mu)\right\}=\mathcal{T}_s(A')=\mathcal{T}_s^*(A).\end{aligned}$$

The proof of (1) is completed. □

The next theorem presents a characterization of the LM-fuzzy preclosed operator.

Theorem 4.3. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A\in L^X$,*

$$\mathcal{T}_p^*(A)=\bigwedge_{x_\lambda\not\leq A}\bigvee_{x_\lambda\not\leq D}\left\{\mathcal{T}^*(D)\wedge\bigwedge_{y_\gamma\not\leq D}\left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)'\right\}. \tag{2}$$

Proof. On the one hand, we have

$$\begin{aligned}\mathcal{T}_p^*(A)=\mathcal{T}_p(A') &= \bigwedge_{x_\lambda\prec A'}\bigvee_{x_\lambda\prec B}\left\{\mathcal{T}(B)\wedge\bigwedge_{y_\mu\prec B}\mathrm{Cl}^{\mathcal{T}}(A')(y_\mu)\right\}\\ &= \bigwedge_{x_\lambda\not\leq A}\bigvee_{x_\lambda\not\leq B'}\left\{\mathcal{T}(B)\wedge\bigwedge_{y_\mu\prec B}\mathrm{Cl}^{\mathcal{T}}(A')(y_\mu)\right\}\\ &= \bigwedge_{x_\lambda\not\leq A}\bigvee_{x_\lambda\not\leq B'}\left\{\mathcal{T}(B)\wedge\bigwedge_{y_\mu\prec B}\bigwedge_{\gamma\not\leq\mu'}\left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)'\right\}\\ &\geq \bigwedge_{x_\lambda\not\leq A}\bigvee_{x_\lambda\not\leq B'}\left\{\mathcal{T}(B)\wedge\bigwedge_{y_\gamma\not\leq B'}\left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)'\right\}\\ &= \bigwedge_{x_\lambda\not\leq A}\bigvee_{x_\lambda\not\leq D}\left\{\mathcal{T}^*(D)\wedge\bigwedge_{y_\gamma\not\leq D}\left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)'\right\}.\end{aligned}$$

On the other hand, we can prove

$$\begin{aligned}
& \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\leq D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
= & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{D' \not\leq (y_\gamma)'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{\exists y_\mu \prec D', y_\mu \not\leq (y_\gamma)'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{\exists y_\mu \prec D', y_\mu \not\leq (y_\gamma)'} \bigvee_{\nu \not\leq \gamma'} \mathrm{Cl}^{\mathcal{T}}(A')(y_\nu) \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{y_\mu \prec D'} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} = \mathcal{T}_s(A') = \mathcal{T}_s^*(A).
\end{aligned}$$

The proof of (2) is completed. □

The following is a characterization of LM-fuzzy preopen operator, which is simpler than Definition 4.1.

Theorem 4.4. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A \in L^X$,*

$$\mathcal{T}_p(A) = \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}. \tag{3}$$

Proof. First we prove

$$\mathcal{T}_p(A) \leq \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.$$

Suppose that there exists $a \in M$ such that $a \prec \mathcal{T}_p(A)$. Then by

$$\mathcal{T}_p(A) \quad = \quad \bigwedge_{x_\lambda \prec A} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} (\mathcal{T}(D'))' \right\}$$

we know that $\forall x_\lambda \prec A$, there exists $B_{x_\lambda} \in L^X$ such that $x_\lambda \prec B_{x_\lambda}$, $\mathcal{T}(B_{x_\lambda}) \geq a$ and $\forall y_\mu \prec B_{x_\lambda}$, $a \leq \bigwedge_{y_\mu \not\leq D \geq A} (\mathcal{T}(D'))'$. Let $B = \bigvee\{B_{x_\lambda} \mid x_\lambda \prec A\}$. Then $A \leq B$, $\mathcal{T}(B) \geq a$ and $\forall y_\mu \prec B$, there exists B_{x_λ} such that $\forall y_\mu \prec B_{x_\lambda}$. This implies

$$\bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\} \geq a.$$

Hence

$$\mathcal{T}_p(A) \leq \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.$$

The inverse of the above inequality is obvious. □

By means of LM-fuzzy interior operator and LM-fuzzy closure operator we can give the other characterizations of LM-fuzzy preopen operator and LM-fuzzy preclosed operator.

Corollary 4.5. *In an LM-fuzzy topological space $(X, \mathcal{T})$, it holds that for any $A \in L^X$,*

$$\mathcal{T}_p(A) = \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}(A)(y_\mu) \right\}, \tag{4}$$

$$\mathcal{T}_p^*(A) = \bigvee_{A \leq D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\leq D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\}. \tag{5}$$

Proof. (4) can be proved from Corollary 2.6 and Theorem 4.4. Based on Theorem 4.3 and analogous to the proof of Theorem 4.4 we can obtain (5) $\square$

5 LM-fuzzy regular open operators

In this section, we shall present the notions of LM-fuzzy regularly open operators and LM-fuzzy regularly closed operators in LM-fuzzy topological spaces.

Definition 5.1. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. For any $A \in L^X$, define a map $\mathcal{T}_r : L^X \to M$ by*

$$\mathcal{T}_r(A) = \mathcal{T}_s(A') \wedge \mathcal{T}(A) = \mathcal{T}_s^*(A) \wedge \mathcal{T}(A).$$

Then $\mathcal{T}_r$ is called the LM-fuzzy regularly open operator corresponding to $\mathcal{T}$, where $\mathcal{T}_r(A)$ can be regarded as the degree to which A is regular open and $\mathcal{T}_r^*(B) = \mathcal{T}_r(B')$ can be regarded as the degree to which B is regularly closed.

Theorem 5.2. *Let $\mathcal{T}_r$ be the LM-fuzzy regularly open operator in LM-fuzzy topological space $(X, \mathcal{T})$. Then*

(1) $\mathcal{T}_r(\emptyset) = \mathcal{T}_r(X) = \top_M$.

(2) $\mathcal{T}_r(A) \leq \mathcal{T}(A)$ *for any* $A \in L^X$.

(3) $\mathcal{T}_r(A \wedge B) \geq \mathcal{T}_r(A) \wedge \mathcal{T}_r(B)$ *for any* $A, B \in L^X$.

Proof. (1) and (2) are obvious. In order to prove (3), we first prove the following inequality.

$$\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right) \geq \bigwedge_{i\in\Omega} \mathcal{T}_s^*(A_i) \text{ for any subfamily } \{A_i \mid i \in \Omega\} \text{ of } L^X. \tag{6}$$

Let $a \in L$ and $a \prec \bigwedge_{i\in\Omega} \mathcal{T}_s^*(A_i)$. Then for any $i \in \Omega$, there exists $B_i \leq (A_i)'$ such that

$$a \prec \mathcal{T}(B_i) \ \text{ and } \ a \prec \bigwedge_{x_\lambda \prec (A_i)'} \bigwedge_{x_\lambda \not\leq D \geq B_i} (\mathcal{T}(D'))'.$$

Hence

$$a \leq \bigwedge_{i\in\Omega} \mathcal{T}(B_i) \leq \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right) \ \text{ and } \ a \leq \bigwedge_{i\in\Omega} \bigwedge_{x_\lambda \prec (A_i)'} \bigwedge_{x_\lambda \not\leq D \geq B_i} (\mathcal{T}(D'))'.$$

By

$$\left\{x_\lambda \mid x_\lambda \prec \left(\bigwedge_{i\in\Omega} A_i\right)'\right\} = \bigcup_{i\in\Omega} \left\{x_\lambda \mid x_\lambda \prec (A_i)'\right\},$$

we have

$$\begin{aligned}
\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right) &= \bigvee_{B\leq\left(\bigwedge_{i\in\Omega} A_i\right)'} \left\{\mathcal{T}(B)\wedge \bigwedge_{x_\lambda\prec\left(\bigwedge_{i\in\Omega} A_i\right)'} \bigwedge_{x_\lambda\not\leq D\geq B} (\mathcal{T}(D'))'\right\} \\
&\geq \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right)\wedge \bigwedge_{i\in\Omega}\bigwedge_{x_\lambda\prec(A_i)'}\bigwedge_{x_\lambda\not\leq D\geq \bigvee_{i\in\Omega} B_i} (\mathcal{T}(D'))' \\
&\geq \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right)\wedge \bigwedge_{i\in\Omega}\bigwedge_{x_\lambda\prec(A_i)'}\bigwedge_{x_\lambda\not\leq D\geq B_i} (\mathcal{T}(D'))' \\
&\geq a.
\end{aligned}$$

This shows $\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right) \geq \bigwedge_{i\in\Omega} \mathcal{T}_s^*(A_i)$.

Since $\mathcal{T}$ is an L-fuzzy topology, it follows that $\mathcal{T}(A\wedge B)\geq\mathcal{T}(A)\wedge\mathcal{T}(B)$. Hence by (6), we obtain

$$\begin{aligned}
\mathcal{T}_r(A\wedge B) &= \mathcal{T}(A\wedge B)\wedge\mathcal{T}_s^*(A\wedge B) \\
&\geq \mathcal{T}(A)\wedge\mathcal{T}(B)\wedge\mathcal{T}_s^*(A)\wedge\mathcal{T}_s^*(B) \\
&= \mathcal{T}_r(A)\wedge\mathcal{T}_r(B).
\end{aligned}$$

(3) is proved. □

Definition 5.3. *A map $f: X\to Y$ between LM-fuzzy topological spaces $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ is called LM-fuzzy almost continuous if $\mathcal{T}_r(U)\leq\mathcal{S}(f_L^{\leftarrow}(U))$ holds for any $U\in L^Y$.*

Obviously an LM-fuzzy continuous map is LM-fuzzy almost continuous. Moreover the following result is also obvious.

Corollary 5.4. *A map $f: X\to Y$ between LM-fuzzy topological spaces $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ is almost continuous if and only if $\mathcal{T}_r^*(U)\leq\mathcal{S}^*(f_L^{\leftarrow}(U))$ for any $U\in L^Y$.*

S.J. Lee and E.P. Lee presented the definitions of the fuzzy r-semiopen set, fuzzy r-preopen and fuzzy r-regularly open set, which rely on level [0,1]-topologies.

Definition 5.5. [12, 13, 14] *Let A be a [0,1]-fuzzy set of a [0,1]-fuzzy topological space $(X,\mathcal{T})$ and $r\in(0,1]$. Then A is said to be*

- ***(1)*** *fuzzy r-semiopen if there is a fuzzy r-open set B such that $B\leq A\leq \mathrm{Cl}(B,r)$.*
- ***(2)*** *fuzzy r-preopen if $A\leq \imath(\mathrm{Cl}(A,r),r)$.*
- ***(3)*** *fuzzy r-regularly open if $A=\imath(\mathrm{Cl}(A,r),r)$.*

Based on Definition 5.5 we can introduce the other definition of LM-fuzzy regular openness.

Definition 5.6. *Let $(X,\mathcal{T})$ be a $[0,1]$-fuzzy topological space. For any $A\in[0,1]^X$, define*

- ***(1)*** $\mathcal{ST}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-semiopen in* $\mathcal{T}_{[r]}\}$.
- ***(2)*** $\mathcal{PT}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-preopen in* $\mathcal{T}_{[r]}\}$.
- ***(3)*** $\mathcal{RT}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-regularopen in* $\mathcal{T}_{[r]}\}$.

In general, $\mathcal{ST} \neq \mathcal{T}_s$, $\mathcal{PT} \neq \mathcal{T}_p$, $\mathcal{RT} \neq \mathcal{T}_r$, these can be seen from the following example.

Example 5.7. Let $X = [0,1]$ and A_1, A_2, A_3 be fuzzy sets defined by

$$A_1(x) = x,\ A_2(x) = 1-x,\ A_3(x) = 0.5,\ \forall x \in [0,1].$$

Define $\mathcal{T} : [0,1]^X \to [0,1]$ by

$$\mathcal{T}(G) = \begin{cases} 1, & \text{if } G = \emptyset, X, \\ 0.8, & \text{if } G = A_1, A_1 \vee A_2, A_1 \wedge A_2, \\ 0.6, & \text{if } G = A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3, \\ 0.1, & \text{if } G = A_2 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $\mathcal{T}$ is a $[0,1]$-fuzzy topology and

$$\mathcal{T}_{[1]} = \{\emptyset, X\};\quad \mathcal{T}_{[0.8]} = \{\emptyset, X, A_1, A_1 \vee A_2, A_1 \wedge A_2\};$$
$$\mathcal{T}_{[0.6]} = \{\emptyset, X, A_1, A_1 \vee A_2, A_1 \wedge A_2, A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3\};$$
$$\mathcal{T}_{[0.1]} = \{\emptyset, X, A_1, A_2, A_1 \vee A_2, A_1 \wedge A_2, A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3\}.$$

It is easy to check that A_1 is a fuzzy 0.8-open set and 0.3-closed set. This implies $\mathcal{RT}(A_1) = 0.3$. By

$$\begin{aligned} \mathcal{T}_s(A_2) &= \bigvee_{B \leq A_2} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq B} (\mathcal{T}(D'))' \right) \\ &= \left(\mathcal{T}(A_1 \wedge A_2) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_2} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2 \wedge A_3) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2 \wedge A_3} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= (0.8 \wedge 0.2) \vee (0.6 \wedge 0.4) \vee (0.1 \wedge 1) = 0.4, \end{aligned}$$

and

$$\mathcal{T}_r(A_1) = \mathcal{T}_s((A_1)') \wedge \mathcal{T}(A_1) = \mathcal{T}_s(A_2) \wedge \mathcal{T}(A_1) = 0.4$$

we know $\mathcal{RT}(A_1) \neq \mathcal{T}_r(A_1)$.

It is easy to check that A_2 is a fuzzy 0.1-open set and 0.8-closed set. This implies $\mathcal{PT}(A_2) = 0.1$. By

$$\begin{aligned} \mathcal{T}_p(A_2) &= \bigvee_{B \geq A_2} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec B} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= \left(\mathcal{T}(A_1 \vee A_2) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2) \wedge \bigwedge_{x_\lambda \prec A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= (0.8 \wedge 0.2) \vee (0.1 \wedge 1) = 0.2. \end{aligned}$$

we know $\mathcal{PT}(A_2) \neq \mathcal{T}_p(A_2)$.

It is easy to check that $A_1 \vee A_3$ is a fuzzy 0.6-open set and 0.6-closed set. This implies $\mathcal{ST}(A_1 \vee A_3) = 0.6$. Hence by the following fact we know $\mathcal{ST}(A_1 \vee A_3) \neq \mathcal{T}_s(A_1 \vee A_3)$.

$$
\begin{aligned}
& \mathcal{T}_s(A_1 \vee A_3) \\
= & \bigvee_{B \leq A_1 \vee A_3} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq B} (\mathcal{T}(D'))' \right) \\
= & \left(\mathcal{T}(A_1 \wedge A_2) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_2} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_2 \wedge A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_2 \wedge A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1 \wedge A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1 \vee A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \vee A_3} (\mathcal{T}(D'))' \right) \\
= & (0.8 \wedge 0.4) \vee (0.6 \wedge 0.4) \vee (0.6 \wedge 0.4) \vee (0.8 \wedge 0.9) \vee (0.6 \wedge 0.4) \vee (0.6 \wedge 1) \\
= & 0.8.
\end{aligned}
$$

Acknowledgements: The project is supported by the National Natural Science Foundation of China (11871097).

Conflict of Interest: The author declares that there are no conflict of interest.

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