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Research Article

On Characterizing Efficient and Properly Efficient Solutions for Multi- Objective Programming Problems in a Complex Space

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Abstract

In this paper, a complex non- linear programming problem with the two parts (real and imaginary) is considered. The efficient and proper efficient solutions in terms of optimal solutions of related appropriate scalar optimization problems are characterized. Also, the Kuhn-Tuckers' conditions for efficiency and proper efficiency are derived. This paper is divided into two independently parts: The first provides the relationships between the optimal solutions of a complex single-objective optimization problem and solutions of two related real programming problems. The second part is concerned with the theory of a multi-objective optimization in complex space.

Keywords: Optimization; Complex multi- objective programming; Efficient solution; Kuhn-Tuckers' conditions; Optimal solution; Decision making

1. Introduction

Multi-objective optimization problems arise when more than one objective function is to be minimized over a given feasible region. Unlike the traditional mathematical programming with a single-objective function, an optimal solution in the sense of one that minimizes all the objective functions simultaneously does not necessarily exist in multi-objective optimization problems, and whence, we are troubled with conflicts among objectives in decision-making problems with multiple objectives. The review of the most important fundamentals in multiobjective optimization (MOO) is introduced by (Emmerich and Deutz, 2018; Gunantora and Ai, 2018) (2018). Jiang and Fan (2020) proposed a method to design thin- film stacks consisting of multiple material types. Roussel et al. (2021) investigate multiobjective Bayesian optimization to find the full pareto front of an accelerator programming problem efficiently.

In the last three decades, many authors have extended a numerous aspects of mathematical programming to complex space, which lead to the importance of this field in many applications as Resistive Network with Sinusoidal Sources, Impedance Matching of Circuits in the Sinusoidal Steady State and Jury- Lee Criterion of Absolute Stability (Abrams and Ben Israel, 1971 and Hanna and Simaan, 1985). Applications of complex programming may be found in Mathematics, engineering, and in many other areas. In earlier works in the field of complex programming problem, all the researchers have considered only the real part of the objective function of the problem as the objective function of the problem neglecting the imaginary part of the objective function, and the constraints of the problem have considered as a cone in the complex space $\mathbb{C}^n$. Abrams (1972) established sufficient conditions for optimal points of the real part of the objective function neglecting the imaginary part. Duca (1978) formulated the vectorial optimization problem in complex space and obtained some necessary and sufficient conditions for a point to be the efficient solution of a problem. Smart and Mond (1991) have shown that the necessary conditions for optimality in polyhedral- cone constrained nonlinear programming problems are sufficient under the assumption of particular form of invexity. Ferrero (1992) considered the finite dimensional spaces making use of separation arguments and the one- to- one correspondence between $\mathbb{C}^n$, and $\mathbb{R}^{2n}$. Malakooti (2010) developed complex method with interior search directions to solve linear and nonlinear programming problems. Youness and Elborolosy (2004) formulated the problem in complex space taking into account the two parts of complex objective function (real and imaginary together) and introduced an extension to necessary optimality conditions in complex programming. Zhang and Xia (2018) proposed two efficient complex- valued optimization methods for solving constrained nonlinear programming problems of real functions in complex variables. Khalifa et al. (2020) characterized the solution of complex nonlinear programming with interval- valued neutrosophic trapezoidal fuzzy parameters. A neuromas authors are studied complex multi- objective programming problem where they are established the optimality conditions, introduced several properties, constructed and provide the weak, strong and strictly converse duality theorems and introduced being a necessary conditions of local weak efficient solution for optimistic optimization problems (see, Huang and Ho, 2021; Huang and Tanaka, 2022; and Hsu and Huang, 2022; Lv et al., 2022).

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This paper aims to study the necessary and sufficient conditions for a point to be an efficient or a properly efficient solution of a multi-objective complex programming problem, when the ordering is taken with respect to a pointed closed convex cone.

The outlay of the paper is constructed as follows: In the next sections, some preliminaries needed in this paper is introduced. Section 3 formulates the complex multi-objective programming problem. Section 4 introduces some of theorem, which described and derived the necessary and sufficient conditions for the complex programming involving Section 5 introduces the Kuhn-Tucker' conditions for existing the efficient and proper efficient solutions. Finally, in section 6 some concluding remarks are reported.

2. Preliminaries

In this section, some definitions, and theorems needed in this paper are recalled.

Definition 1. A non- empty set U *in* $\mathbb{C}^m$ (complex $\times\, n$) is a convex if it is closed in a convex combination, i,e., $\theta U + (1-\theta)U \subseteq U$, for $0 \le \theta \le 1$.

Definition 2. (Berman, 1973). A non- empty U in $\mathbb{C}^m$ is a cone if it is closed in a non- negative scalar multiplication, i.e., $\theta U \subseteq U$ for $\theta \ge 1$.

Remark 1. Let the cone $\{(w_1, w_2) \in \mathbb{C}^{2m}: w_2 = \overline{w_1}\}$ is denoted by P.

Definition 3.(Berman, 1973)**.** A hyperplane H in $\mathbb{C}^m$ is a set of the form

$H = \{x \in \mathbb{C}^m : \mathrm{Re}\, v^H x = \alpha\}$, where v non- zero vector in $\mathbb{C}^m$ and $\alpha \in \Re$.

Definition 4. (Berman, 1973). A polyhedral cone U in $\mathbb{C}^m$ is a convex cone generated by many vectors finitely, i.e., a set of the form $U = A\Re_+^k = \{Ax: x \in \Re_+^k\}$, for some $k \in \mathbb{Z}$ and $A \in \mathbb{C}^{n\times k}$.

Definition 5. (Berman, 1973). The dual U° of a non-empty set $U \subset \mathbb{C}^m$ is defined as

$$U^\circ = \{y \in \mathbb{C}^m: x \in U \Rightarrow \mathrm{Re}\, y^H x \ge 0\}.$$

Proposition 1. Let $U \subset \mathbb{C}^m$ be a closed convex cone, then $\widetilde{U} = \{0\}$ if and only if U is a real linear space of $\mathbb{C}^m$,i.e., U is a closed under real linear combination.

Theorem 1. (Berman, 1973). Let U be a closed convex cone in $\mathbb{C}^m$**,**then U is a pointed cone if and only if U° is solid. In this case *int* U° (interior of U°) is given by

$$\mathrm{int}\, U^\circ = \{y \in U^\circ: 0 \ne x \in U \Rightarrow \mathrm{Re}\, y^H x > 0\}.$$

It is clear that, for a closed convex cones U, U is solid iff U° is pointed. In this case,

$$\mathrm{int}\, U = \{x \in U: 0 \ne y \in U^\circ \Rightarrow \mathrm{Re}\, y^H x > 0\}.$$

Definition 6. Let M be a closed convex cone in $\mathbb{C}^m$, the largest subspace which is contained in M is $M \cap (-M)$, while the smallest subspace contains M is $M - M$. A convex cone is pointed if $M \cap (-M) = \{0\}$.

Corollary 1. (Berman, 1973). Let M and U be closed convex cones in $\mathbb{C}^n$, then

$$cl(M^\circ + U^0) = (M \cap U)^\circ.$$

Theorem 2. (Mond and Craven, 1977). Let M be a closed convex cone in $\mathbb{C}^m$ and $a \in M$. Then $y \in (M(a))^\circ$ iff $y \in M^\circ$ and $Re\; y^H = 0$.

Theorem 3. (Mond and Greenblatt, 1975; Rockafellar, 1970).

1. A set M is a polyhedron cone in $\mathbb{C}^m$ if and only if it is the intersection of finite number of closed half spaces in $\mathbb{C}^m$, each containing the origin on its boundary, i.e., $M = \bigcap_{k=1}^{p} H_{u_k}$, where $H_{u_k} = \{z \in \mathbb{C}^m: Re\; z^H u_k \ge 0\}$, for some vectors $u_1, u_2, \dots, u_p \in \mathbb{C}^m$ and integer $p > 0$. This is equivalent to M is a polyhedron cone if there is a positive integer p and a matrix $B \in \mathbb{C}^{p\times m}$ such that
$M = \{z \in \mathbb{C}^m: \mathrm{Re}\, z \ge 0\}$ and in this case $M^\circ = \{w \in \mathbb{C}^m: \exists\, u \in \mathbb{C}_+^m, w = B^H u\}$,
2. A polyhedral cone is a closed convex cone,
3. M is a polyhedral cone if and only if M° is a polyhedral cone,
4. The sum of polyhedral cones is polyhedral, and
5. The Cartesian product of polyhedral cones is a polyhedral cone.

Definition 7. (Craven and Mond, 1973).The functions $f: P \to \mathbb{C}$ and $g: P \to \mathbb{C}^n$ are said to be differentiable at the point $(z^*, \overline{z}^*) \in P$, if for every $(z, \overline{z}) \in P$,

$$f(z,\overline{z}) - f(z^*,\overline{z}^*) = \nabla_z f(z^*,\overline{z}^*)(z - z^*) + \nabla_{\overline{z}} f(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*) + o(\|z - z^*\|),$$ and

$$g(z,\overline{z}) - g(z^*,\overline{z}^*) = \nabla_z g(z^*,\overline{z}^*)(z - z^*) + \nabla_{\overline{z}} g(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*) + o(\|z - z^*\|),$$ where $\nabla_z f(z^*,\overline{z}^*)$ and $\nabla_{\overline{z}} f(z^*,\overline{z}^*)$ are the lower vectors of partial derivatives $\frac{\partial f(z,\overline{z}^*)}{\partial w_1^i}$ and $\frac{\partial f(z,\overline{z}^*)}{\partial w_2^i}$ respectively, While $D_z g(z^*,\overline{z}^*)$ and $D_{\overline{z}} g(z^*,\overline{z}^*)$ are the matrices $m \times n$ whose elements are $\frac{\partial g(z,\overline{z}^*)}{\partial w_1^i}$ and $\frac{\partial g(z,\overline{z}^*)}{\partial w_2^i}$ respectively, and $\frac{o(\|z-z^*\|)}{\|z-z^*\|} \to 0 as\;\; z \to z^*$.

Defintion 8. (Craven and Mond, 1973). Let $f: \mathbb{C}^n \to \mathbb{C}$ and $E \subset \mathbb{C}^n$, then $f(z)$ is called analytic in E, if in some neighborhood of every point E, it may be represented by an absolutely convergent power series about that point in the n complex variable.

Theorem 4. (Craven and Mond, 1973). Let M be a polyhedron cone in $\mathbb{C}^m, f: \mathbb{C}^{2n} \to \mathbb{C}, and\ g: \mathbb{C}^{2n} \to \mathbb{C}^m$ be two analytic functions in a neighborhood of a qualified point $(z^*, \overline{z}^*)$. Then a necessary condition for $(z^*, \overline{z}^*)$ to be a local minimum of problem(2) is that there is a vector $u \in (M(z^*))^\circ \subset M^\circ$ such that

$$\nabla_z f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} f}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^T \overline{D_{\overline{z}} g}(z^*, \overline{z}^*) = 0, \quad \text{and}$$

$$Re\ u^H g(z^*, \overline{z}^*) = 0.$$

3. Complex programming Problem statement

Consider the following multiobjective complex programming problem

$$\min f(z, \overline{z}) = (f_1(z, \overline{z}), f_2(z, \overline{z}), \ldots, f_p(z, \overline{z}))$$
$$\text{Subject to} \qquad (1)$$
$$(z, \overline{z}) \in G = \{(z, \overline{z}) \in Q: g(z, \overline{z}) \in M\}.$$

Where, $f_i: \mathbb{C}^{2n} \to \mathbb{C}, i = 1, 2, \ldots, p, p \geq 2$, as well as $g: \mathbb{C}^{2n} \to \mathbb{C}^m$ are differentiable functions, and M is a closed convex cone on $\mathbb{C}^m$.

The concept of the efficient solution, for the above complex problem, with respect to a complex domination structure of the decision maker is introduced, and a characterization of those solutions is given. Also, Kuhn-Tucker conditions for efficiency and proper efficiency of problem (1) are derived.

4. Efficient Solutions Concept in Complex Space

As we have mentioned, in a single-objective optimization problem, the meaning of optimality is clear. Whereas, in contrast, an optimal solution that minimizes all the objective functions simultaneously, can rarely be expected to exist in multi-objective optimization problems, since the objectives usually conflict with one another. Instead of optimality, the notion of efficiency.

Definition 9. A feasible point $(z^*, \overline{z}^*)$ is said to be an efficient solution of problem (1) with respect to a pointed closed convex cone (domination structure) $U \subset \mathbb{C}^p$ if there is no other feasible $(z, \overline{z})$ such that

$$f(z^*, \overline{z}^*) - f(z, \overline{z}) \in U \backslash \{0\}.$$

In other words, $(z^*, \overline{z}^*)$ is an efficient solution if and only if

$$(f(G) - f(z^*, \overline{z}^*)) \cap (-U) = \{0\}.$$

In addition, the efficient set of a non- empty set $Q \subset \mathbb{C}^p$ with respect to a cone $E \subset \mathbb{C}^p$ is given as in the following definition.

Definition 10. Let $E \subset \mathbb{C}^p$ be a cone, then the set of efficient elements of a set $Q \subset \mathbb{C}^p$ with respect to E is given by

$$D(Q, E) = \{\check{q} \in Q: \text{thers is no } q \neq \check{q} \in Q \text{ such that } \check{q} \in q + E\}.$$

From this definition, it follows that a point $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to U if and only if $(z^*, \overline{z}^*) \in D(f(G), U)$.

Proposition 2. Let $E \subset \mathbb{C}^p$ be a non- empty pointed convex cone, then

$$D(Q, E) = E(Q + E, E).$$

Proof (Smart and Mond, 1991).

4.1. Characterization of efficient solutions

In this subsection, we characterize the efficient solutions in terms of optimal solutions of related appropriate scalar optimization problems. A method of characterizing efficient solutions is via secularization by vectors in the polar cone U° of a domination structure cone U.

Consider the following scalar optimization problem

$$\min Re\ \zeta^H f(z, \overline{z})$$
$$\text{Subject to} \qquad (2)$$
$$(z, \overline{z}) \in G, \text{ for some } \zeta \in U^\circ.$$

Now, let us introduce the state with proof for the two theorems in order to study the relationships between the efficient solutions of the complex multi- objective problem (1) with respect to U and the optimal solutions of the related scalar optimization problem (2).

Theorem 5. Let $U \subset \mathbb{C}^p$ be a pointed closed convex cone and $f(G) + U$ be a convec set. If $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to U, then there is a $0 \neq \zeta \in U^\circ$ such that $(z^*, \overline{z}^*)$ is an optimal solution of problem (2).

Proof. Assume that $(z^*, \overline{z}^*)$ be an efficient solution of problem (1) with respect to U, i.e., $f(z^*, \overline{z}^*) \in D(f(G), U)$, it follows that

$f(z^*, \overline{z}^*) \in D(f(G) + U, U)$ (**Proposition 2**). So, $(f(G) + U - f(z^*, \overline{z}^*)) \cap (-U) = \{0\}$. Since from the convexity of the two sets $f(G) + U$ and U, there is a hyperplane separating between them, i.e., there is a non- zero vector $\zeta \in \mathbb{C}^p$ such that:

$$Re\ \zeta^H (f(z, \overline{z}) + s - f(z^*, \overline{z}^*)) \geq 0; \ \forall\ (z, \overline{z}) \in G, s \in U,$$
and (3)

$$Re\ \zeta^H(-s') \leq 0; \ \forall s' \in U. \qquad (4)$$

From (4), it follows that $\zeta \in U^\circ$, and by setting $s = 0$ in inequality (3), we have

$$Re\ \zeta^H f(z, \overline{z}) \geq Re\ \zeta^H f(z^*, \overline{z}^*); \ \forall\ (z, \overline{z}) \in G.$$

We conclude that, $(z^*, \overline{z}^*)$ is an optimal solution of problem (2).

Theorem 6. A point $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to a pointed closed convex cone $U \subset \mathbb{C}^p$ if there is a $\zeta \in U^\circ$ such that $(z^*, \overline{z}^*)$ is an optimal solution of problem (2) and one of the following two conditions holds:

1. $\zeta \in int\, U^{\circ}$, or
2. $(z^*, \overline{z}^*)$ is a unique optimal solution of problem (2).

Proof. Since $(z^*, \overline{z}^*)$ solves problem (2) for some $0 \neq \zeta \in U^{\circ}$, then

$$Re\, \zeta^H \left(f(z,\overline{z}) - f(z^*, \overline{z}^*)\right) \geq 0;\ \forall\, (z,\overline{z}) \in G. \quad (5)$$

Suppose that $(z^*, \overline{z}^*)$ is not an efficient solution for problem (1) with respect to U, then there is $(\check{z}, \check{\overline{z}}) \in G$ such that:

$$f(z^*, \overline{z}^*) - f(\check{z}, \check{\overline{z}}) \in U \backslash \{0\}.$$

If the **condition 1** satisfied, then

$\mathrm{Re}\, \zeta^H \left(f(z,\overline{z}) - f(z^*, \overline{z}^*)\right) \geq 0$, which contradicts (5).
If the condition2 holds, then inequality (5) becomes
$Re\, \zeta^H \left(f(z,\overline{z}) - f(z^*, \overline{z}^*)\right) \geq 0;\ \forall\, (z,\overline{z}) \in G.$ While $(z^*, \overline{z}^*)$ is not efficient, which implies to the existence of $(\check{z}, \check{\overline{z}}) \in G$ such that:

$$Re\, \zeta^H \left(f(z^*, \overline{z}^*) - f(\check{z}, \check{\overline{z}})\right) \geq 0$$, contradiction.

4. 2. Kuhn- Tucker's conditions for efficiency

As we observe, the objective function of problem (2) is the real part of the objective function of problem (1) and the optimality conditions of them have discussed by many authors. Therefore, necessary and sufficient conditions for efficiency due to the Kuhn- Tucker are as in the same analogous to those for optimality of a single objective real objective functions.

Theorem 9. Assume that M is a polyhedron cone in $\mathbb{C}^n$; $f: \mathbb{C}^{2n} \to \mathbb{C}$, $g: \mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions at a feasible point $(z^*, \overline{z}^*)$ at which the Kuhn- Tucker constraint qualification holds, and $f(G) + U$ is convex set. Then, a necessary condition for $(z^*, \overline{z}^*)$ to be efficient solution for problem (1) with respect to a pointed closed convex cone $U \subset \mathbb{C}^p$ is there is $0 \neq \zeta \in U^{\circ}$ and $u \in \left(M(z^*, \overline{z}^*)\right)^{\circ} \subset M^{\circ}$ such that

$$\left(\zeta^H D_z f(z^*, \overline{z}^*) + \zeta^H \overline{D_{\overline{z}} f}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^H \overline{D_{\overline{z}} g}(z^*, \overline{z}^*)\right) \geq 0, \text{ and} \quad (6)$$

$$\mathrm{Re}\, u^H g(z^*, \overline{z}^*) = 0. \quad (7)$$

Proof. Since $(z^*, \overline{z}^*)$ is an efficient solution for problem (1) with respect to U, then from **Theorem 3,** there is a $0 \neq \zeta \in U^{\circ}$ such that $(z^*, \overline{z}^*)$ solves problem (1). Consequently from Theorem4, there is $u \in \left(M(z^*, \overline{z}^*)\right)^{\circ}$ such that

$\nabla_z \zeta^H f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} \zeta^H} f(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^H \overline{D_{\overline{z}} g}(z^*, \overline{z}^*) = 0,$ and
$(\mathrm{Re}\, u^H g(z^*, \overline{z}^*) = 0.$
Since ζ is a constant vector, then equations (6) and (7) yield.

Theorem 10. Assume that $f: \mathbb{C}^{2n} \to \mathbb{C}$, $g: \mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions, f is convex with respect to a point closed convex cone $U \subset \mathbb{C}^p$ and g is concave with respect to a closed convex cone $\subset \mathbb{C}^m$. Then a sufficient condition for $(z^*, \overline{z}^*)$ to be an efficient solution for problem (1) with respect to U is the existence of $\zeta^{*H} \in int\, U^{\circ}$ and $u^* \in \left(M(z^*, \overline{z}^*)\right)^{\circ} \subset M^{\circ}$ such that

$$\boldsymbol{\zeta^{*H} D_z f(z^*, \overline{z}^*) + \zeta^{*H} \overline{D_{\overline{z}} f}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^T \overline{D_{\overline{z}} g}(z^*, \overline{z}^*) = 0}, \text{ and} \quad (8)$$

$$\mathrm{Re}\, u^H g(z^*, \overline{z}^*) = 0. \quad (9)$$

Proof. At first, since $f(z, \overline{z})$ is convex with respect to U, that is for any $(z, \overline{z}) \in P$,

$$f(z,\overline{z}) - f(z^*, \overline{z}^*) - D_z f(z^*, \overline{z}^*)(z - z^*) - D_{\overline{z}} f(z^*, \overline{z}^*)(\overline{z} - \overline{z}^*) \in U. \quad (10)$$

Since $\zeta^* \in int\, U^{\circ}$, then from (9) and for any $(z, \overline{z}) \in P$, we have

$$\mathrm{Re}\, \left(\zeta^{*H} f(z,\overline{z}) - \zeta^{*H} f(z^*, \overline{z}^*) - \nabla_z \zeta^{*H} f(z^*, \overline{z}^*)(z - z^*) - \nabla_{\overline{z}} \zeta^{*H} f(z^*, \overline{z}^*)(\overline{z} - \overline{z}^*)\right) \geq 0. \quad (11)$$

This means that
$\mathrm{Re}\, \zeta^{*H} f(z, \overline{z}) = 0$ is convex with respect to $\mathfrak{R}_+$.
At second, equation (8) is equivalent to

$$\nabla_z \zeta^H f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} \zeta^H} f(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^H \overline{D_{\overline{z}} g}(z^*, \overline{z}^*) = 0. \quad (12)$$

Hence, from (9) and (12) , $(z^*, \overline{z}^*)$ is an optimal solution for the problem

$$\min \mathrm{Re}\ \zeta^{*H} f(z, \overline{z})$$
Subject to
$$(z, \overline{z}) \in G.$$

Consequently, Theorem 5 implies to $(z^*, \overline{z}^*)$ is an efficient solution for problem (1) with respect to U.

5. Proper Efficiency

In this section, strengthened efficient solutions; called properly efficient solutions is introduced. In real multi-objective optimization problems, a number of various definitions to proper efficiency with respect to a domination cone(For instance, Geoffrion's concept (Chankong and Haimes, 1983) and Geoffrion, J. Borwein's notion of propernes (Borwein, 1977) . Henig (1982) introduced the concepts of local and global properness; the first is equivalent to Benson's definition and the second to Borwein's. D. Duca (1980) formlated a vectorial programming problem in complex space as

$$\min \mathrm{Re}\, f(z, \overline{z}) = (\mathrm{Re}\, f_1(z, \overline{z}), \mathrm{Re}\, f_2(z, \overline{z}), \ldots, \mathrm{Re}\, f_l(z, \overline{z}))$$
$$\text{Subject to} \quad (13)$$
$$(z, \overline{z}) \in G = \{(z, \overline{z}) \in Q: g(z, \overline{z}) \in M\}.$$

Definition 11. (Proper efficient). A point $(z^*, \overline{z}^*)$ is said to be a properly efficient solution of problem (13) if it is efficient and there is a real $N > 0$ such that for each $i \in \{1, 2, \dots, l\}$ and each $(z, \overline{z}) \in G$ satisfying

$\operatorname{Re}\left(f_i(z^*, \overline{z}^*) - f_i(z, \overline{z})\right) > 0$, there exists at least $j \neq i$ such that

$\operatorname{Re}\left(f_i(z, \overline{z}) - f_i(z^*, \overline{z}^*)\right) \leq N \operatorname{Re}\left(f_j(z, \overline{z}) - f_j(z^*, \overline{z}^*)\right)$

Remark 2. An efficient solution which is not proper is said to be improperly efficient.

Remark 3. The proper efficiency of the problem (1) in the case that U is a complex polyhedral cone. In fact, if U is polyhedral, i.e.,

$$U = \{z \in \mathbb{C}^p : Re\, b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}, \qquad (14)$$

Then

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U$, for $(z, \overline{z})$ and $(z^*, \overline{z}^*) \in G$ is equivalent to

$\operatorname{Re} b_i f(z, \overline{z}) \geq \operatorname{Re} b_i f(z^*, \overline{z}^*)$, for i=1, 2,…, l

Therefore, by putting $f_i(z, \overline{z}) = b_i f_i(z, \overline{z}), i = 1, 2, \dots, l$, the problem (1) can be identified with the p -real objective optimization problem (13)

Definition 12. Let $U = \{z \in \mathbb{C}^p : \operatorname{Re} b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$. A point $(z^*, \overline{z}^*)$ is said to be a properly efficient solution of problem (1) with respect to U if it is efficient with respect to U if it is efficient with respect to U and there is a real $N > 0$ such that for each $i \in \{1, 2, \dots, l\}$ and each $(z, \overline{z}) \in G$ satisfying

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U_i = \{z \in \mathbb{C}^p : \operatorname{Re} b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$ there exists at least $j \neq i$ such that

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U_{ij} = \{z \in \mathbb{C}^p : \operatorname{Re}\left(b_i + N b_j\right) z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$.

Proposition 3. Let U defined in (14) , then the point $(z^*, \overline{z}^*)$ is a properly efficient solution of problem (13) with

$$f(z, \overline{z}) = \left(f_i(z, \overline{z})\right), f_i(z, \overline{z}) = b_i f(z, \overline{z}), i = 1, 2, \dots, l \qquad (15)$$

If and only if $(z^*, \overline{z}^*)$ is properly efficient solution of problem (1) with respect to U.

5.1. Characterization of Properly efficient solutions

Now, in the case of U is polyhedron which is defined as in (14), and from theorem3 it follows that for any $\zeta \in U^\circ$, there exists $\vartheta \in \Re^l, \vartheta \geq 0$ such that $\zeta = b^H \vartheta$, $b = [b_1\ b_2 \dots b_l]^T$. Therefore, problem (14) can be rewritten ad follows

$$\begin{aligned} &\min Re\, \vartheta^T b f(z, \overline{z}) \\ &\text{Subject to} \qquad (16) \\ &(z, \overline{z}) \in G, for\ some\ \vartheta \in \Re^l, \vartheta \geq 0. \end{aligned}$$

Theorem 7. Let U defined in (14), f be a convex function with respect to U on P and g be concave function with respect to a closed convex cone M on P. If $(z^*, \overline{z}^*)$ is a properly efficient solution for problem (1) with respect to U, then there is $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$, with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (16).

Proof. If $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U, then it is properly efficient for the problem (13) accompanied with (8).

On the other hand, since $f(z, \overline{z})$ is a convex with respect to U, one can easily find that $Re\, bf(z, \overline{z})$ is convex with respect to $\Re_+^l$ on P, this is leads to the existence of $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (9).

Theorem 8. Let U defined in (14) and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (16), then $(z^*, \overline{z}^*)$ is a properly efficient for problem (13) with respect to U.

Proof. If $(z^*, \overline{z}^*)$ is a solution for problem (16), then $(z^*, \overline{z}^*)$ is a properly efficient for problem (6) satisfying equation (8), and hence $(z^*, \overline{z}^*)$ is properly efficient for problem (1) with respect to U.

5.2. Kuhn- Tucker's conditions for proper efficiency

In this section, the conditions for proper efficiency are established in the case of U is a polyhedral cone.

Theorem 11. Assume that U defined in (14), M is a polyhedral cone in $\mathbb{C}^m$; $f: \mathbb{C}^{2n} \to \mathbb{C}$, $g: \mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions at a feasible point $(z^*, \overline{z}^*)$ at which the Kuhn- Tucker's constraint qualification holds, f is convex on P with respect to U, g is concave on P with respect to M. Then a necessary condition for $(z^*, \overline{z}^*)$ to be a properly efficient solution for problem (1) with respect to U is that there is $u \in \left(M\left(g(z^*, \overline{z}^*)\right)\right)^\circ$ and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that

$$\vartheta^T b \nabla_z f(z^*, \overline{z}^*) + \overline{\vartheta^T b \nabla_{\overline{z}}}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - \overline{u^T D_{\overline{z}} g}(z^*, \overline{z}^*) = 0, \text{ and} \qquad (17)$$

$$\operatorname{Re} u^H g(z^*, \overline{z}^*) = 0 \qquad (18)$$

Proof. Since $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U, then from theorem7, there is a $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (9). From theorem4, there exist a $u \in \left(M\left(g(z^*, \overline{z}^*)\right)\right)^\circ$ such that

$\nabla_z \vartheta^T b f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} \vartheta^T b}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - \overline{u^T D_{\overline{z}} g}(z^*, \overline{z}^*) = 0$, and

$\operatorname{Re} u^H g(z^*, \overline{z}^*) = 0$.

Theorem 12. Assume that U defined in (7), M is a closed convex cone in $\mathbb{C}^m$; f is an analytic function and convex on P with respect to U, g is an analytic function and concave on P with respect to M. Then a sufficient condition for $(z^*, \overline{z}^*) \in G$ to be properly efficient solution for the problem (1) with respect to U is that there is $u \in M^\circ$ and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that the equalities (17) and (18hold.

Proof. Since f is a convex function with respect to $U, Re\, bf(z, \overline{z})$ is convex with respect to $\Re_+^l$ on P. It follows from $\vartheta > 0$ that $Re \vartheta^T\, bf(z, \overline{z})$ is convex with respect to $\Re_+$ on P. Therefore, theorem4 implies to $(z^*, \overline{z}^*)$ is a solution of problem (9), that $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U.

Table 1
List of symbols

Symbols	Meaning
$\mathbb{C}^{m\times n}$	Complex $m \times n$ matrix
A	Matrix $\mathbb{C}^{m\times n}$
A^T	Transpose of A
$\overline{A}$	Conjugate of A
A^H	Conjugate transpose
Re x	Real part
Int U	The interior of U
$\mathfrak{R}_+^k$	Non- negative orthant of $\mathfrak{R}^n$, $\mathfrak{R}_+^k = \{x \in \mathfrak{R}^n : x_j \geq 0, j = \overline{1,n}\}$.

6. Conclusions and Future Works

In this paper, the necessary and sufficient conditions for a point to be an efficient or a properly efficient solution of a multi-objective complex programming problem have studied, when the ordering has taken with respect to a pointed closed convex cone**.** The future of this work summarized as in the following points:

A. Developing the duality theory in both single-objective and multi-objective optimization problems.
B. Deriving the optimality conditions under other types of generalized convexity such as E-convexity and invexity.
C. Deriving the saddle point optimality criteria without differentiability requirements.
D. Study of fractional programming problems in both single-objective and multi-objective programming problems.
E. Study of multiplicative programming problems.
F. Study of stability in parametric programming problems over complex space.
G. Extending the various concepts of proper efficiency to complex space and establishing the relationships between them.

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