Complete Closest-Target Based Directional FDH Measures of Efficiency in DEA

Mahmood Mehdiloozad[a,*], Israfil Roshdi[b]
[a] *MSc, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran*
[b] *PhD Candidate, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

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Abstract

In this paper, we aim to overcome three major shortcomings of the FDH (Free Disposal Hull) directional distance function through developing two new, named Linear and Fractional CDFDH, complete FDH measures of efficiency. To accomplish this, we integrate the concepts of similarity and FDH directional distance function. We prove that the proposed measures are translation invariant and unit invariant. In addition, we present effective enumeration algorithms to compute them. Our proposed measures have several practical advantages such as: (a) providing closest Pareto-efficient observed targets (b) incorporating the decision maker's preference information into efficiency analysis and (c) being flexible in computer programming. We illustrate the newly developed approach with a real world data set.
Keywords: DEA, FDH, Efficiency, Closest Target, FDH Directional Distance Function.

1. Introduction

Data envelopment analysis (DEA), originally developed by Charnes et al. (1978) and later extended by Banker et al. (1984), is a non-parametric linear programming-based method to evaluate the relative efficiency of a set of homogeneous decision making units (DMUs). The relative comparison in DEA is made with reference to a production possibility set (PPS) constructed from the set of observed DMUs by assuming several postulates. One of the assumed postulates in constructing the conventional DEA PPSs is the convexity assumption. Therefore, in evaluating a given inefficient DMU by the conventional DEA models, the obtained projection point may be a virtual activity (i.e., be an unobserved DMU). However, in most practical applications, virtual activities may have not actual existence. In such cases, the production technology does not satisfy the convexity assumption and, therefore, assuming it is meaningless. By relaxing the convexity assumption, Deprins et al. (1984) proposed an extension of the conventional technologies, called Free Disposal Hull (FDH) non-convex technology. One approach for computing the value of FDH measures is using the common mixed integer linear programming (MILP) techniques. However, solving an MILP is not efficient, from computational point of view. Thus, Tulkens (1993) presented effective enumeration algorithms, based on vector dominance reasoning, to compute the traditional input and output efficiency measures developed by Farrell (1957). Up to now, many papers in DEA literature have been published studying the FDH approach from both application and theoretical perspectives. Some articles relating to the application of the FDH models are Ruiz-Torres and Lopez (2004), Ching-Kuo (2007), Amin and Hosseini Shirvani (2009), Witte et al. (2010), Halkos and Tzeremes (2010), Blancard et al. (2011) and Alimardani et al. (2012). The theoretical aspect of the FDH approach has been also explored in Kerstens and Vanden Eeckaut (1999), Thrall (1999), Cherchye et al.(2000), Agrell and Tind (2001), Cherchye et al. (2001), Leleu (2006, 2009), Keshvari and Dehghan Hardoroudi (2008) and Alirezaee and Khanjani Shiraz (2010). Among the previous researchers, Cherchye et al. (2001) using the directional distance function, recently introduced by Chambers et al. (1996; 1998), developed the FDH directional distance function (we call it DFDH) to estimate the FDH technical inefficiency. They, further, by extending the Tulkens's (1993) algorithms presented an enumeration algorithm for computing the FDH inefficiency of DMUs. As we point out in this paper, the DFDH suffers from a number of major shortcomings including

(i) Not providing an efficiency score for the DMU under evaluation.

* Corresponding author E-mail address: m.mehdiloozad@gmail.com

(ii) Dealing only with proportionate improvements in data and, thus, not taking account the non-zero slacks as sources of inefficiency.

As well as the efficiency score, as a practical outcome, the most powerful piece of the obtained information by a DEA analysis is the set of Pareto-efficient projection points for the DMU under evaluation. The coordinates of a projection point can be interpreted as the “target” levels of operation of inputs and outputs. The obtained targets give an indication of how improving the assessed DMU perform efficiently. Therefore, the more the assessed DMU is close to the targets, the less it needs practical effort to be efficient. In fact, this idea is the foundation of the concept of “similarity” specified in the DEA literature (Silva et al., 2003; Aparicio et al., 2007). This is justification of finding nearest targets. To find closest observed targets in relation to the FDH technology, Silva et al. (2003) proposed an approach using the BRZW efficiency measure (Brockett et al., 1997). The targets determined by the DFDH are not necessarily the closest observed targets whereas this property is in contrast with the concept of “similarity” specified in the literature. Therefore, the third shortcoming of the DFDH is

(iii) Not providing closest observed target.

The purpose of this paper is to overcome all the above-mentioned shortcomings of the DFDH. To this purpose, at first we formulate two complete, named Linear and Fractional FDFDH, FDH models. Although these models take the non-zero slacks into account, the targets determined by them have the maximum distance from the DMU under assessment. Thus, modifying the presented models, we develop two new, named Linear and Fractional CDFDH, models that provide closest observed Pareto-efficient targets as well as efficiency scores. To determine the optimal solutions of the Linear and Fractional FDFDH / CDFDH models, we present efficient enumeration algorithms. The presented measures are complete, translation invariant and unit invariant. Our proposed models, as well as theoretical properties, have several practical advantages such as: (a) providing closest Pareto-efficient observed targets (b) taking the decision maker’s preference information into efficiency analysis and target setting and (c) being flexible in computer programming.

The remainder of this paper unfolds as follows. The next section briefly describes the directional distance function and the DFDH. In Section 3, we discuss about the shortcomings of the DFDH and formulate the Linear and Fractional FDFDH / CDFDH models. In addition, we discuss the properties and advantages of the proposed models. Section 4 provides two enumeration algorithms for determining the optimal solutions of our models. Section 5, using an illustrative together with an application example, sketches the proposed models and their properties, and compares them with the FDH directional distance function. Finally, conclusions are given in the last section.

2. Background

First, we introduce the necessary notations and define basic concepts used in this article. Throughout this paper, we deal with n observed DMUs, DMUj $(j = 1, \dots, n)$, with m inputs $(i = 1, \dots, m)$ and s outputs $(r = 1, \dots, s)$. The input and output vectors of DMUj, respectively are denoted by $x_j = (x_{1j}, \dots, x_{mj})^T$ and $y_j = (y_{1j}, \dots, y_{sj})^T$ where $x_j \geq 0$, $x_j \neq 0$, $y_j \geq 0$ and $y \neq 0$. Further, we consider DMUo as the DMU under evaluation.

2.1 FDH Production Possibility Set

One of the first steps in DEA, after identifying inputs and outputs and gathering corresponding data, is choosing an appropriate technology, i.e., determining the PPS. The PPS, T, is the set of all feasible input-output vectors is given by the following production technology:

$$T = \{(x, y) | x \text{ can produce } y\}. \quad (1)$$

Under the preliminary assumptions of inclusion of observed DMUs, variable returns to scale (VRS), strong (free) disposability of inputs and outputs the unique non-empty FDH technology, generated from n observed DMUs, $\text{DMU}_j = (x_j, y_j)$, $j = 1, \dots, n$, is deduced as follows:

$$T_{FDH} = \left\{ (x, y) \in \mathbb{R}^{m+s}_{\geq 0} \middle| \begin{array}{l} x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0,1\}, \forall j \end{array} \right\} \quad (2)$$

The non-convex Free Disposal Hull (FDH) technology, T_{FDH}, is one of the popular PPSs in DEA. The primary motivation in developing this technology is that linear combinations of the observed DMUs may be meaninglessness, in practical application. To remove these combinations, Deprins et al. (1984) developed the FDH technology, relaxing the convexity assumption. In is worth noting that, relaxing the convexity makes the FDH technology as tight as possible.

2.2 The Directional Distance Function

The directional distance function, recently introduced by Chambers et al. (1996, 1998), generalizes the traditional Shephard distance functions (Shephard, 1970), and is well suited to the task of providing a measure of technical efficiency in the full input-output space. This distance function projects a given input-output vector, (x, y), radially from itself to the frontier of PPS, T, in a pre-assigned direction vector $g = (-g^-, g^+) = (-\mathbb{R}^+_m) \times \mathbb{R}^+_s$, and is defined as:

$$\vec{D}_T(x, y; -g^-, g^+) = Max\{\beta | (x - \beta g^-, y + \beta g^+) \in T\} \quad (3)$$

Given the technology and the direction vector, the directional distance function measures the maximum simultaneous expansion of outputs and contraction of inputs, along a path previously determined by the given direction vector. It is important to note that

$\vec{D}_T(x, y; -g^-, g^+)$ provides a complete characterization of the technology, i.e.

$$\vec{D}_T(x, y; -g^-, g^+) \geq 0 \iff (x, y) \in T \quad (4)$$

2.3 The FDH Directional Distance Function

By formulating the directional distance function relative to (2), we have the mathematical formulation of the DFDH (Cherchye et al., 2001) as

$$\begin{aligned} \beta^* = &Max \quad \beta \\ s.t. \quad &\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} - \beta g_i^-, \quad i = 1, \dots, m, \\ &\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} + \beta g_r^+, \quad r = 1, \dots, s, \\ &\textstyle\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \in \{0,1\}, \quad j = 1, \dots, n, \\ &s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s. \end{aligned} \quad (5)$$

The above model is a mixed integer linear programming (MILP) problem in which the variables λ_j, $j = 1, \dots, n$, are binary variables. In this model, since $\sum_{j=1}^{n} \lambda_j = 1$ and $\lambda_j \in \{0,1\}$ for $j = 1, \dots, n$, only one of the binary variables λ_j $(j = 1, \dots, n)$ will be unit valued, while all the other variables will be 0 in the optimal solution. Therefore, in the model (5) the reference point is chosen among the existing DMUs instead of their combinations. In other words, the FDH model (5) compares the under assessment DMU with an observed unit.

The model (5) can be solved via the common MILP techniques. However, Cherchye et al. (2001) proposed an enumeration algorithm to compute the optimal solution for (5). Their algorithm is based on an alternative equivalent characterization of the FDH technology that is provided using monotone hulls of the observed DMUs. The monotone hull of DMUj is defined as follows:

$$M(j) = \{(x, y) | x \geq x_j, y \leq y_j\} \quad (6)$$

From this definition, the FDH technology can be approximated as:

$$T_{FDH} = \cup_{j=1}^{n} M(j) \quad (7)$$

Based on this approximation, the model (5) is rewritten as

$$\beta^* = \text{Max}_{j=1,\dots,n} \left\{ \text{Max} \left\{ \beta \left| \begin{matrix} x_{ij} \leq x_{io} - \beta g_i^-, i = 1, \dots, m \\ y_{rj} \geq y_{ro} + \beta g_r^+, r = 1, \dots, s \end{matrix} \right. \right\} \right\} \quad (8)$$

Since we have suppose that $g^- > 0$ and $g^+ > 0$, so according to Cherchye et al.'s (2001) enumeration algorithm the optimal objective of (5) can be computed as

$$\begin{aligned} \beta^* &= \underset{j=1,\dots,n}{\text{Max}} \{\beta_j\} \\ \beta_j &= \text{Min}_{\substack{i=1,\dots,m, \\ r=1,\dots,s}} \left\{ \frac{1}{g_i^-}(x_{io} - x_{ij}), \ \frac{1}{g_r^+}(y_{rj} - y_{ro}) \right\}, \forall j \end{aligned} \quad (9)$$

3. Complete Directional FDH Measures of Efficiency

As mentioned in the introductory section, the DFDH, (5), suffers from some serious shortcomings. In this section, we thoroughly discuss about these shortcomings and attempt to remedy them, by proposing new solution approaches.

Shortcoming 1 of the DFDH: In the model (5), β^*, in general, cannot be interpreted as an efficiency index for any arbitrary direction vector. A way of avoiding this shortcoming is imposing one of the following primary conditions on the direction vector g

$$\text{Max}_{j=1,\dots,n} \left\{ \frac{x_{io}}{g_i^-} \right\} \leq 1 \quad (10)$$

$$\underset{j=1,\dots,n}{\text{Max}} \left\{ \frac{\overline{x}_i - \underline{x}_i}{g_i^-} \right\} \leq 1, \overline{x}_i = \underset{j=1,\dots,n}{\text{Max}} \{x_{ij}\}, \underline{x}_i = \text{Min}_{j=1,\dots,n} \{x_{ij}\} \quad (11)$$

which assures that $\beta^* \leq 1$ and, thereby, $1 - \beta^*$ can be interpreted as an efficiency index. For example, each of the following direction vectors satisfies the condition (10):

$$g_i^- = x_{io}, g_r^+ = y_{ro}, i = 1, \dots, m, r = 1, \dots, s \quad (12)$$

$$\begin{cases} g_i^- = \overline{x}_i = \text{Max}_{j=1,\dots,n} \{x_{ij}\}, \forall \, i, \\ g_r^+ = \overline{y}_r = \text{Max}_{j=1,\dots,n} \{y_{rj}\}, \forall \, r \end{cases} \quad (13)$$

In addition, the following direction vector satisfies the condition (11):

$$\begin{cases} g_i^- = \overline{x}_i - \underline{x}_i = \text{Max}_{j=1,\dots,n} \{x_{ij}\} - \text{Min}_{j=1,\dots,n} \{x_{ij}\}, \forall \, i, \\ g_r^+ = \overline{y}_r - \underline{y}_r = \text{Max}_{j=1,\dots,n} \{y_{rj}\} - \text{Min}_{j=1,\dots,n} \{y_{rj}\}, \forall \, r \end{cases} \quad (14)$$

3.1 Complete Furthest-Target Based Directional FDH Measures of Efficiency

Shortcomings 2 of the DFDH: The model (5) radially (proportionately) projects the given DMU onto the frontier of T_{FDH}. Thus, the DFDH fails to take account the non-zero input and output slacks as sources of inefficiency that makes the projected point not to be necessarily Pareto-efficient. To create a visual representation, consider the FDH technology depicted in Fig 1. This technology is constructed by the Pareto-efficient, A, B, C, D, and inefficient, E and F, DMUs for the simplest case of single input and single output. As evident in the figure, maximum proportional decrease in input and increase in output of the inefficient unit F in the direction of g is achieved on M(C). By these improvements, the unit F is projected onto the boundary unobserved point F', and the efficient unit C is determined as the reference DMU to the unit F. However, the value of s^- is not contributed in evaluation of the unit F.

Relative to the radial DEA models, the non-radial ones have higher discriminatory power in evaluating DMUs. In addition, as noted by Silva et al. (2003), "the non-convex nature of the FDH efficient frontier usually results in higher slack values than those obtained in convex technologies…." Therefore, developing complete efficiency measures for the FDH technology is particularly important. In considering this, we introduce the Linear and Fractional complete furthest-target based directional FDH, named the Linear and Fractional FDFDH, models as follows:

Linear FDFDH

$$\begin{aligned}
\tau = Max\ & \frac{1}{m}\sum_{i=1}^{m}\beta_i^- + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+ \\
s.t.\ & \sum_{j=1}^{n}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \in \{0,1\}, \quad j = 1,\dots,n, \\
& \beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s.
\end{aligned} \tag{15}$$

Fractional FDFDH

$$\begin{aligned}
\rho_F = Min\ & \frac{1-\frac{1}{m}\sum_{i=1}^{m}\beta_i^-}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r^+} \\
s.t.\ & \sum_{j=1}^{n}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \in \{0,1\}, \quad j = 1,\dots,n, \\
& \beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s
\end{aligned} \tag{16}$$

Here, the vector g represents the pre-assigned direction vector, which satisfies in (10) or (11). The variables, β_i^- $(i = 1,\dots,m)$ and β_r^+ $(r = 1,\dots,s)$ represent the individual rates of contraction and expansion in the ith input and the rth output of DMUo in the direction of g; also, the objective functions of (15) and (16) seek to jointly maximize the values of them.

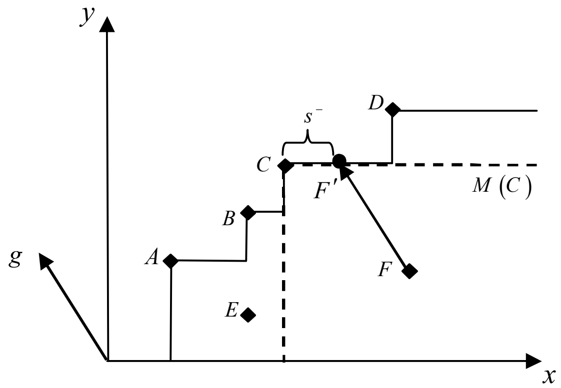

Fig 1. FDH Directional distance function

Corresponding to the models (15) and (16), we introduce two efficiency indexes ρ_F and ρ_L. The latter index is defined as follows:

$$\rho_L := \left[1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^{-*}\right] \times \left[1 + \frac{1}{s}\sum_{r=1}^{s}\beta_r^{+*}\right]^{-1} \tag{17}$$

Where $(\lambda_j^*,\ \beta_i^{-*},\ \beta_r^{+*}, i = 1,\dots,m, r = 1,\dots,s, j = 1,\dots,n)$ is an optimal solution of (15). From the relations (10) and (11), it can be easily verified that the introduced indexes satisfy the property of efficiency requirement i.e., $0 \leq \rho_F \leq 1$ and $0 \leq \rho_L \leq 1$. That is, they are well defined as the efficiency measures.

Definition1. DMU_o is F-efficient (L-efficient) if and only if $\rho_F = 1$ $(\rho_L = 1)$.

This condition is equivalent to $\beta_i^{-*} = \beta_r^{+*} = 0, i = 1,\dots,m, r = 1,\dots,s$, in each optimal solution for the model (15) and model (16).

3.2. Complete Closest-Target Based Directional FDH Measures of Efficiency

Shortcoming 3 of the DFDH: A serious drawback of the DFDH is that its identified target is not necessarily the closest target from the DMU under evaluation. However, from a practical viewpoint, finding closest Pareto-efficient targets for an inefficient DMU is of major importance. Because the obtained targets give an indication of how improving the inefficient DMU to perform efficiently; thus, the more the targets are close to a DMU, the less DMU needs practical effort to be efficient.

As mentioned in the preceding subsection, the obtained targets from the Linear and Fractional FDFDH are the furthest ones. Therefore, these models do not remove the above-mentioned shortcoming. In considering this, to avoid this shortcoming, we modify them and introduce the Linear and Fractional complete closest-target based directional FDH, named the Linear and Fractional CDFDH, models as follows:

Linear CDFDH	Fractional CDFDH
$$\begin{aligned} \tau^C = Min \quad & \frac{1}{m}\sum_{i=1}^{m}\beta_i^- + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+ \\ s.t. \quad & \sum_{j\in E}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m, \\ & \sum_{j\in E}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s, \\ & \sum_{j\in E}\lambda_j = 1, \\ & \lambda_j \in \{0,1\}, \quad j = 1,\dots,n, \\ & \beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s. \end{aligned} \quad (18)$$	$$\begin{aligned} \rho_F^C = Max \quad & \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^-}{1 + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+} \\ s.t. \quad & \sum_{j\in E}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m, \\ & \sum_{j\in E}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s, \\ & \sum_{j\in E}\lambda_j = 1, \\ & \lambda_j \in \{0,1\}, \quad j = 1,\dots,n, \\ & \beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s. \end{aligned} \quad (19)$$

In the above models, E is the set of all F-efficient (L-efficient) DMUs. The guiding idea of formulating the models (18) and (19), to determine the closest targets, is minimizing the distance of the given inefficient DMUo from all the F-efficient (L-efficient) DMUs dominating DMUo. Corresponding to the models (18) and (19), we introduce two new efficiency indices, ρ_F^C and ρ_L^C, where the former is defined as follows:

$$\rho_L^C := \left[1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^{-*}\right] \times \left[1 + \frac{1}{s}\sum_{r=1}^{s}\beta_r^{+*}\right]^{-1} \qquad (20)$$

Similar to ρ_F and ρ_L, the indexes ρ_F^C and ρ_L^C satisfy the property of efficiency requirement and can be interpreted as efficiency measures. Relative to these measures, we define the "efficiency" as follows:

Definition 1. DMU_o is said to be FC-efficient (LC-efficient) if and only if $\rho_F^C = 1$ ($\rho_L^C = 1$).

The following relationships are held among the proposed models:

(R1). DMU_o is F-efficient if and only if it is L-efficient, i.e., $\rho_F = 1$ if and only if $\rho_L = 1$.

(R2). DMU_o is FC-efficient if and only if it is LC-efficient, i.e., $\rho_F^C = 1$ if and only if $\rho_L^C = 1$.

(R3). DMU_o is Pareto-efficient if and only if $\rho_F = 1$ if and only if $\rho_F^C = 1$.

(R4). $\rho_F \leq \rho_L$, $\rho_L^C \leq \rho_F^C$ and $\rho_F \leq \rho_F^C$.

3.3 Properties of the Linear and Fractional DFDH / CDFDH Models

In this section, we study the properties of the proposed models.

P1. Completeness (Cooper et al., 1999)

Our proposed measures are "complete", in the sense that they are non-oriented and take account all the non-zero slacks as sources of inefficiency.

P2. Straightforward interpretation

Each of the efficiency indexes ρ_L, ρ_F, ρ_L^C and ρ_F^C can be interpreted as the product of two separate components of the input efficiency, $\theta_I = 1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^{-*}$, and the output efficiency, $\theta_O = \left[1 + \frac{1}{s}\sum_{r=1}^{s}\beta_r^{+*}\right]^{-1}$. This interpretation gives a better explanation of the efficiency of the under assessment DMU.

P3. Translation invariance (Ali and Seiford, 1990; Lovell and Pastor, 1995; Pastor, 1996; Cooper et al., 1999)

The property of translation invariance assures that translating the original input and output data has no influence on the optimal solutions. By choosing the direction vector (14), the models (15), (16), (18) and (19) will be translation invariant.

To demonstrate this, suppose that $\tilde{x}_{ij} = x_{ij} + v_i$ $(i = 1,\dots,m)$ and $\tilde{y}_{rj} = y_{rj} + u_r$ $(r = 1,\dots,s)$, respectively indicate the translated ith input and rth output of DMUj. Obviously, the original direction (14) will not be changed for the translated data. Therefore, from the constraint $\sum_j \lambda_j = 1$, it follows that our models are translation invariant.

This property makes our models free from the primary positivity assumption on the data whereby they are able to appropriately deal with negative data. In this case, by adding suitable constants to the affected input or output rows we can transform them to positive valued data and use the transformed data in our models.

P4. Units invariance (Cooper et al., 1999; Lovell and Pastor, 1995)

By choosing a direction vector such that g_i^- and g_r^+ respectively have the same units of measurement as the ith input and the rth output, the models (15), (16), (18) and (19) will be unit invariant. Since, g_i^- and the ith input have the same units of measurement, when we rescale the ith input by the scalar $\alpha > 0$, the ith component of the direction vector is converted to αg_i^-, accordingly. Observe that, by multiplying the constraint $\sum_j \lambda_j(\alpha x_{ij}) = \alpha x_{io} - \beta_i^-(\alpha g_i^-)$ by $\frac{1}{\alpha}$, the same constraint of the original model is obtained. For example, the vectors (12) and (13) satisfy this property.

This property indicates that we can rescale each input or output with an arbitrary scalar, without any affection on the optimal solutions of our models.

P5. Alternative optima invariance (Cooper et al., 1999)

The efficiency indices ρ_F and ρ_F^C are invariant to alternative optima. However, the indices ρ_L and ρ_L^C fail to satisfy this property.

P6. Taking account the DM's preference information

In some practical cases, the DM may do not equally prefer the efficient DMUs. In such a case, the DM needs to take account the priorities. According to the preference orders of inputs and outputs given by the DM, we can flexibly modify the vector g. Indeed, the values of the modified direction vector, g','s components describe the relative importance of inputs and outputs. Let the non-zero weights, $w_i, i = 1, \ldots, m$, and $v_r, r = 1, \ldots, s$, respectively are associated with the priorities given by the DM to the inputs and outputs such that the larger the w_i (v_r), the more important the ith input (rth output) is. After incorporating these weights in our models, coefficients of the variables β_i^- and β_r^+, in the objective function will be w_i and v_r, respectively. Therefore, the components of the modified direction vector, g', should be $g_i^{-\prime} = \xi_i g_i^-$ and $g_r^{+\prime} = \psi_r g_r^+$, where $\xi_i = \frac{1}{w_i}$ and $\psi_r = \frac{1}{v_r}$. This implies that if an input (output) has a greater importance, it should be attached a greater weight or equivalently a small direction's component. By considering (10), we must have $\xi_i \geq 1,\ i = 1, \ldots, m$, equivalently $w_i \leq 1,\ i = 1, \ldots, m$[1]. We will clearly exemplify this property by an illustrative example in the next section.

P7. Flexibility in computer programming

A practical advantage of our models is that by writing a computer code for one of them, changing just the direction vector's inputs in this program is enough to achieve new scores and targets associated with a new direction vector. This capability of our models greatly helps the DM to make a more accurate evaluation, by considering several direction vectors, when he cannot rely on an assessment depending only on a specific direction vector. In this case, by running the program for the direction vectors given by the DM, for example, the average of the obtained scores can be proposed to the DM as a final score for a given DMU_o.

4. Enumeration

The models (15) and (18) are MILP problems while the models (16) and (19) are mixed integer non-linear programming problems. Nonetheless, each of the models (16) and (19) can be easily converted into a MILP problem using a transformation technique, similar to the "Charnes-Cooper transformation" (Charnes and Cooper, 1962), with assurance that the optimal value of the converted model will also be optimal for the original model. These MILP problems can be solved using the common MILP techniques. However, computing the optimal solutions of them by enumeration algorithms is superior to solving them via MILP solvers, from the computational point of view. Here, we present new enumeration algorithms for estimating the proposed efficiency measures.

Consider a given DMU_o. DMU_o is weakly dominated by DMU_j if and only $x_j \leq x_o$ and $y_j \geq y_o$ with strict inequality holding in at least one input or output component. Let W_o be the set of all DMUs, which weakly dominate DMU_o. The sets $W_j, j = 1, \ldots, n$, and accordingly the set of all L-efficient DMUs, E, can be readily characterized by comparing the values of inputs and outputs of the DMUs. Let $E_o = W_o \cap E$ i.e., E_o is the set of all efficient DMUs, which weakly dominate DMU_o. Obviously, DMU_o is L-efficient if and only if $E_o = \emptyset$. If $E_o \neq \emptyset$., then, to determine the furthest and closest observed targets for DMU_o we operate as follows.

To find the optimal solutions of the models (15) and (16), for any $j \in E_o$ we define:

$$\sigma_{jo}^L = Max\left\{\frac{1}{m}\sum_{i=1}^{m}\beta_i^- + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+ \,\middle|\, \begin{array}{l}\beta_i^- = \frac{1}{g_i^-}(x_{io} - x_{ij}), \forall i,\\ \beta_r^+ = \frac{1}{g_r^+}(y_{rj} - y_{ro}), \forall r\end{array}\right\}$$

$$= \frac{1}{m}\sum_{i=1}^{m}\frac{1}{g_i^-}(x_{io} - x_{ij}) + \frac{1}{s}\sum_{r=1}^{s}\frac{1}{g_r^+}(y_{rj} - y_{ro}) \quad (21)$$

$$\sigma_{jo}^F = Max\left\{\frac{1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^-}{1 + +\frac{1}{s}\sum_{r=1}^{s}\beta_r^+} \,\middle|\, \begin{array}{l}\beta_i^- = \frac{1}{g_i^-}(x_{io} - x_{ij}), \forall i,\\ \beta_r^+ = \frac{1}{g_r^+}(y_{rj} - y_{ro}), \forall r\end{array}\right\}$$

$$= \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{1}{g_i^-}(x_{io} - x_{ij})}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{1}{g_r^+}(y_{rj} - y_{ro})} \quad (22)$$

After that, according to (7), we calculate the optimal objectives of the Linear and Fractional FDFDH models, (15) and (16), as follows:

$$\tau = \mathrm{Max}_{j \in E_o}\{\sigma_{jo}^L\} \quad (23)$$

$$\rho_F = \mathrm{Min}_{j \in E_o}\{\sigma_{jo}^F\} \quad (24)$$

Now, let $(\beta_i^{-*},\ \beta_r^{+*}, i = 1, \ldots, m, r = 1, \ldots, s)$ be the optimal solution of (21) that is correspond to (23). Then, the efficiency score ρ_L can be computed from (17).

Similarly, the optimal objectives of the models (18) and (19) can be computed via the following relations:

$$\tau^C = \mathrm{Min}_{j \in E_o}\{\sigma_{jo}^L\} \quad (25)$$

$$\rho_F^C = \mathrm{Max}_{j \in E_o}\{\sigma_{jo}^F\} \quad (26)$$

The inputs and outputs levels of the L-efficient DMUs that satisfy in (23) or (24) / (25) or (26) indicate the furthest / closest observed efficient target levels for DMU_o.

1 If the given weights do not satisfy in these conditions, the normalized (dividing by $\mathrm{Max}_{i=1,\ldots,m}\{w_i\}$) form of them will satisfy in these conditions.

5. Real World Example

Example1. To provide an application of the proposed approach, we discuss the efficiency assessment of 15 university departments, denoted by D1, D2,…, D15, with three inputs and three outputs as follows:

Table 2 shows the input-output data, taken from Soleimani-damaneh and Mostafaee (2009).
Table 3 represents the efficiency scores, projection points, and reference points obtained from the DFDH, Fractional FDFDH, and Fractional CDFDH models where the direction vector (13) is used in them.

Table 1
Titles of inputs and outputs of Example 2

Input1:	Budget	**Output1**	Average of the scores of the students,
Input2:	Area of cultural, educational, and research space	**Output2**	Number of the students accepted for graduate studies (in the past two years),
Input3:	Number of books in the library	**Output3**	Satisfaction of students, staff, and professors

Table 2
Data related to the application with real world data (Example 1)

	I_1	I_2	I_3	O_1	O_2	O_3
D1	26	20	10	15.5	8	26
D2	20	15	7	17.2	5	25
D3	22	10	9.5	14.3	8	23
D4	15	12	8.4	14	5	20
D5	30	22	10	12	3	20
D6	35	15	11	16.3	4	22
D7	35	25	12	12	2	18
D8	34	24	12	12	2	19
D9	20	16	9.5	14.3	10	28
D10	22	17	10	13.5	8	26
D11	24	19	8	15	9	29
D12	18	20	12	16	8	23
D13	28	20	10.1	14.2	6	20
D14	30	12	9	14	5	20
D15	25	15	10	17	3	29

Table 3
Data related to the application with real world data (Example 2)

	DFDH			Fractional FDFDH			Fractional CDFDH		
	Score	*Proj.*	*Ref.*	*Score*	*Proj.*	*Ref.*	*Score*	*Proj.*	*Ref.*
D1	1.0000	D1	D1	1.0000	D1	D1	1.0000	D1	D1
D2	1.0000	D2	D2	1.0000	D2	D2	1.0000	D2	D2
D3	1.0000	D3	D3	1.0000	D3	D3	1.0000	D3	D3
D4	1.0000	D4	D4	1.0000	D4	D4	1.0000	D4	D4
D5	0.8276	P5	D2	0.5919	D9	D9	0.7175	D1	D1
D6	1.0000	D6	D2	0.6874	D2	D2	0.6874	D2	D2
D7	0.7586	P7	D2	0.4565	D2	D2	0.5825	D1	D1
D8	0.7931	P8	D2	0.4787	D2	D2	0.6044	D1	D1
D9	1.0000	D9	D9	1.0000	D9	D9	1.0000	D9	D9
D10	0.9600	P10	D9	0.8630	D9	D9	0.8630	D9	D9
D11	1.0000	D11	D11	1.0000	D11	D11	1.0000	D11	D11
D12	1.0000	D12	D12	1.0000	D12	D12	1.0000	D12	D12
D13	0.9600	P13	D11	0.6957	D9	D9	0.8427	D1	D1
D14	1.0000	D14	D4	0.8405	D4	D4	0.8405	D4	D4
D15	1.0000	D15	D15	1.0000	D15	D15	1.0000	D15	D15

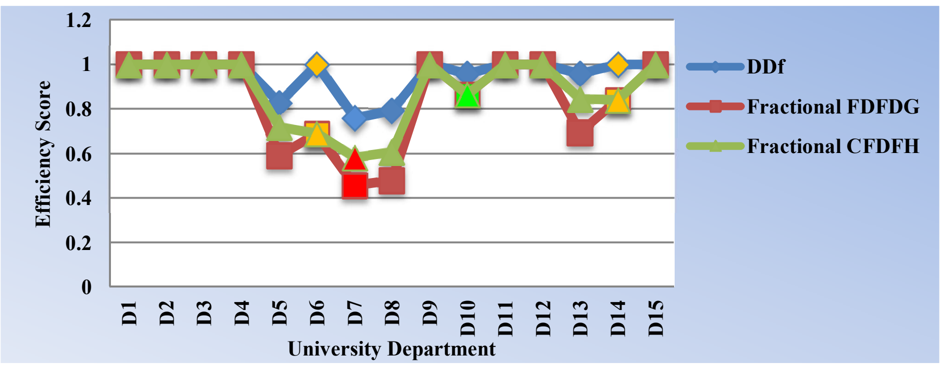

Fig. 2. Comparison of departments' efficiencies

Comparing the results obtained by the DFDH, Fractional DFDH and Fractional CDFDH models, we make the following observations:

- The number of the efficient departments determined by the DFDH is more than that determined by the Fractional DFDH and CDFDH models.
- The departments D1, D2, D3, D4, D9, D11, D12, and D15 are characterized as both F-efficient and FC-efficient units. (See Fig. 2)

The departments D6 and D14 are determined efficient by the DFDH model and are projected on themselves. In addition, the obtained reference point for each of them is different from themselves. However, the Fractional FDFDH and CDFDH models determine these departments as inefficient units and recognize D14 better than D6. This happen because the Fractional FDFDH/CDFDH models take the non-zero slacks into efficiency measurement, unlike the DFDH model. (See Fig. 2)

- Between the inefficient departments, the department D10 is determined with the best performance, by the Fractional FDFDH and Fractional CDFDH models. Furthermore, the inefficient department D7 has the worst efficiency between all departments. (See Fig. 2)
- In evaluating the inefficient departments D10 and D13 by the DFDH model, they have identical performance and the corresponding efficiency score is equal to 0.9600. However, in evaluating them by the Fractional FDFDH and CDFDH models, D10 performs better than D13.
- The Fractional FDFDH model ranks the inefficient departments as D10, D14, D13, D6, D5, D8, and D7 whereas the Fractional CDFDH model gives the different order D10, D13, D14, D5, D6, D8, and D7. The cause of this happening is that the Fractional FDFDH/ CDFDH models estimate the efficiency score with respect to the furthest/closest F-efficient units.
- As we expect, the efficiency scores obtained from the Fractional CDFDH model are greater or equal than that obtained from the Fractional FDFDH model i.e., $\rho_F \leq \rho_F^C$. In addition, the efficiency scores obtained from the Fractional FDFDH and CDFDH models are less or equal to than that obtained from the DFDH model i.e., $\rho_F \leq 1-\beta^*$ and $\rho_F^C. \leq 1-\beta^*$. (See Fig. 2)
- In evaluating the inefficient departments D5, D6, D7, D8, D10, D13 and D14 by the Fractional FDFDH, the departments **D9**, D2, **D2**, **D2**, D9, **D9** and D4 are the corresponding *furthest observed targets*. However, in evaluating them by the Fractional CDFDH, the departments **D1**, D2, **D1**, **D1**, D9, **D1** and D4 are the corresponding *closest observed targets*.
- The obtained furthest and closest targets for each of the inefficient departments D6, D10 and D14 are the same. However, the Fractional FDFDH and CDFDH models give different departments as the furthest and closest observed targets for the other inefficient departments.
- In evaluating the departments D5, D7, D8, D10, and D13 by the Fractional FDFDH and the Fractional CDFDH models, the observed departments are determined as the corresponding furthest and closest projection points. However, the DFDH model gives the following unobserved units P5, P7, P8, P10, and P13 as the associated projection points:

$$\mathrm{P5} = (23.9655, 17.6897, 7.9310, 14.9655, 4.7241, 25.0000),$$
$$\mathrm{P7} = (26.5517, 18.9655, 9.1034, 16.1517, 4.4138, 25.0000),$$
$$\mathrm{P8} = (26.7586, 18.8276, 9.5172, 15.5586, 4.0690, 25.0000),$$
$$\mathrm{P10} = (20.6000, 16.0000, 9.5200, 14.1880, 8.4000, 27.1600),$$
$$\mathrm{P13} = (26.6000, 19.0000, 9.6200, 14.8880, 6.4000, 21.1600),$$
$$\mathrm{P14} = (30.0000, 12.0000, 9.0000, 14.0000, 5.0000, 20.0000).$$

In sum, the Fractional CDFDH model, as well as efficiency scores and DMUs classification, provide the closest observed efficient targets for the inefficient departments; thus, it is superior to the DFDH model in removing their deficiencies and drawbacks.

6. Conclusion

In this study, we have carefully examined the FDH directional distance function and pointed out its major shortcoming. Then, providing suitable remedies, we have tried to resolve them. First, by imposing a primary condition on the direction vector, we introduced an efficiency index associated to the DFDH. Next, extending the DFDH, we developed two furthest-target based directional, Linear and Fractional FDFDH, models. Then, modifying these models, we presented two closest-target based, named Linear and Fractional CDFDH, models. The Linear and Fractional CDFDH models successfully overcome all shortcomings of the DFDH and have several desirable theoretical and practical properties. These properties include: (1) complete efficiency requirement (2) straightforward interpretation (3) translations invariance (4) unit invariance (5) alternative optima invariance (6) incorporating the DM's preference information and (7) flexibility in computer programming. In such practical applications, which the virtual activities have not actual existence, the Linear and Fractional CDFDH models are very useful due to comparing performance of the inefficient DMUs with the observed efficient DMUs. Furthermore, these models, by finding the closest observed targets for an inefficient DMU, provide an approach for how improving the inefficient DMU with the lowest effort to make it efficient.

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