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Solving linear and nonlinear optimal control problem using modified adomian decomposition method

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Abstract

First Riccati equation with matrix variable coefficients, arising in optimal and robust control approach, is considered. An analytical approximation of the solution of nonlinear differential Riccati equation is investigated using the Adomian decomposition method. An application in optimal control is presented. The solution in different order of approximations and different methods of approximation will be compared respect to accuracy. Then the Hamilton-Jacobi-Belman (HJB) equation, obtained in nonlinear optimal approach, is considered and an analytical approximation of the solution of it using the Adomian decomposition method is presented.

Keywords: Adomian decomposition method; Riccati differential equation; optimal control; Hamilton-Jacobi-Belman equation

1. Introduction

Riccati equations arise in optimal and robust control theory and it is a nonlinear, time-variant matrix coefficient equation. For solving this equation no analytical method exists. A method for solving this equation numerically is discretization of it in time domain and substitution of derivation operator with discrete approximation and finding solution in each iteration. But this method is very sensitive to sample time ΔT in discretization and may be unstable for some ΔT . Also, Hamilton-Jacobi-Belman equations obtained in nonlinear optimal control and for solving them, two approaches exist. In first approach, we discrete given system and with using dynamic programming find optimal control signal. In second approach, we use given continuous system and reach to HJB equation. Then use approximation for solving this equation and finding optimal control signal. In this paper we start with linear, time-invariant system and apply optimal control to this system. With using calculus of variations, we reach to a Riccati equation. Then we apply Adomian decomposition method for analytical solving this equation and compare solution of this method with different order approximations. Then we consider a nonlinear system and apply nonlinear optimal control to this system. With using dynamic programming, we reach to a HJB equation. Finally, we apply the Adomian Decomposision method for solving this equation.

Adomian decomposition method is a approximated approach for solving nonlinear differential equations by substitution of nonlinear parts of equation with Adomian polynomials and use a step by step method for finding solutions [1]. This method is a powerful approach in nonlinear differential equations and accuracy of it depends on number of used partial solutions. Also, solution of this method has a fast convergence to exact solution generally. In recent years, some modifications on this method have been presented [5,6]. Modification of method is in quality of computation of Adomian polynomials. These modifications affects on convergence of method. In some papers [2,3], Adomian decomposition method used in nonlinear optimal control and non-quadratic cost functions optimal control. In [7,8,9] modified Adomian Decomposition method has been used in linear optimal control. In these papers, linear optimal problem formulated and solved by using modified Adomian Decomposition method and results show that this method is powerful in

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this field. We expand results of these papers for nonlinear optimal problem in the presented paper. Structure of paper is as follows: In section 2, a brief description of optimal control will be presented. In section 3, a brief description of nonlinear optimal control will be presented. In section 4, Adomian decomposition method for solving differential equations will be described. In section 5, we apply Adomian method to Riccati equation and find solution. Then three examples will be presented and method will be applied to them and solutions will be compared. Also, we apply Adomian Decomposition method to HJB equation and find solution. In section 6, we have conclusion and suggestion of future works.

2. Linear Optimal Control

In this section we have a brief description of optimal control. First consider the following linear time-invariant system in state space realization [4]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_u u(t) \quad , \quad x(0) = x_0 \\ y(t) = C_y x(t) \end{cases} \tag{1}$$

System has no disturbance input. Suppose that A is a $n \times n$ matrix, $x(t)$ a $n \times 1$ state space vector, $y(t)$ output vector and $u(t)$ is control signal. Our propose is control of the above system and finding control signals subject to minimizing the following cost function:

$$J(u,y) = \frac{1}{2} y^T(t_f) H_y y(t_f) + \frac{1}{2}\int_0^{t_f} (y^T(t) Q_y y(t) + u^T(t) R u(t))dt \tag{2}$$

In this cost function, Q, R and H_y are positive definite and symmetric with appropriate dimensions. Now we want to rewrite $J(u,y)$ according to $x(t)$. Substitution of $y(t) = C_y x(t)$ in (2) results:

$$J(u,x) = \frac{1}{2} x^T(t_f) H x(t_f) + \frac{1}{2}\int_0^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t))dt \tag{3}$$

That in (3) we have:

$$H = C_y^T H_y C_y$$

$$Q = C_y^T Q_y C_y$$

That H and Q are positive semi-definite and symmetric matrices. Therefore, a constrained optimization problem is obtained with system dynamic equations constrains. With using Lagrange coefficients method and adding a constrained equation to cost function (3), we convert it to an unconstrained problem as follows:

$$J_a(x,u,p(t)) = J(x,u) + \int_0^{t_f} p^T(t)(Ax(t) + B_u u(t) - \dot{x}(t))dt \tag{4}$$

In (4) $p(t)$ is Lagrange coefficient vector or co-state. With using calculus of variations and simplifying the problem the following equations result:

$$\begin{cases} p^T(t_f) = x^T(t_f) H \\ \dot{p}(t) = -x^T(t) Q - p^T(t) A \\ u(t) = -R^{-1} B_u^T p(t) \\ \dot{x}(t) = Ax(t) + B_u u(t) \end{cases} \tag{5}$$

If delete $u(t)$ in (5), we have:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} A & -B_u R^{-1} B_u^T \\ -Q & -A \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = Z \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \quad ; \quad \begin{cases} x(0) = x_0 \\ p(t_f) = Hx(t_f) \end{cases} \tag{6}$$

ove system is corresponding Hamiltonian system for (1) and (3). Solution of (6) in $t = t_f$ with using state transient matrix will be:

$$\begin{pmatrix} x(t_f) \\ p(t_f) \end{pmatrix} = e^{Z(t_f - t)} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t_f - t) & \phi_{12}(t_f - t) \\ \phi_{21}(t_f - t) & \phi_{22}(t_f - t) \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \tag{7}$$

We have:

$$p(t) = [\phi_{22}(t_f - t) - H\phi_{12}(t_f - t)]^{-1} \times [H\phi_{11}(t_f - t) - \phi_{21}(t_f - t)]x(t) \tag{8}$$

or:

$$p(t) = P(t)x(t) \tag{9}$$

That:

$$P(t) = [\phi_{22}(t_f - t) - H\phi_{12}(t_f - t)]^{-1} \times [H\phi_{11}(t_f - t) - \phi_{21}(t_f - t)] \tag{10}$$

If we derive from (9) and substitute from (6) then simplify it, we have:

$$\begin{cases} -\dot{P}(t) = P(t)A + A^T P(t) + Q - P(t) B_u R^{-1} B_u^T P(t) \\ P(t_f) = H \end{cases} \tag{11}$$

This equation called "Riccati Equation" and is a nonlinear time-variant differential equation. Because $Q, R \geq 0$ and are symmetric, global existence of solutions is guaranteed. It has two solutions that positive semi-definite solution ($P(t) \geq 0$) is desirable. Optimal control signal obtained as follows:

$$u_{opt}(t) = -R^{-1} B_u^T P(t) x(t) = -K(t) x(t)$$
$$K(t) = R^{-1} B_u^T P(t) \tag{12}$$

2.1. Example

Consider the following linear scalar time-invariant system:

We want to find $u(t)$ such that minimize the following cost Function:

$$J = \frac{1}{2} 8x^2(10) + \frac{1}{2}\int_0^{10} (3x^2(t) + u^2(t))dt$$

First we have:

$A=1, B_u=1, t_f=10, H=8, Q=3, R=1$

Organize Hamiltonian matrix Z:

$$Z=\begin{pmatrix}1 & -1\\ -3 & -1\end{pmatrix}$$

State transient matrix obtained as:

$$e^{Zt}=\begin{pmatrix}\frac{3}{4}e^{2t}+\frac{1}{4}e^{-2t} & -\frac{1}{4}e^{2t}+\frac{1}{4}e^{-2t}\\ -\frac{3}{4}e^{2t}+\frac{3}{4}e^{-2t} & \frac{1}{4}e^{2t}+\frac{3}{4}e^{-2t}\end{pmatrix}$$

Using (10), (11) and (12) will result the following optimal gain:

$$K(t)=\frac{27e^{2(10-t)}+5e^{-2(10-t)}}{9e^{2(10-t)}-5e^{-2(10-t)}}$$

It is clear that optimal feedback law is a nonlinear time-variant vector. If time horizon tends to infinity, optimal gain will be tending to $\frac{27}{9}$. This is steady state value of optimal gain.

3. Nonlinear Optimal Control

In this section we have a brief description of nonlinear optimal control. First consider the following nonlinear system in state space realization:

$$\dot{x}(t)=a(x(t),u(t),t) \tag{13}$$

In above system, $x(t)$ is state vector, $u(t)$ is control signal. Our purpose is control of system and finding control signal such that minimize the following cost function:

$$J=h(x(t_f),t_f)+\int_{t_0}^{t_f} g(x(\tau),u(\tau),\tau)d\tau \tag{14}$$

In this cost function, h and g are arbitrary convex functions and t_f is final time of system operation. With using dynamic programming approach, we introduce a new variable as:

$$\underset{t\le\tau\le t_f}{J(x(t),t,u(\tau))}=h(x(t_f),t_f)+ \underset{t_0\le t\le t_f}{\int_t^{t_f} g(x(\tau),u(\tau),\tau)d\tau} \tag{15}$$

Suppose that we have:

$$\begin{aligned}V(x(t),t)&=J^*(x(t),t)\\ &=\underset{\substack{u(\tau)\\ t\le\tau\le t_f}}{Min}\left\{h(x(t_f),t_f)+\int_t^{t_f} g(x(\tau),u(\tau),\tau)d\tau\right\}\end{aligned} \tag{16}$$

Therefore, we have:

$$V(x(t),t)=\underset{\substack{u(\tau)\\ t\le\tau\le t_f}}{Min}\left\{h(x(t_f),t_f)+\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau+\int_{t+\Delta t}^{t_f} g(x(\tau),u(\tau),\tau)d\tau\right\} \tag{17}$$

According to principle of optimality, we have:

$$V(x(t),t)=\underset{\substack{u(\tau)\\ t\le\tau\le t+\Delta t}}{Min}\left\{\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau+V(x(t+\Delta t),t+\Delta t)\right\} \tag{18}$$

Therefore, with using Taylor series we have:

$$V(x(t),t)=\underset{\substack{u(\tau)\\ t\le\tau\le t+\Delta t}}{Min}\left\{\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau+V(x(t),t)+\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial x}[x(t+\Delta t)-x(t)]+H.O.T\right\} \tag{19}$$

If suppose Δt be small enough then $\tau\to t$ and we have:

$$V(x,t)=\underset{u(t)}{Min}\left\{g\Delta t+V(x,t)+\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial x}a(x(t),u(t),t)\Delta t+O(\Delta t)\right\} \tag{20}$$

By divide both side of (20) by Δt, we have:

$$-\frac{\partial V}{\partial t}=\underset{u(t)}{Min}\left\{g(x(t),u(t),t)+\frac{\partial V}{\partial x}a(x(t),u(t),t)\right\} \tag{21}$$

This nonlinear time-variant differential equation called “H.J.B equation”. We have the following boundary condition:

$$J^*(x(t_f),t_f)=V(x(t_f),t_f)=h(x(t_f),t_f) \tag{22}$$

By introducing the Hamiltonian function as follows:

$$H(x,u,V_x,t)=g(x,u,t)+\frac{\partial V}{\partial x}a(x,u,t)$$

We have:

$$H(x,u^*,V_x,t)=\underset{u(t)}{Min}\,H(x,u,V_x,t) \tag{23}$$

Therefore by substitution of Hamiltonian function (23) in (21), we have:

$$-\frac{\partial V}{\partial t}=H(x,u^*(x,V_x,t),V_x,t) \tag{24}$$

3.1. Example

Consider the following system:

$$\dot{x}(t)=x(t)+u(t)$$

Suppose that we consider the following cost function for this system:

$$J=\frac{1}{4}x^2(T)+\int_0^T\frac{1}{4}u^2(t)dt$$

Corresponding Hamiltonian function will be:

$$H(x,u,V_x,t)=\frac{1}{4}u^2(t)+V_x(x,t)[x+u]$$

For finding u^*, we have:

$$\frac{\partial H}{\partial u}=\frac{1}{2}u+V_x(x,t)=0$$

Therefore we obtain:

$$u^*=-2V_x(x,t)$$

Because $\frac{\partial^2 H}{\partial u^2}=\frac{1}{2}>0$, u^* is a minimum and acceptable. Now, by substitution u^* in HJB equation, we have the following equation:

$$-V_t=-V_x^2+V_x x \quad ; \quad V(x(T),T)=\frac{1}{4}x^2(T)$$

Our goal in section 6 is solving this equation using the Adomian Decomposition method and then finding optimal control signal u^*.

4. Adomian Decomposition Method

In this section we have a brief description of Adomian method. Suppose that we have a nonlinear differential equation in the form of [1]:

$$Lu + Ru + Nu = g(x) \qquad (25)$$

Where L is the highest order derivative which assumed to be easily invertible, R the linear differential operator of less order than L, Nu represents the nonlinear parts and g is the input part. Using inverse operator L^{-1} to both side of (25), we obtain:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \qquad (26)$$

$f(x)$ will be produced after integration from $g(x)$ and using given initial conditions. In this regard, the nonlinear operator $N(u) = F(u)$ is usually represented by an infinite series of the so-called Adomian polynomials as follows:

$$F(u) = \sum_{j=0}^{\infty} A_j \qquad (27)$$

The polynomials A_j are produced for all of nonlinearities so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The modified decomposition method defines the solution $u(x)$ by the series $u = \sum_{n=0}^{\infty} u_n$, that components $u_0, u_1, u_2, \ldots$ are usually determined recursively from the following equations:

$$\begin{cases} u_0 = f_0(x) \\ u_{i+1} = f_{i+1}(x) - L^{-1}(Ru) - L^{-1}(A_i) \end{cases} \qquad (28)$$

And $f(x)$ can be expressed in the Taylor series $f(x) = \sum_{k=0}^{\infty} f_k(x)$. We have:

$$\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1\left(\frac{d}{du_0}\right) f(u_0) \\ A_2 &= u_2 \frac{d}{du_0} f(u_0) + \left(\frac{u_1^2}{2!}\right)\left(\frac{d^2}{du_0^2}\right) f(u_0) \\ A_3 &= u_3\left(\frac{d}{du_0}\right) f(u_0) \\ &+ u_1 u_2\left(\frac{d^2}{du_0^2}\right) f(u_0) + \left(\frac{u_1^3}{3!}\right)\left(\frac{d^3}{du_0^3}\right) f(u_0) \end{aligned} \qquad (29)$$

And so on.

4.1. Example

Consider the following nonlinear differential equation [1]:

$$\frac{du}{dt} - u^2 = 0 \quad , \quad u(0) = 1$$

We use Adomian method for this problem. It is found that:

$$u = \sum_{n=0}^{\infty} u_n = u(0) + L^{-1} \sum_{n=0}^{\infty} A_n$$

$$u_0 = u(0) = 1$$

$$u_1 = L^{-1}(1) = t$$

$$u_2 = t^2$$

$$u_3 = t^3$$

and $u = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ is the exact solution.

5. Brief Description of Method

In this section we describe the application of Adomian method for solving Riccati and HJB equations. In 5.1, we describe method in linear optimal control case and in 5.2; nonlinear optimal control case will be considered.

5.1. Linear Optimal Control Case

According to (11), we have a nonlinear matrix equation with two solutions. The positive definite solution is acceptable. First, we introduce variables $\tau = t_f - t$ and $B = B_u R^{-1} B_u^T$. Then we have:

$$\begin{cases} \dot{P}(\tau) = P(\tau)A + A^T P(\tau) + Q - P(\tau)BP(\tau) \\ P(0) = H \end{cases} \qquad (30)$$

Let now $L = \frac{d}{d\tau}$, so we have $LP = \dot{P}$ and $NP = PBP$, that N is the nonlinear operator. With substitution of mentioned parameters, (30) becomes:

$$\dot{P} = PA + A^T P + Q - NP \qquad (31)$$

And in terms of inverse operator $L^{-1} = \int_0^{\tau} [.] dx$:

$$P = L^{-1}(PA + A^T P) + L^{-1}Q - L^{-1}NP \qquad (32)$$

Now, we can apply the Adomian decomposition method mentioned in the previous section to (32) and find the solution. Suppose that solution is $P = \sum_{n=0}^{\infty} P_n$ and writing nonlinear part in the form of the below Adomian polynomials:

$$NP = \sum_{n=0}^{\infty} A_n \qquad (33)$$

Therefore (32) becomes:

$$\begin{aligned} \sum_{n=0}^{\infty} P_n &= H + L^{-1}\Big(\sum_{n=0}^{\infty} P_n A + \\ & A^T \sum_{n=0}^{\infty} P_n\Big) + L^{-1}Q - L^{-1} \sum_{n=0}^{\infty} A_n \end{aligned} \qquad (34)$$

Thus we can find the components of solution (P_n) as:

$$
\begin{aligned}
P_0 &= H + L^{-1}Q \\
P_1 &= L^{-1}(P_0 A + A^T P_0) - L^{-1}A_0 \\
P_2 &= L^{-1}(P_1 A + A^T P_1) - L^{-1}A_1 \\
&. \\
P_n &= L^{-1}(P_{n-1}A + A^T P_{n-1}) - L^{-1}A_{n-1}
\end{aligned}
\tag{35}
$$

Now, we shall produce A_n polynomials for completion of method. There is a step by step method for finding A_n as:

$$
\begin{aligned}
A_0 &= P_0 B P_0 \\
A_1 &= P_1 B P_0 + P_0 B P_1 \\
A_2 &= P_2 B P_0 + P_1 B P_1 + P_0 B P_2 \\
&. \\
A_n &= \sum_{i=0}^{n} P_i B P_{n-i}, \qquad n \ge 0
\end{aligned}
\tag{36}
$$

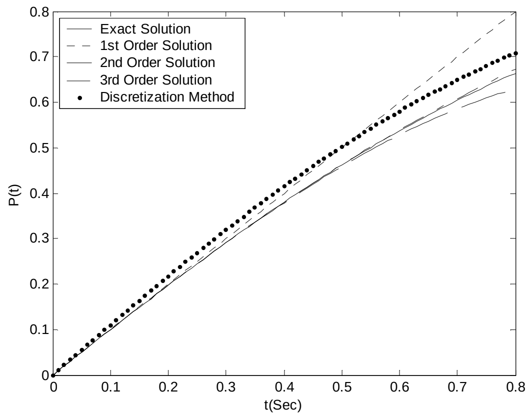

Fig. 1. Exact and approximated solutions of Riccati equation mentioned in example 5-1-1

Thus with substitution (36) in (35) we have the following step by step equations:

$$
\begin{aligned}
P_0 &= H + L^{-1}Q \\
P_1 &= L^{-1}(P_0 A + A^T P_0) - L^{-1}P_0 B P_0 \\
P_2 &= L^{-1}(P_1 A + A^T P_1) - L^{-1}\{P_0 B P_1 + P_1 B P_0\} \\
&. \\
P_n &= L^{-1}(P_{n-1}A + A^T P_{n-1}) - L^{-1}\left\{\sum_{i=0}^{n-1} P_i B P_{n-i}\right\}, \quad n \ge 1
\end{aligned}
\tag{37}
$$

Therefore with computing partial solution P_n and calculation of the sum of them, we can find approximated response with a desirable accuracy. It is clear that when we use more terms of partial solution, the obtained response is more accurate. Now, we use the mentioned algorithm for a typical example.

5.1.1. Example

Consider the following Ricatti equation:

$$
\begin{cases} \dot{P}(t) = -P^2(t) + 1 \\ P(0) = 0 \end{cases}
$$

In this case we have:

$A = 1, B = -1, Q = 1, H = 0$

Exact solution of this equation is:

$$
P(t) = \frac{e^{2t} - 1}{e^{2t} + 1}
$$

If we use Adomian decomposition method for this equation, we have:

$$
\begin{aligned}
P_0 &= H + L^{-1}(Q) = t \\
P_1 &= L^{-1}(P_0 A + A^T P_0) + L^{-1}P_0 B P_0 = L^{-1}(-t^2) = -\frac{1}{3}t^3 \\
P_2 &= L^{-1}(P_1 A + A^T P_1) + L^{-1}(P_1 B P_0 + P_0 B P_1) = \frac{2}{15}t^5
\end{aligned}
$$

Therefore we consider $\Phi_n = \sum_{i=0}^{n-1} P_i, \quad n \ge 1$ as partial solution of Riccati equation. So:

$$
\begin{aligned}
\Phi_1 &= t \\
\Phi_2 &= t + t^2 - \frac{1}{3}t^3 \\
\Phi_3 &= t + t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5
\end{aligned}
$$

We plot exact and approximated solutions of the Riccati equation in figure 1. Also the approximated solution of this equation with the discretization method is plotted in this figure. From fig.1, it is clear that by increasing the number of Adomian partial solutions, accuracy of solution increases.

Now we calculate error of solutions in $t = 0.8Sec$ for comparing them. We have absolute error in different cases as follows:

- Case1: 0.1360
- Case 2: 0.0347
- Case 3: 0.009
- Case 4(Discretization Method): 0.0443

This confirms that by increasing the partial sum of the solution, error reduces. Also it is considered that the Adomian method is better than the discretization method. Because, error of approximation in cases 2 and 3 is less than error of discretization method. Now we consider another example for this purpose.

5.1.2. Example

Consider the following Ricatti equation:

$$
\begin{cases} \dot{P}(t) = 2P(t) - P^2(t) + 1 \\ P(0) = 0 \end{cases}
$$

In this case we have:

$A = 1, B = -1, Q = 1, H = 0$

Exact solution of this equation is:

$$
P(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))
$$

If we use Adomian decomposition method for this equation, we have:

$$
\begin{aligned}
P_0 &= H + L^{-1}(Q) = t \\
P_1 &= L^{-1}(P_0 A + A^T P_0) + L^{-1}P_0 B P_0 = L^{-1}(2t) + L^{-1}(-t^2) = t^2 - \frac{1}{3}t^3 \\
P_2 &= L^{-1}(P_1 A + A^T P_1) + L^{-1}(P_1 B P_0 + P_0 B P_1) = \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5
\end{aligned}
$$

Therefore we consider $\Phi_n = \sum_{i=0}^{n-1} P_i, \quad n \geq 1$ as partial solution of Riccati equation. So:

$$\Phi_1 = t$$

$$\Phi_2 = t + t^2 - \frac{1}{3}t^3$$

$$\Phi_3 = t + t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5$$

We plot exact and approximated solutions of the Riccati equation in Fig. 2. Also the approximated solution of this equation with discretization method is plotted in this figure.

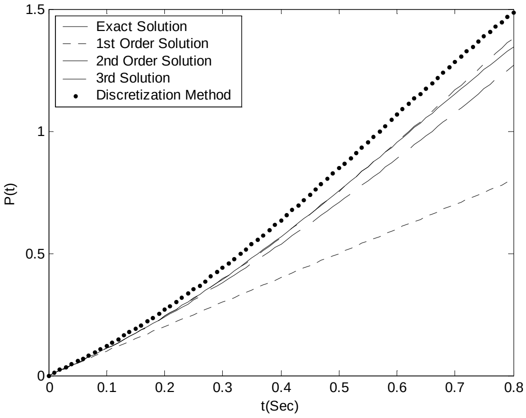

Fig. 2. Exact and approximated solutions of Riccati equation mentioned in example 5-1-2

From Fig. 2 it is clear that accuracy of the solution increases with using more number of Adomian partial solutions in our response. Now, we calculate absolute error of solution in $t = 0.8Sec$ for comparing them. We have absolute error in different cases as follows:

- Case1: 0.5464
- Case 2: 0.0771
- Case 3: 0.0349
- Case 4(Discretization Method): 0.1402

This confirms that by increasing partial sum of solution, error reduces. Also error in cases 2 and 3 is less than relative error in the discretization method. Therefore the Adomian method has a better accuracy than discretization method.

Therefore, we used introduced method for solving the Riccati equation and considered application of it. Also the effect of the number of partial solutions is considered in accuracy of the solution.

5.1.3. Example

Consider the following state space system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Suppose that relevant weight matrixes in riccati equation are:

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; Q = R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore with using proposed method in section 5, we have the following results:

$$P_0 = H + L^{-1}(Q) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

$$P_1 = L^{-1}(P_0 A + A^T P_0) + L^{-1} P_0 B P_0 = \begin{pmatrix} 0 & 2t^2 \\ 2t^2 & 0 \end{pmatrix}$$

$$P_2 = L^{-1}(P_1 A + A^T P_1) + L^{-1}(P_1 B P_0 + P_0 B P_1)$$

$$= \begin{pmatrix} 4t^3 & \frac{10}{3}t^3 + \frac{3}{4}t^4 \\ \frac{10}{3}t^3 + \frac{3}{4}t^4 & 4t^3 + \frac{4}{3}t^4 + \frac{2}{15}t^5 \end{pmatrix}$$

And so on.

Therefore we consider $\Phi_n = \sum_{i=0}^{n-1} P_i, \quad n \geq 1$ as partial solution of Riccati equation. So:

$$\Phi_0 = P_0 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

$$\Phi_1 = P_0 + P_1 = \begin{pmatrix} t & 2t^2 \\ 2t^2 & t + 2t^2 + \frac{1}{3}t^3 \end{pmatrix}$$

$$\Phi_2 = P_0 + P_1 + P_2$$

$$= \begin{pmatrix} t + 4t^3 & 2t^2 + \frac{10}{3}t^3 + \frac{3}{4}t^4 \\ 2t^2 + \frac{10}{3}t^3 + \frac{3}{4}t^4 & t + 2t^2 + \frac{13}{3}t^3 + \frac{4}{3}t^4 + \frac{2}{15}t^5 \end{pmatrix}$$

We plot exact and approximated solution of riccati equation ($P_{22}(t)$) in Fig. 3. Also eigen values of exact and approximated solution are plotted in Fig. 4 and 5. From Fig. 3, it is clear that accuracy of solution increased by adding more terms of adomian's polynomials. Also from Fig.4 and 5, it is considered that eigen values of approximated solution of Riccati equation tend to eigen values of exact solution of this equation by increasing terms of adomian's polynomials in solution. The values of exact and approximated solutions of Riccati equation in $t = 0.2Sec$ are:

$$P\big|_{t=0.2Sec} = \begin{pmatrix} 0.2484 & 0.1230 \\ 0.1230 & 0.3326 \end{pmatrix}$$

$$\Phi_1\big|_{t=0.2Sec} = \begin{pmatrix} 0.2000 & 0.0800 \\ 0.0800 & 0.2827 \end{pmatrix}$$

$$\Phi_2\big|_{t=0.2Sec} = \begin{pmatrix} 0.2320 & 0.1079 \\ 0.1079 & 0.3168 \end{pmatrix}$$

And values of eigen values of exact and approximated solutions of Riccati equation in $t = 0.2Sec$ are:

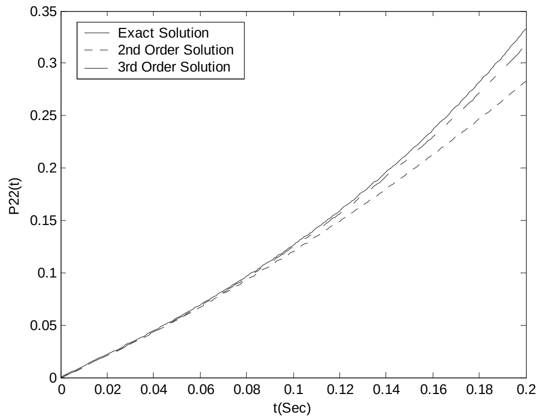

Fig. 3. Exact and approximated solutions of Riccati equation mentioned in example 5-1-3

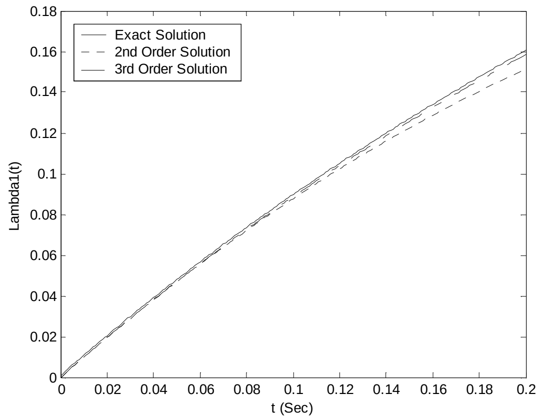

Fig. 4. Exact and approximated 1st eigen value of Riccati equation solution mentioned in example 5-1-3

$$\lambda(P(t))\big|_{t=0.2\,Sec} = \begin{pmatrix} 0.1609 \\ 0.4209 \end{pmatrix}$$

$$\lambda(\Phi_1(t))\big|_{t=0.2\,Sec} = \begin{pmatrix} 0.1513 \\ 0.3314 \end{pmatrix}$$

$$\lambda(\Phi_2(t))\big|_{t=0.2\,Sec} = \begin{pmatrix} 0.1585 \\ 0.3903 \end{pmatrix}$$

This is clear that by increasing the order of approximation, we have a more accurate solution.

5.2. Nonlinear Optimal Control Case

According to (24) we have a nonlinear, time variant differential equation regard to $V(x,t)$. First, we introduce variable $\tau = t_f - t$. Therefore, we have:

$$\frac{\partial V}{\partial \tau} = H(x, u^*(x, V_x, \tau), V_x, \tau) \qquad (38)$$

and consequently, initial condition will be as follows:

$$J^*(x(0),0) = V(x(0),0) = h(x(0),0) \qquad (39)$$

Let now $L_\tau = \dfrac{\partial}{\partial \tau}$, so we have:

$$L_\tau V = H(x, u^*(x, V_x, \tau), V_x, \tau) \qquad (40)$$

If we find u^* from (23), we can suppose that:

$$u^*(x, V_x, \tau) = f(V_x, x, \tau) \qquad (41)$$

With substitution (41) to (40), we have:

$$L_\tau V = H(x, f(x, V_x, \tau), V_x, \tau) \qquad (42)$$

Therefore, we can rewrite (42) as follows:

$$L_\tau V = R(V_x) + N(V_x) \qquad (43)$$

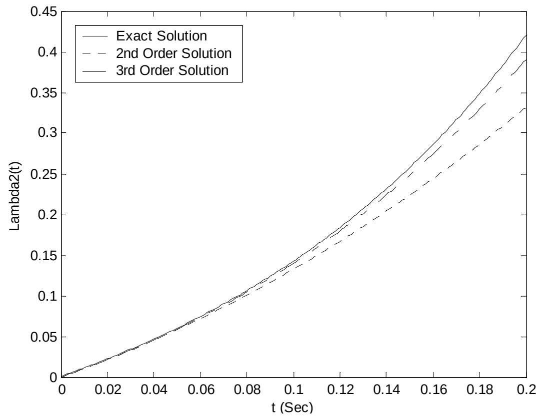

Fig. 5. Exact and approximated 2nd eigen value of Riccati equation solution mentioned in example 5-1-3

That $R(V_x)$ is linear part and $N(V_x)$ is nonlinear part of $H(x, f(x, V_x, \tau), V_x, \tau)$. We apply inverse operator $L_\tau^{-1} = \int_0^\tau [.] d\tau$ to both side of (43). Therefore, we have:

$$L_\tau^{-1} L_\tau V = L_\tau^{-1} R(V_x) + L_\tau^{-1} N(V_x) \qquad (44)$$

Then, we consider $V(x,\tau)$ as follows:

$$V(x,\tau) = \sum_0^\infty V_n \qquad (45)$$

Also, the nonlinear term of (44) will be equate to $\sum_0^\infty A_n$, that A_n are Adomian polynomial and shall be computed according to nonlinear part format. By substitution of mentioned terms in (44), we have:

$$\sum_0^\infty V_n = V_0 + L_\tau^{-1} L_x R(\sum_0^\infty V_n) + L_\tau^{-1}(\sum_0^\infty A_n) \qquad (46)$$

In (46), V_0 identified as follows:

$$V_0 = h(x(0),0) \qquad (47)$$

By using (28), we can compute V_i as follows:

$$\begin{cases} V_1 = L_\tau^{-1} L_x R V_0 + L_\tau^{-1} A_0 \\ V_2 = L_\tau^{-1} L_x R V_1 + L_\tau^{-1} A_1 \\ V_3 = L_\tau^{-1} L_x R V_2 + L_\tau^{-1} A_2 \\ . \\ . \\ V_{n+1} = L_\tau^{-1} L_x R V_n + L_\tau^{-1} A_n \end{cases} \qquad (48)$$

And Adomian polynomial A_n can be compute as follows:

$$\begin{aligned}
&A_0 = N(V_{0x})\\
&A_1 = V_{1x}(\frac{d}{dV_{0x}})N(V_{0x})\\
&A_2 = V_{2x}\frac{d}{dV_{0x}}N(V_{0x}) + (\frac{V_{1x}^2}{2!})(\frac{d^2}{dV_{0x}^2})N(V_{0x})\\
&A_3 = V_{3x}(\frac{d}{dV_{0x}})N(V_{0x}) +\\
&V_{1x}V_{2x}(\frac{d^2}{dV_{0x}^2})N(V_{0x}) + (\frac{V_{1x}^3}{3!})(\frac{d^3}{dV_{0x}^3})N(V_{0x})
\end{aligned} \tag{49}$$

And so on.

Therefore, $V_n(x,\tau) = \sum_{i=0}^{n} V_i$ is partial solution of HJB equation and we can improve accuracy of solution by increasing the number of partial solution.

5.2.1. Example

Suppose that we have the following HJB equation:

$$\begin{cases} V_t = x^2 - (1/4)(V_x)^2 \\ V(x,0) = 0 \end{cases}$$

If we use mentioned method for this equation, we have:

$$L_t V = x^2 - (1/4)(V_x)^2$$

And then, we have:

$$L_t^{-1} L_t V = L_t^{-1} x^2 - L_t^{-1}(1/4)(V_x)^2$$

Since the left side is $V(x,t) - V(x,0) = V(x,t)$, we have:

$$\sum_{n=0}^{\infty} V_n = V_0 - (1/4)L_t^{-1}\sum_{i=0}^{\infty} A_n$$

We let $V_n(x,t) = \sum_{i=0}^{n} V_i$, that $V_0 = L_t^{-1}x^2 = x^2 t$. It is clear that nonlinear part is $(V_x)^2$. Therefore, we can compute Adomian polynomials using (49) as follows:

$$\begin{aligned}
&A_0 = V_0^2\\
&A_1 = 2V_0V_1\\
&A_2 = V_1^2 + 2V_0V_2
\end{aligned}$$

And so on. Consequently, we have:

$$\begin{aligned}
&V_1 = -(1/4)L_t^{-1}(V_{0x})^2 = -(1/4)L_t^{-1}(4x^2t^2) = -x^2t^3/3\\
&V_2 = -(1/4)L_t^{-1}(2V_{0x}V_{1x}) = (2/15)x^2t^5\\
&V_3 = -(1/4)L_t^{-1}(V_{1x}^2 + 2V_{0x}V_{2x})
\end{aligned}$$

So that we can compute closed form for solution as follows:

$$\begin{aligned}
&V(x,t) = x^2(t - t^3/3 + 2t^5/15 - \dots)\\
&V(x,t) = x^2 \tanh t, \qquad |t| < \pi/2
\end{aligned}$$

And therefore we calculated solution of HJB equation with using mentioned method.

6. Conclusion

We introduced a new method for solving Riccati and HJB equations using Adomian Decomposition method. It was considered that increasing in number of partial solutions causes decreasing in error of approximated solution. For future works, we can use this method with some modification for solving HJI equations. Also we can use this method for analyzing singular perturbation systems.

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