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Application of Optimization Algorithm to Nonlinear Fractional Optimal Control Problems

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Abstract

In this study, an optimization algorithm based on the generalized Laguerre polynomials (GLPs) as the basis functions and the Lagrange multipliers is presented to obtain approximate solution of nonlinear fractional optimal control problems. The Caputo fractional derivatives of GLPs is constructed. The operational matrices of the Caputo and ordinary derivatives are introduced. The established scheme transforms obtaining the solution of such problems into finding the solution of algebraic systems of equations by approximating the state and control variables using the mentioned basis functions. The method is very accurate and is computationally very attractive. Examples are included to provide the capacity of the proposal method.

Keywords: Generalized Laguerre polynomials; Nonlinear fractional optimal control problems; Optimization algorithm; Operational matrix; Coefficients and parameters.

1.Introduction

Optimal control problems (OCPs) have recently been investigated in few studies. Postavaru and Toma [1] presented a computational method based on the fractional-order hybrid of block-pulse functions and Bernoulli polynomials for solving Fractional optimal control problems (FOCPs). Heydari and Razzaghi [2] considered the piecewise Chebyshev cardinal functions as an appropriate family of basis functions to construct a numerical method for solving a category of FOCPs.

Tricaud and Chen [3] introduced rational approximation for solving a wide class of FOCPs. Li et al. [4] investigated a spectral Petrov-Galerkin method for an OCPs governed by a two-sided space-fractional diffusion-advection-reaction equation. Wang et al. [5] used linear conforming finite element method in space and piecewise constant discontinuous Galerkin method in time for a control constrained distributed OCPs subject to a time fractional diffusion equation with non-smooth initial data. Kheyrinataj and Nazemi [6] described an artificial intelligence approach using neural networks to solve a class of delay OCPs of fractional order with equality and inequality constraints. Hoseini et al. [7] applied an approximate technique based on fractional shifted Vieta-Fibonacci functions for solving a type of FOCPs. Mohammadi and Hassani [8] used generalized polynomials for solving two-dimensional variable-order FOCPs. Zaky [9] applied a Legendre collocation method for distributed-order FOCPs. Lima [10] investigated the solution of FOCPs by using the orthogonal collocation method and the multi-objective optimization stochastic fractal search algorithm. Fakharian and Hamidi Beheshti [11] used Adomian decomposition method for solving linear and nonlinear OCPs. Hadizadeh and Amiraslani [12] constructed a numerical algorithm based on Adomian decomposition method for the nonlinear feedback operators for the time-variant optimal control with nonquadratic criteria. Fakharian et al. [13] applied Adomian decomposition method to solve the Hamilton-Jacobi-Bellman equation arising in nonlinear optimal problem. Phuong Dong et al. [14] presented a general formulation for the OCPs to

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a class of fuzzy fractional differential systems relating to SIR and SEIR epidemic models. Also, they investigated these epidemic models in the uncertain environment of fuzzy numbers with the rate of change expressed by granular Caputo fuzzy fractional derivatives of order $\beta\epsilon[0,1]$. Li et al. [15] investigated a sensitivity analysis of OCPs for a class of systems described by nonlinear fractional evolution inclusions on Banach spaces. Nemati et al. [16] applied the Ritz spectral method to solve a class of FOCPs. The developed numerical procedure is based on the function approximation by the Bernstein polynomials along with fractional operational matrix usage. Ghanbari and Razzaghi [17] introduced an alternative numerical method based on fractional-order Chebyshev wavelets for solving variable-order FOCPs. Marzban [18] provided a new framework based on a hybrid of block-pulse functions and Legendre polynomials for the numerical examination of a special class of scalar nonlinear FOCPs involving delay. Rezazadeh and Avazzadeh [19] formulated a numerical method based on using shifted discrete Legendre polynomials and collocations method to approximate the solution of two-dimensional OCPs with a fractional parabolic partial differential equation constraint in the Caputo type. Hassani et al. [20] proposed hybrid method based on the transcendental Bernstein series and the generalized shifted Chebyshev polynomials for two dimensional nonlinear variable order FOCPs.

The optimization method plays a significant role in signal and image processing, control theory, physics, engineering, chemistry and mathematics. Heydari and Atangana [21] proposed an optimization scheme based on the Lagrange multipliers scheme for solving variable-order space-time mobile-immobile advection-dispersion equation involving derivatives with non-singular kernels. Pakdaman et al. [22] approximated the solution of fractional differential equations by using the fundamental properties of artificial neural networks for function approximation. Soradi-Zeid [23] introduced an optimization algorithm, called King, for solving variable order FOCPs. Heydari and Avazzadeh [24] applied an optimization method through the Legendre wavelets for solving variable-order fractional Poisson equation. Dehestani et al. [25] used fractional-Lucas optimization method for evaluating the approximate solution of the multi-dimensional fractional differential equations. S M et al. [26] introduced an optimization-based physics-informed neural network scheme for solving fractional differential equations. Hassani et al. [27] solved the nonlinear systems of fractional-order partial differential equations using an optimization technique based on generalized polynomials. Dahaghin and Hassani [28] proposed an optimization method based on the generalized polynomials for nonlinear variable-order time fractional diffusion-wave equation. Hassani et al. [29] proposed an optimization method standing on a basis formed by the transcendental Bernstein series for solving nonlinear variable-order fractional functional boundary value problems. Alam Khan et al. [30] used bat optimization algorithm for computing the approximate solution of fractional order Helmholtz equation, with Dirichlet boundary conditions. Idiri et al. [31] used the parametric optimization method to find optimal control laws for fractional systems. Kheyrinataj and Nazemi [32] applied fractional Chebyshev functional link neural network-optimization method for solving delay FOCPs with Atangana-Baleanu derivative.

In the current paper we focus on a class of FOCPs with the Caputo fractional derivative in a dynamical system and propose a new direct computational method based on the new families of basis functions namely Generalized Laguerre polynomials (GLPs) to obtain an approximate solution for them. The problem formulation is as follows:

$$\min \mathcal{J}[w] = \int_{\circ}^{1} \mathscr{F}(t, v(t), w(t))dt \tag{1}$$

with the fractional dynamical system:

$$ {}_{0}^{C}D_{t}^{\vartheta} v(t) = \mathscr{G}(t, v(t), w(t)), \quad q-1 < \vartheta < q,\ t\epsilon[0,1], \tag{2}$$

and the initial conditions:

$$\begin{aligned} &v(0) = a_0, \\ &v'(0) = a_1, \dots, v^{(q-1)}(0) = a_{q-1}, \end{aligned} \tag{3}$$

where q is a positive integer, a_j for $j = 0, 1, \dots, q-1$ are real constants, $\mathscr{F}$and $\mathscr{G}$are continuous functions, assuming that the function $\mathscr{F}(t, v(t), w(t))$ is considered bounded below and ${}_{0}^{C}D_{t}^{\vartheta} v(t)$ denotes the fractional derivative of order ϑ in the Caputo sense.

The present paper is devoted to proposing an optimization algorithm based on new basis functions, generalized Laguerre polynomials (GLPs), for solving a family of FOCPs. First, the functions $v(t)$ and $w(t)$ are approximated based on the GLPs in terms of the unknown coefficients and parameters. We generate a new operational matrices with the help of GLPs and the Caputo derivative. Then, by substituting the estimated values of $v(t)$ and $w(t)$ into the cost functional $\mathcal{J}$, an algebraic equation in terms of the unknown coefficients and parameters is achieved to be optimized. Imposing the necessary condition for optimality on the mentioned equation, a system of algebraic equations is obtained. By solving this system, the unknown coefficients are calculated. The method is illustrated by means of some examples and the numerical approximations compared with the analytical solutions. The method is simple to implement and accurate.

The rest of the paper is organized as follows. In the next section, a brief overview of the fractional calculus, some basic definitions of Caputo fractional derivative, Laguerre polynomials (LPs), GLPs, function approximation and convergence analysis are given. In Section 3, an optimization algorithm is offered for solving FOCPs. Section 4 is devoted to several examples to display the applicability and the efficiency the proposed method. Section 5 provides the conclusion.

2.Fundamental Definition

Here, we provide a brief review of fractional calculus, LPs and GLPs which shall be used in the proposed scheme.

2.1.Caputo Fractional Derivative

Definition 2.1. (see $[33-35]$ and references there in) Let $u(t)$ be differentiable function, and let $\vartheta_i \epsilon (n-1, n]$ be the order of derivative. Then Caputo fractional derivative is defined as follows:

$$ {}_{0}^{C}D_{t}^{\vartheta} u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\vartheta_i)} \displaystyle\int_{\circ}^{t} (t-\zeta)^{(n-1-\vartheta_i)} u^{(n)}(\zeta) d\zeta, & n-1 < \vartheta_i < n, \\ \dfrac{d^n u(t)}{dt^n} & \vartheta_i = n, \end{cases} \quad (4)$$

where $\Gamma(.)$ denotes the gamma function.

Collary 2.2. From the definition above for $k \epsilon \mathbb{N} \cup \{0\}$, we have

$$ {}_{0}^{C}D_{t}^{\vartheta} t^k = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k-\vartheta_i+1)} t^{k-\vartheta_i} & n \leq k, \\ 0 & n > k, \end{cases} \quad (5)$$

Where $\vartheta_i \epsilon (n-1, n]$.

2.2. Laguerre Polynomials

Definition 2.3. (see $[36-38]$ and references there in) The LPs, $L_n(t)$, are the solutions to second order linear differential equation of $ty'' + (1-t)y' + ny = 0$, $n \epsilon \mathbb{N}$.

Definition 2.4. (see $[36-38]$ and references there in) The representation of power series for LPs, $L_n(t)$, is provided by

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(n)!}{(k)!\,(n-k)!} t^k . \quad (6)$$

The first LPs are given by:

$$L_0(t) = 1,$$
$$L_1(t) = -t + 1,$$
$$L_2(t) = \frac{1}{2}(t^2 - 4t + 2),$$
$$L_3(t) = \frac{1}{6}(-t^3 + 9t^2 - 18t + 6).$$

In general, the considered function $u(t)$ with the first $n+1$ LPs terms is approximated such that

$$u(t) \simeq A^T B \Xi_n(t), \quad (7)$$

where

$$B = \begin{pmatrix} b_{0,0}\, b_{0,1}\, b_{0,2} \ldots b_{0,n} \\ b_{1,0}\, b_{1,1}\, b_{1,2} \ldots b_{1,n} \\ \vdots \quad \vdots \quad \vdots \; \vdots \; \vdots \\ b_{n,0} b_{n,1} b_{n,2} \ldots b_{n,n} \end{pmatrix}, \quad (8)$$

$$A^T = [a_0 \;\; a_1 \;\; \ldots \;\; a_n], \qquad \Xi_n(t) = [1 \;\; t \;\; t^2 \;\; \ldots t^n]^T.$$

and

$$b_{ij} = \begin{cases} \dfrac{(-1)^j}{j!} \dfrac{(i)!}{(j!)(i-j)!}, & i \geq j, \\ 0, & i < j. \end{cases} \quad (9)$$

2.3. Generalized Laguerre Polynomials

Definition 2.5. The GLPs, $\mathcal{L}_m(t)$, are formed with a change of variable. Accordingly, t^i is changed to $t^{i+\eta_i}$, $(i+\eta_i > 0)$, on the LPs and defined as

$$\mathcal{L}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{(m!)}{(k!)(m-k)!} t^{k+\eta_k}, \quad (10)$$

where η_k indicate control parameters. If $\eta_k = 0$, then GLPs fully coincide with classical LPs.

The expansions of the states and controls, $v(t)$ and $w(t)$, in the terms of GLPs are respectively shown in the form of matrices

$$v(t) \simeq P^T \mathscr{R}\, \square(t), \qquad w(t) \simeq Q^T \mathscr{S}\, \phi(t), \tag{11}$$

where

$$\mathscr{R} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{q,0} & r_{q,1} & r_{q,2} & \dots & r_{q,q} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{m_1,0} & r_{m_1,1} & r_{m_1,2} & \dots & r_{m_1,q} & \dots & r_{m_1,m_1} \end{pmatrix},$$

$$\mathscr{S} = \begin{pmatrix} s_{0,0} & 0 & 0 & 0 & \dots & 0 \\ s_{1,0} & s_{1,1} & 0 & 0 & \dots & 0 \\ s_{2,0} & s_{2,1} & s_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{m_2,0} & s_{m_2,1} & s_{m_2,2} & s_{m_2,3} & \dots & s_{m_2,m_2} \end{pmatrix}, \tag{12}$$

$$P^T = [p_0 \;\; p_1 \;\; \dots \;\; p_{m_1}], \qquad Q^T = [q_0 \;\; q_1 \;\; \dots \;\; q_{m_2}],$$

$$\square(t) \triangleq [v_0(t) \;\; v_1(t) \;\; \dots v_{m_1}(t)]^T, \tag{13}$$

$$\phi(t) \triangleq [\phi_0(t) \;\; \phi_1(t) \;\; \dots \phi_{m_2}(t)]^T,$$

and

$$r_{i,j} = \frac{(-1)^j}{j!}\frac{(i)!}{(j!)(i-j)!}, \quad i = 2,3,\dots,m_1, \quad j = 0,1,\dots,m_1,$$
$$s_{i,j} = \frac{(-1)^j}{j!}\frac{(i)!}{(j!)(i-j)!}, \quad i = 1,2,\dots,m_2, \quad j = 0,1,\dots,m_2, \tag{14}$$

$$v_j(t) = \begin{cases} t^j, & j = 0,1,\dots,q-1, \\ t^{j+\eta_j}, & j = q, q+1, \dots, m_1, \end{cases} \tag{15}$$

$$\phi_j(t) = t^{j+\zeta_j}, \qquad j = 0,1,\dots,m_2, \tag{16}$$

with η_j and ζ_j denoting the control parameters.

2.4. Oprational Matrices

In the literature there are lot of articles in which we can see the derivation of OMs of differentiation in the Caputo sense. In this section we will derive the OMs of fractional derivatives and ordinary derivatives based on the GLPs.

The fractional derivatives of orders $q-1 < \vartheta \leq q$, of $\square(t)$ can be written as

$${}_0^C D_t^{\vartheta} \square(t) = \mathcal{D}_t^{(\vartheta)} \square(t), \tag{17}$$

where $\mathcal{D}_t^{(\vartheta)}$ is the $(m_1+1) \times (m_1+1)$ OMs of fractional derivatives of order ϑ defined by

$$\mathcal{D}_t^{(\vartheta)} = t^{-\vartheta} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{\Gamma(q+1+\eta_q)}{\Gamma(q+1-\vartheta+\eta_q)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{\Gamma(q+2+\eta_{q+1})}{\Gamma(q+2-\vartheta+\eta_{q+1})} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{\Gamma(m_1+1+\eta_{m_1})}{\Gamma(m_1+1-\vartheta+\eta_{m_1})} \end{pmatrix}.$$

The first-order derivative of $\square(t)$ can be written as:

$$\frac{d\square(t)}{dt} = \mathcal{D}_t^{(1)} \square(t), \tag{19}$$

where the $(m_1+1) \times (m_1+1)$ matrix $\mathcal{D}_t^{(1)}$ is called the operational matrix of ordinary derivative and its elements can be computed as follows:

$$\mathcal{D}_t^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{t} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{t} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{q-1}{t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{q+1+\eta_q}{t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{m_1+\eta_{m_1}}{t} \end{pmatrix}.$$

Generally, the r-order derivative operational matrix of $\mathcal{D}$ can be expressed as follows:

$$\frac{d^r\square(t)}{dt^r} = \mathcal{D}_t^{(r)} \square(t). \tag{19}$$

2.5. Function Approximation

Let $\mathbf{X} = L^2[0,1]$ and assume that $\square(t)$, be the vector defined in Eq. (20),
$\mathbf{Y}_m = span\{v_0(t), v_1(t) \;\; \dots v_{m_1}(t)\}$ and $\tilde{x}$ be an arbitrary element in $\mathbf{X}$. Since $\mathbf{Y}_m$ is a finite dimensional vector subspace of $\mathbf{X}$, $\tilde{v}$ has a unique best approximation out of $\mathbf{Y}_m$ such as $v_0 \epsilon \mathbf{Y}_m$, that is

$$\forall\, \hat{v} \epsilon \mathbf{Y}_m, \;\; \|\tilde{v} - v_0\|_2 \leq \|\tilde{v} - \hat{v}\|_2 .$$

Since $v_0 \epsilon \mathbf{Y}_m$, there exist the unique coefficients $P^T = [p_0 \;\; p_1 \;\; \dots \;\; p_{m_1}]$, such that

$$v(t) \simeq v_0(t) = P^T \mathscr{R}\; \square(t).$$

2.6. Convergence Analysis

The following theorem will be useful in subsequent results.

Theorem 2.6. Let $f: [0,1] \to \mathbb{R}$ be a function, $f \in C^{m+1}[0,1]$ and $\sup_{0 \leq t \leq 1} |f^{(m+1)}(t)|$. If there exist $p_0, p_1, \dots, p_m \in \mathbb{R}$ and $m > 0$ such that for the function

$$v(t) \simeq P^T \mathscr{R}\; \square(t),$$

we have

$\|f-v\|_2 = dist\,(f,v)$,

Then the error bound is presented as follows:

$\|f - P^T \mathscr{R}\ \psi(t)\ \|_2 < \frac{M}{(m+1)!\,\sqrt{2m+3}}$,

where

$\|f - P^T \mathscr{R}\ \psi(t)\ \|_2 = inf\{\|f-y\ \|_2 : y \in \mathbf{Y}_m\}$.

$proof$. Let $q(t) = \sum_{i=0}^{m} \frac{f^{(i)}(0)}{i!} t^i$, in this case:

$\|f - P^T \mathscr{R}\ \psi(t)\ \|_2 \leq \|f - q\ \|_2$.

We notice that $t^k \in \mathbf{Y}_m$, which implies that $q(t) \in \mathbf{Y}_m$. Now, by the Taylor theorem for all $0 \leq t \leq 1$, we have:

$$f(t) = q(t) + \frac{f^{(m+1)}(\eta_t)}{(m+1)!} t^{m+1}, \quad 0 \leq \eta_t \leq t.$$

Therefore

$$|f(t) - q(t)| \leq \frac{M}{(m+1)!} t^{m+1},$$

and

$$\begin{aligned}\|f-q\ \|_2 &\leq \frac{M}{(m+1)!}\left(\int_0^1 t^{2m+2}\right)^{\frac{1}{2}}\\ &= \frac{M}{(m+1)!}\left(\frac{1}{2m+3}\right)^{\frac{1}{2}}\\ &= \frac{M}{(m+1)!\sqrt{2m+3}}.\end{aligned}$$

Remark 1. Note that the same results for $w(t)$ can be investigated.

3.Algorithm of the Solution for FOCPs

Now, we use the above obtained results to solve the FOCPs (1) with the fractional dynamical system (2) subject to the initial conditions (3). To this end, we approximate the state variable, i.e. $v(t)$ and the control variable, i.e. $w(t)$ by $\psi(t)$ and $\phi(t)$, respectively as follows:

$$\begin{aligned}&v(t) \simeq P^T \mathscr{R}\ \psi(t),\\ &w(t) \simeq Q^T \mathscr{S}\ \phi(t),\end{aligned} \tag{20}$$

where

$$P^T = [p_0\ \ p_1\ \ \dots\ \ p_{m_1}], \qquad Q^T = [q_0\ \ q_1\ \ \dots\ \ q_{m_2}],$$

are unknown vectors that called vectors of the free coefficients which should be computed and $\psi(t)$ and $\phi(t)$, are the vectors which defined in Eqs. (13).

At this stage, by Eq. (20), we approximate ${}_0^C D_t^{\vartheta} v(t)$ in terms of the operational matrix of fractional derivative of GLPs as follows:

$${}_0^C D_t^{\vartheta} v(t) \simeq P^T \mathscr{R} \mathcal{D}_t^{(\vartheta)}\ \psi(t). \tag{21}$$

By substituting Eqs. (20) into Eq. (1), the performance index $\mathscr{S}$ is approximated as:

$$\mathscr{S}[w] = \mathscr{S}[\mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}] = \int_0^1 \mathscr{F}(t, P^T, \mathscr{R}\psi(t), Q^T \mathscr{S}\phi(t))dt \triangleq \mathscr{F}(\mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}), \tag{22}$$

where $\mathscr{K}$ and $\mathscr{L}$ are unknown control parameters vectors with elements η_j's and ζ_j's, respectively.

Also, by substituting Eqs. (20) and (21) into the fractional dynamical system (2), we have:

$$P^T \mathscr{R} \mathcal{D}_t^{(\vartheta)}\ \psi(t) - \mathscr{G}(t, P^T, \mathscr{R}\psi(t), Q^T \mathscr{S}\phi(t)) \triangleq \tilde{\mathscr{G}}(t, \mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}) \simeq 0. \tag{23}$$

Furthermore, by taking the collocation points $t_i = \frac{i}{\hat{m}-1}$ for $i = q, q+1, \dots, \hat{m}-1$, where $\hat{m} = \min(m_1, m_2)$ into Eq. (23), we obtain the following system of algebric equations:

$$\begin{aligned}&\Lambda_i \triangleq \hat{\mathscr{G}}(t_i, \mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}) \simeq 0,\\ &i = q, q+1, \dots, \hat{m}-1.\end{aligned} \tag{24}$$

In addition by the initial conditions (3) and considering Eq. (19), we obtain the following system of algebraic equations:

$$\begin{aligned}&\Lambda_i \triangleq P^T \mathscr{R} \mathcal{D}_t^{(\vartheta)}\ \psi(t) - \mathrm{a}_i = 0,\\ &i = 0, 1, \dots, q-1.\end{aligned} \tag{25}$$

Now, assume that

$$\mathcal{J}^* = \mathscr{S}[\mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}, \lambda] = \mathcal{J}[\mathscr{R}, \mathscr{S}, \mathscr{K}, \mathscr{L}] + \Lambda\lambda, \tag{26}$$

where $\Lambda = [\Lambda_0\ \Lambda_1\ \dots\ \Lambda_{\hat{m}-1}]$ and $\lambda = [\lambda_0\ \lambda_1\ \dots\ \lambda_{\hat{m}-1}]^T$, is the unknown Lagrange multiplier.

Finally, the necessary conditions for the extermum are given by the following system of nonlinear algebraic equations:

$$\begin{aligned}&\partial\mathcal{J}^*/\partial\mathscr{R} = 0, \quad \partial\mathcal{J}^*/\partial\mathscr{S} = 0, \quad \partial\mathcal{J}^*/\partial\mathscr{K} = 0,\\ &\partial\mathcal{J}^*/\partial\mathscr{L} = 0, \quad \partial\mathcal{J}^*/\partial\lambda = 0.\end{aligned} \tag{27}$$

The above system of nonlinear algebraic equations can be solved using Maple or Matlab software packages. Finally, by determining the free coefficients and control parameters, we can determine a good approximate solutions for $v(t)$ and $w(t)$ using Eq. (20).

The algorithm of the proposed method is as follows:

Algorithm

Input: $m_1, m_2, q-1 < \vartheta \le q, a_j \ (j = 0,1,\dots,q-1)$ and the functions $\mathscr{F}$ and $\mathscr{G}$

Step 1: Define the basis functions $v_j(t)$ and $\phi_j(t)$ by Eqs. (15) and (16).

Step 2: Construct GLPs vectors □(t) and $\phi(t)$ using Eqs. (13).

Step 3: Define the unknown matrices $P^T = [p_i]_{1\times m_1} + 1$ and $Q^T = [q_i]_{1\times m_2} + 1$.

Step 4: Compute the fractional and ordinary operational matrices $\mathcal{D}_t^{(\vartheta)}$ and $\mathcal{D}_t^{(i)}$ using Eqs. (17) and (18).

Step 5: Compute the equation $\mathcal{J}[w] \triangleq \hat{\mathscr{F}}(t_i, \mathscr{R}.\mathscr{S}.\mathscr{K}.\mathscr{L})$ using (22).

Step 6: Compute the system of nonlinear algebraic equations using Eq. (23).

Step 7: Determine the free coefficients and control parameters using Eq. (27).

Output: The approximate solution: $v(t) \simeq P^T \mathscr{R}$ □(t) and $w(t) \simeq Q^T \mathscr{S}\,\phi(t)$.

4.Illustrative Test Problems

In this section, we show the performance of the proposed method. We compute the error by

$E_v(t_j) = |\, P^T \mathscr{R}$ □$(t) - v(t_j)|, \qquad t_j \in [0,1]$,

and

$E_w(t_j) = |\, Q^T \mathscr{S}\,\phi(t) - w(t_j)|, \qquad t_j \in [0,1].$

During this part, in order to show the efficiency and validity of the suggested numerical method, we solve two test problems.

Example 1. Consider the following FOCPs:

$$\min \mathcal{J}[w] = \int_0^1 [v^2(t) + w^2(t) + 2t^{\frac{3}{2}}v(t) - 2(1 - t^{\frac{3}{2}})w(t)]dt$$

subject to the fractional dynamical system:

$${}_0^C D_t^{\frac{3}{2}} v(t) = \frac{3\sqrt{\pi}}{4}\big(v(t) - w(t)\big), \qquad 1 < \vartheta \le 2,$$

and the initial conditions $v(0) = v'(0) = 0$.

The problem's analytical solution is given by $v(t) = -t^{\frac{3}{2}}$ and $w(t) = 1 - t^{\frac{3}{2}}$. The minimum value of the objective index $\mathcal{J}$ is attained at $\mathcal{J} = -0.7$. To solve this problem, we use the proposed method with different values of m_1 and m_2. The approximate values for the performance index J are summarized in Table 1. The runtime for the proposed method at different values of m_1 and m_2 are reported in Table 2. The absolute error values for the states and controls are reported in Table 3 with $m_1 = m_2 = 3$. Plots of the absolute errors are shown in Fig. 1 and 2 for the states and controls with $m_1 = m_2 = 3$.

Example 2. Consider the following FOCPs:

$$\min \mathcal{J}[w] = \int_0^1 \left[\left(v(t) - t^{\frac{5}{2}}\right)^4 (1 + t^2)\left(w(t) + t^6 - \frac{15\sqrt{\pi}}{8\Gamma\left(\frac{7}{2} - \vartheta\right)}\, t^{\frac{5}{2} - \vartheta}\right)^2\right] dt,$$

subject to the fractional dynamical system:

$${}_0^C D_t^{\vartheta} v(t) = t v^2(t) + w(t), \qquad 1 < \vartheta \le 2.$$

and the initial conditions $v(0) = v'(0) = 0$.

For this problem, the values $v(t) = t^{\frac{5}{2}}$ and $w(t) = \frac{15\sqrt{\pi}}{8\Gamma\left(\frac{7}{2} - \vartheta\right)}\, t^{\frac{5}{2} - \vartheta} - t^6$ are the minimizing solutions for the state and

Table 1
The values of $\mathcal{J}$ at different alternatives of $m_i, i = 1,2$ for Example 1.

$m_i,\ i = 1,2$	$\mathcal{J}$
$m_1 = m_2 = 2$	-0.70000000
$m_1 = m_2 = 3$	-0.70000000
$m_1 = 4, \quad m_2 = 3$	-0.70000000
$m_1 = m_2 = 4$	-0.70000000

Table 2
The runtime (in seconds) of the proposed method for different values of m_1 and m_2.

parameters	$m_1 = m_2 = 2$	$m_1 = m_2 = 3$	$m_1 = 4,\ m_2 = 3$	$m_1 = m_2 = 5$
CPU time	1.16	2.74	4.27	7.61

Table 3
The $E_v\,(t_i)$, $E_w\,(t_i)$ with $m_1 = m_2 = 3$ for Example 1.

t_i	$E_v\,(t_i)$	$E_w\,(t_i)$
0.1	2.2926E − 12	1.3725E − 10
0.2	7.0964E − 12	2.6323E − 11
0.3	1.0087E − 11	4.4701E − 11
0.4	1.0386E − 11	6.6935E − 11
0.5	8.1560E − 12	5.1324E − 11
0.6	4.1211E − 12	1.2594E − 11
0.7	6.0814E − 13	3.1783E − 11
0.8	4.6274E − 12	6.1710E − 11
0.9	6.2730E − 12	5.4425E − 11

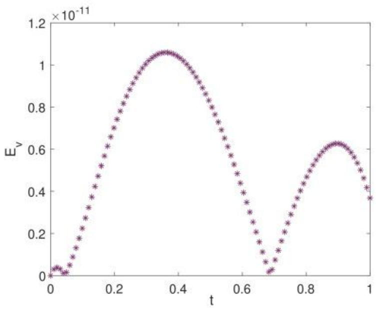

Fig. 1. The absolute error function using the GLPs for the state variable, $v(t)$, with $m_1 = m_2 = 3$ in Example 1.

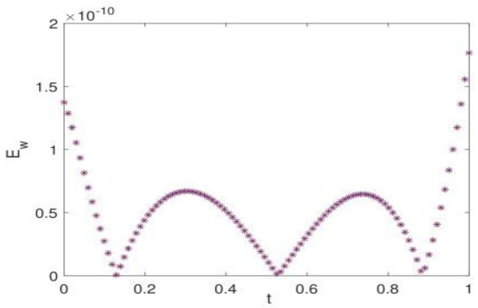

Fig. 2. The absolute error function using the GLPs for the control variable, $w(t)$, with $m_1 = m_2 = 3$ in Example 1.

control variables, respectively, and the performance index $\mathcal{J}$ has the minimum value of 0. To solve this problem, we use the proposed method with different values of m_1 and m_2 for some different ϑ. The approximate values for the performance index $\mathcal{J}$ are summarized in Table 4. The runtime for the proposed method at different values of m_1 and m_2 with $\vartheta = 1.25$ are reported in Table 5. The absolute error values for the states and controls are reported in Table 6 with $m_1 = m_2 = 3$ and $\vartheta = 1.25$. Plots of the absolute errors are shown in Fig. 3 and 4 for the states and controls with $m_1 = m_2 = 3$ and $\vartheta = 1.25$. According to the obtained numerical results as reported in the related tables and figures, it can be observed that the numerical and exact solutions are in close agreement where the accuracy of the approximate solutions is enhanced as the number of basis functions increases.

Table 4
The values of $\mathcal{J}$ at different alternatives of $m_i, i = 1,2$ and ϑ (the exact optimal value is zero), for Example 2.

$\vartheta = 1.25$		$\vartheta = 1.85$	
$m_i, i = 1,2$	$\mathcal{J}$	$m_i, i = 1,2$	$\mathcal{J}$
$m_1 = 2$, $m_2 = 3$	8.7598E − 09	$m_1 = m_2 = 2$	2.5148E − 09
$m_1 = m_2 = 3$	3.5738E − 09	$m_1 = 2$, $m_2 = 3$	7.2537E − 10
$m_1 = 3$, $m_2 = 4$	9.1533E − 10	$m_1 = 4$, $m_2 = 3$	1.9416E − 10
$m_1 = m_2 = 5$	2.4420E − 10	$m_1 = m_2 = 5$	7.5486E − 11

Table 5
The runtime (in seconds) of the proposed method with $\vartheta = 1.25$ for different values of m_1 and m_2.

parameters	$m_1 = 2$, $m_2 = 3$	$m_1 = m_2 = 3$	$m_1 = 3$, $m_2 = 4$	$m_1 = m_2 = 5$
CPU time	2.26	3.48	5.31	8.64

Table 6
The $E_v\ (t_i)$, $E_w\ (t_i)$ with $m_1 = m_2 = 3$ and $\vartheta = 1.25$ for Example 2.

t_i	$E_v\ (t_i)$	$E_w\ (t_i)$
0.1	5.3557E − 10	2.1321E − 09
0.2	1.1547E − 09	2.5265E − 09
0.3	1.2926E − 09	7.7580E − 11
0.4	9.6869E − 10	2.1829E − 09
0.5	4.0390E − 10	2.1393E − 09
0.6	1.4497E − 10	2.7203E − 10
0.7	4.7787E − 10	1.5762E − 09
0.8	5.0945E − 10	1.3657E − 09
0.9	3.0888E − 10	8.0321E − 10

Example 3. Consider the following FOCPs [39]:

$$\min \mathcal{J}[w] = \int_0^1 \left[(v(t) - t^2)^2 (1 + t^2)\left(w(t) + t^4 - \frac{2t^{2-\vartheta}}{\Gamma(3-\vartheta)}\right)^2\right] dt,$$

subject to the fractional dynamical system:

$$ {}_0^C D_t^{\vartheta} v(t) = t^2 v(t) + w(t), \qquad 1 < \vartheta \leq 2,$$

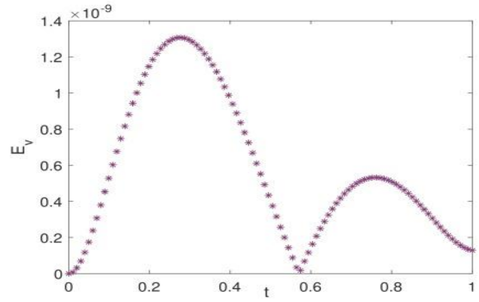

Fig. 3. The absolute error function using the GLPs for the state variable, $v(t)$, with $m_1 = m_2 = 3$ and $\vartheta = 1.25$, in Example 2.

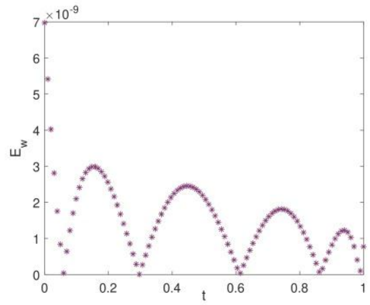

Fig. 4. The absolute error function using the GLPs for the control variable, $w(t)$, with $m_1 = m_2 = 3$ and $\vartheta = 1.25$, in Example 2.

and the initial conditions $v(0) = v'(0) = 0$.
For this problem, the values $v(t) = t^2$ and $w(t) = \frac{2t^{2-\vartheta}}{\Gamma(3-\vartheta)} - t^4$ are the minimizing solutions for the state and control variables, respectively, and the performance index $\mathcal{J}$ has the minimum value of 0. In

[39], a numerical method based on the fractional shifted Vieta-Fibonacci functions is applied to find the numerical solution of this problem. The absolute errors with $\vartheta = 1.1$ and different values for $m_1$and m_2 obtained from the GLPs method are shown in Table 7 and compared with those previously reported in [39]. Plots of the approximate solution and absolute errors are shown in Fig. 5, 6, 7 and 8 for the states and controls with $m_1 = m_2 = 6$ and $\vartheta = 1.1$, respectively. Table 7 and Figs. 5, 6, 7 and 8 indicate that highly accurate approximate solutions are provided by the GLPs method.

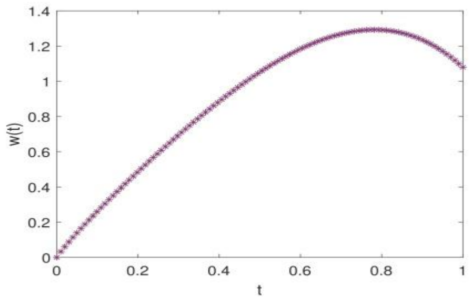

Fig. 6. The approximate solution using the GLPs for the control variable, $w(t)$, with $m_1 = m_2 = 6$ and $v = 1.1$ in Example 3.

Table 7
The values of absolute error using the GLPs method with $v = 1.1$ and the proposed method in [39] for Example 3.

t_i	$E_v\ (t_i)$ (method in [39])	$E_w\ (t_i)$ (method in [39])	$E_v\ (t_i)$ (GLPs method)	$E_w\ (t_i)$ (GLPs method)
	$N = 6$	$N = 6$	$m_1 = m_2 = 6$	$m_1 = m_2 = 6$
0.1	$1.1e-03$	$3.1e-03$	$9.8e-11$	$5.3e-13$
0.3	$8.8e-04$	$7.5e-03$	$1.4e-10$	$2.1e-12$
0.5	$6.3e-04$	$9.5e-03$	$5.2e-12$	$8.6e-12$
0.7	$2.5e-03$	$7.7e-03$	$6.5e-11$	$1.8e-11$
0.9	$3.0e-03$	$1.2e-02$	$1.0e-11$	$1.0e-12$

5. Conclusion

This study introduces a novel approach that combines the GLPs as a basis and Lagrange multipliers to solve FOCPs with the fractional dynamical system. By adopting the GLPs basis and operational matrices of fractional derivatives, the main problem was reduced to the solution of a system of algebraic equations. Two numerical examples are presented to demonstrate the effectiveness and validity of the algorithm. This method shows that with fewer number of basis functions we can obtain the approximate the solutions. The presented tables and graphs show comparisons of exact solutions and approximate solutions with errors.

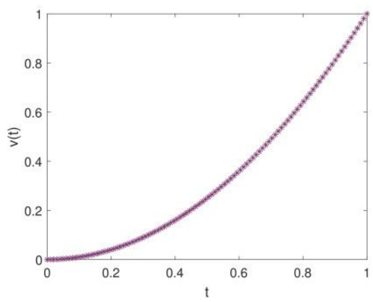

Fig. 5. The approximate solution using the GLPs for the state variable, $v(t)$, with $m_1 = m_2 = 6$ and $v = 1.1$ in Example 3.

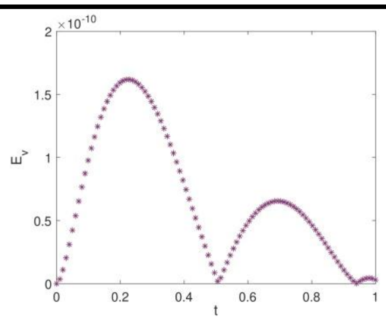

Fig. 7. The absolute error function using the GLPs for the state variable, $v(t)$, with $m_1 = m_2 = 6$ and $v = 1.1$ in Example 3.

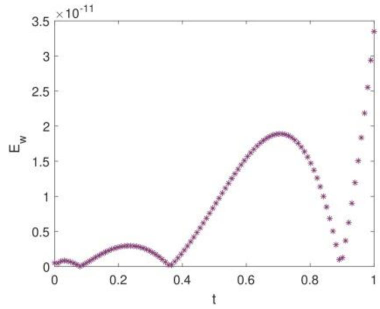

Fig. 8. The absolute error function using the GLPs for the control variable, $w(t)$, with $m_1 = m_2 = 6$ and $v = 1.1$ in Example 3.

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical approval is not applicable for this article. **Informed consent** was obtained from all authors before the article began.

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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