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Application of Numerical Iterative Methods for Solving Benjamin-Bona-Mahony Equation

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Abstract

In this paper, a generalized Benjamin-Bona-Mahony equation (BBM) is solved by using the Adomian's decomposition method (ADM), modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM) and homotopy analysis method (HAM). The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords: *Generalized Benjamin-Bona-Mahony equation, Adomian decomposition method , Modified Adomian decomposition method , Variational iteration method , Modified variational iteration method , Homotopy analysis method*.

1.Introduction

The generalized Benjamin-Bona-Mahony equation has a higher-order nonlinearity of the form

$$u_t + u_x + au^n u_x + u_{xxt} = 0, \quad n \geq 1, \tag{1}$$

where a is constant. The case $n = 1$ corresponds to the BBM equation, which was first proposed in 1972 by Benjamin et al. [5]. This equation is an alternative to the Korteweg-de Vries (KdV) equation and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The BBM equation is well known in physical applications. This equation models long wave in a nonlinear dispersive system. The solution of the BBM equation exhibits definite soliton-like behavior that is not explainable by any known theory [24]. The BBM equation is used in the analysis of the surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma, a coustic-gravity wave in compressible fluids and a coustic waves in anharmonic crystals. Where $n = 2$, the BBM equation is called the modified BBM equation (mBBM). A lot of works have been done in order to find the numerical solution of this equation. For example [37,1,7,8,17,29,38,26,39,30,19,40,31,9,27,20,12,35,25,11,41,42], variational iteration method [36,28,34], homotopy analysis method [2]. In this work, we develop the ADM, MADM, VIM, MVIM and HAM to solve the Eq.(1) with the initial conditions as follows:

$$u(x,0) = f(x), \quad u_{xx}(x,0) = g(x). \tag{2}$$

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The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. Finally, the numerical example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq.(1), by integrating one time from Eq.(1) with respect to t and using the initial conditions we obtain,

$$u(x,t) = -F(x,t) - \int_0^t D\big(u(x,t)\big)\,dt - \int_0^t H\big(u(x,t)\big)\,dt, \quad (3)$$

where,

$$D^i\big(u(x,t)\big) = \frac{\partial^i u(x,t)}{\partial x^i}, \quad i = 1,2,$$

$$F(x,t) = f(x) + g(x) + D^2\big(u(x,t)\big),$$

$$H\big(u(x,t)\big) = a u^n(x,t) D\big(u(x,t)\big).$$

In Eq.(3), we assume $F(x,t)$ is bounded for all $J=[0,T]$ $(T \in \mathbb{R})$.

The terms $D(u(x,t))$, $H(u(x,t))$ are Lipschitz continuous with $|D(u) - D(u^*)| \le L_1|u - u^*|$, $|H(u) - H(u^*)| \le L_2|u - u^*|$, and

$\alpha := T(L_1 + L_2)$,

$\beta := 1 - T(1-\alpha)$,

$\gamma := 1 - TL\,\alpha$.

2. The Iterative Methods

2.1. Description Of The MADM And ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$\mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} = \mathrm{g}_1, \quad (4)$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g_1 is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4), and using the iven conditions we obtain

$$(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \ u \quad (5)$$

where the function $f_1(x)$ represents the terms arising from integrating the source term g_1. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \quad (6)$$

where $A_n, n \ge 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}\Big[\frac{d^m}{d\lambda^n}\Big[N\Big(\sum_{i=0}^{\infty}\lambda^i u_i\Big)\Big]\Big]_{\lambda=0} \quad (7)$$

Adomian polynomials were introduced in [6,10,32] as

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0), \ldots \end{aligned} \quad (8)$$

2.1.1 Adomian decomposition method The standard

decomposition technique represents the solution of $u(x,t)$ in (4) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \quad (9)$$

where the components $u_0, u_1, \ldots$ are usually determined recursively by

$$u_0 = -F(x,t)$$

$$u_1 = -\int_0^t A_0(x,t)dt - \int_0^t B_0(x,t)dt,$$

$$u_{n+1} = -\int_0^t A_n(x,t)\,dt - \int_0^t B_n(x,t)dt, \quad n \geq 0. \tag{10}$$

Substituting (8) into (10) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (9) follows immediately.

2.1.2. The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [33]. The modified forms was established based on the assumption that the function $F(x,t)$ can be divided into two parts, namely $F_1(x,t)$ and $F_2(x,t)$. Under this assumption, we set

$$F(x,t) = F_1(x,t) + F_2(x,t) \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part F_1 be assigned to the zeroth component u_0, whereas the remaining part F_2 be combined with the other terms given in (11) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= -F_1(x,t), \\ u_1 &= -F_2(x,t) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &-L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1 \end{aligned} \tag{12}$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (12) as follows:

$$u_0 = -F_1(x,t),$$

$$\begin{aligned} u_1 &= -F_2(x,t) - \int_0^t A_0\,(x,t)dt - \int_0^t B_0\,(x,t)dt \\ &- \int_0^t A_n\,(x,t)dt - \int_0^t B_n\,(x,t)dt. \end{aligned} \tag{13}$$

The operators $D(u)$, $H(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$D(u) = \sum_{i=0}^{\infty} A_i,$$

$$H(u) = \sum_{i=0}^{\infty} B_i.$$

where A_i and B_i are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [8]:

$$\begin{aligned} A_n &= D(s_n) - \sum_{i=0}^{n-1} A_i, \\ B_n &= H(s_n) - \sum_{i=0}^{n-1} B_i. \end{aligned} \tag{14}$$

Where the partial sum is $s_n = \sum_{i=0}^{n} u_i(x,t)$.

2.2. Description of the VIM and MVIM

In the VIM [13-16], we consider the following nonlinear differential equation:

$$Lu + Nu = g_1, \tag{15}$$

where L is a linear operator, N is a nonlinear operator and $g_1$is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,)) + N(u_n(x,\tau)) - g_1(x,\tau)\}d\tau, \quad n \geq 0, \tag{16}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The

zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be

obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1), according to the VIM, we can write iteration formula (16) as follows:

$$u_{n+1}(x,t) = u_n x,t) + L_t^{-1}(\lambda[u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))dt + \int_0^t H(u_n(x,t))dt]), \quad n \geq 0. \tag{18}$$

Where,

$$L_t^{-1}(.) = \int_0^t (.)\, d\tau$$

To find the optimal λ, we proceed as

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta L_t^{-1}(\lambda[u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))\, dt + \int_0^t H(u_n(x,t))\, dt]). \tag{19}$$

From Eq.(19), the stationary conditions can be obtained as follows:

$\lambda' = 0$ and $1 + \lambda' = 0$.

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$, and by substituting in (18), the following iteration formula is obtained.

$$u_0(x,t) = -F(x,t),$$
$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}(u_n(x,t) + F(x,t) + \int_0^t D(u_n(x,t))\, dt + \int_0^t H(u_n(x,t))\, dt), \quad n \geq 0. \tag{20}$$

To obtain the approximation solution of Eq.(1), based on the MVIM [3,4,23], we can write the following iteration formula:

$$u_0(x,t) = -F(x,t), \tag{21}$$

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\int_0^t D\left(u_n(x,t) - u_{n-1}(x,t)\right) dt + \int_0^t H\left(u_n(x,t) - u_{n-1}(x,t)\right) dt\right), \quad n \geq 0.$$

Relations (20) and (21) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

2.3. Description of the HAM

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function, and x is an independent variable. let $u_n(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$.

Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \hat{H}[\phi(x,t;q); u_0(x,t)H_1(x,t), h, q]. \quad (22)$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q] = 0, \quad (23)$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \quad (24)$$

When $q = 0$, the zero-order deformation Eq.(24) becomes

$$\phi(x;0) = u_0(x,t), \quad (25)$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(24) is equivalent to

$$\phi(x,t;1) = u(x,t). \quad (26)$$

Thus, according to (25) and (26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in

homotopy [21,22].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \quad (27)$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (27) of $\phi(x,t;q)$ converges at $q = 1$; then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \quad (28)$$

From Eq.(27), we can write Eq.(24) as follows

$$(1-q)L[\phi(x,t,q)-u_0(x,t)]=(1-q)L[\sum_{m=1}^{\infty}u_m(x,t)q^m]=q\,h\,H_1(x,t)N[\phi(x,t,q)] \Rightarrow L[\sum_{m=1}^{\infty}u_m(x,t)q^m]-q\,L[\sum_{m=1}^{\infty}u_m(x,t)q^m]=q\,h\,H_1(x,t)N[\phi(x,t,q)] \quad (29)$$

By differentiating (29) m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty}u_m(x,t)q^m]-q\,L[\sum_{m=1}^{\infty}u_m(x,t)q^m]\}^m=\{q\,h\,H_1(x,t)N[\phi(x,t,q)]\}^m = m!\,L[u_m(x,t)-u_{m-1}(x,t)]=h\,H_1(x,t)\,m\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0} \quad (30)$$

Therefore,

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH_1(x,t)\Re_m(u_{m-1}(x,t)),$$

where,

$$\Re_m(u_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \quad (31)$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Note that the high-order deformation Eq.(30) is governing the nonlinear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1), according to HAM, let

$$N[u(x,t)]=u(x,t)+F(,t)+\int_0^t D(u(x,t))\,dt+\int_0^t H(u(x,t))\,dt,$$

so,

$$\Re_m(u_{m-1}(x,t))=u_{m-1}(x,t)+F(x,t)+\int_0^t D(u_{m-1}(x,t))\,dt+\int_0^t H(u_{m-1}(x,t))\,dt, \quad (32)$$

Substituting (32) into (30)

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH_1(x,t)[u_{m-1}(x,t)+\int_0^t D(u(x,t))\,dt+\int_0^t H(u(x,t))\,dt+(1-\chi_m)F(x,t)]. \quad (33)$$

We take an initial guess $u_0(x,t)=-F(x,t)$, an auxiliary nonlinear operator $Lu=u$, a nonzero auxiliary parameter $h=-1$, and auxiliary function $H_1(x,t)$. This is substituted into (33) to give the recurrence relation

$$u_0(x,t)=-F(x,t),\quad u_{n+1}(x,t)=-\int_0^t D(u_n(x,t))\,dt+\int_0^t H(u_n(x,t))\,dt,\quad n\ge 1. \quad (34)$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$$
$$=-F(x,t)$$
$$+\sum_{n=1}^{\infty}\left(-\int_0^t D\big(u_n(x,t)\big)\,dt\right.$$
$$\left.+\int_0^t H\big(u_n(x,t)\big)\right) \quad (35)$$

Which is the method of successive approximations. If

$$|\,u_n(x,t)|<1,$$

then the series solution (35) convergence uniformly.

3. Existence and convergency of iterative methods

Theorem 3.1. Let $0<\alpha<1$, then BBM equation (1), has a unique solution.

Proof. Let u and u^* be two different solutions of (3) then

$$|u-u^*|=\left|\int_0^t D\big(u(x,t)\big)\,dt - \int_0^t H\big(u(x,t)\big)\,dt\right|$$
$$\le \int_0^t |\,D\big(u(x,t)\big)-D(u^*(x,t))|\,dt + \int_0^t H\big(u(x,t)\big) - H(u^*(x,t))|dt$$
$$\le T(L_1+L_2)\,|u-u^*| = \alpha\,|u-u^*|\,.$$

From which we get $(1-\alpha)\,|u-u^*|\le 0$. Since $0<\alpha<1$, then $|u-u^*|=0$. Implies $u=u^*$ and completes the proof.

Theorem 3.2. The series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ of problem(1) using MADM convergence when $0<\alpha<1$,$|u_1(x,t)|<\infty$.

Proof. Denote as $(C[J],\|.\|)$ the Banach space of all continuous functions on J with the norm $\|f(t)\|=\max|f(t)|$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n\ge m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\|s_n-s_m\| = max_{\forall t\in J}|s_n-s_m|$$
$$= max_{\forall t\in J}\left|\sum_{i=m+1}^{n}u_i(x,t)\right|$$
$$= max_{\forall t\in J}\left|\sum_{i=m+1}^{n}(-\int_0^t A_{i-1}dt - \int_0^t B_{i-1}\,dt\right|$$
$$= max_{\forall t\in J}\left|-\int_0^t (\sum_{i=m}^{n-1}A_i)\,dt - \int_0^t \left(\sum_{i=m}^{n-1}B_i\right)dt\right|.$$

From [8], we have

$$\sum_{i=m}^{n-1}A_i = D(s_{n-1})-D(s_{m-1}),$$
$$\sum_{i=m}^{n-1}B_i = H(s_{n-1}-1)-H(s_{m-1}).$$

So,

$$\|s_n-s_m\| = max_{\forall t\in J}\left|-\int_0^t [D(s_{n-1}) - D(s_{m-1})]\,dt - \int_0^t [H(s_{n-1}) - H(s_{m-1})]\,dt\right| \le$$
$$\int_0^t |\,D(s_{n-1})-D(s_{m-1})|\,dt + \int_0^t H(s_{n-1}) - H(s_{m-1})|\,dt$$
$$\le \alpha\|s_n-s_m\|\,.$$

Let $n=m+1$, then

$$\|s_n-s_m\| \le \alpha\|s_m-s_{m-1}\|$$
$$\le \alpha^2\|s_{m-1}-s_{m-2}\| \le \cdots$$
$$\le \alpha^m\|s_1-s_0\|\,.$$

From the triangle inquality we have

$$\|s_n - s_m\| \le \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \le [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-m-1}]\|s_1 - s_0\| \le \alpha^m[1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}]\|s_1 - s_0\| \le \alpha^m \left[\frac{1-\alpha^{n-m}}{1-\alpha}\right] \|u_1(x,t)\| .$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\|s_n - s_m\| \le \frac{\alpha^m}{1-\alpha} max_{\forall t \in J} |u_1(x,t)| .$$

But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete.

Theorem 3.3. The solution $u_n(x,t)$ obtained from the relation (20) using VIM converges to the exact solution of the problem (1) when $0 < \alpha < 1$ and $0 < \beta < 1$.

Proof.

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}([u_n(x,t) + F(x,t) + \int_0^t D\big(u_n(x,t)\big)\, dt + \int_0^t H\big(u_n(x,t)\big)\ dt]) \quad (36)$$

$$u(x,t) = u(x,t) - L_t^{-1}([u(x,t) + F(x,t) + \int_0^t D\big(u(x,t)\big)\, dt + \int_0^t H(u(x,t)))\ dt]) \quad (37)$$

By subtracting relation (36) from (37),

$$u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - L_t^{-1}(u_n(x,t) - u(x,t) - \int_0^t D\big(u_n(x,t)\big) - D\big(u(x,t)\big)] - \int_0^t [H\big(u_n(x,t)\big)\, dt - H(u(x,t))\, dt]\, dt),$$

if we set, $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - |e_n(x,t^*)| = max_t\, |e_n(x,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$e_{n+1}(x,t) = e_n(x,t) + L_t^{-1}(-e_n(x,t) + \int_0^t [D\big(u_n(x,t)\big) - D\big(u(x,t)\big)]\, dt + \int_0^t [H\big(u_n(x,t)\big) - H\big(u(x,t)\big)]\, dt)$$

$$\le e_n(x,t) + L_t^{-1}[-e_n(x,t) + L_t^{-1}|e_n(x,t)|(T(L_1 + L_2)] \le e_n(x,t) - Te_n(x,\eta) + T(L_1 + L_2)L_t^{-1}\, L_t^{-1}\, |e_n(x,t)| \le (1 - T(1-\alpha)|e_n(x,t^*)|,$$

where $0 \le \eta \le t$. Hence, $e_{n+1}(x,t) \le \beta|e_n(x,t^*)|$.

Therefore,

$$\|e_{n+1}\| = max_{\forall t \in J}|e_{n+1}| \le \beta max_{\forall t \in J}|e_n| \le \beta\|e_n\|.$$

Since $0 < \beta < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete.

Theorem 3.4. The solution $u_0(x,t)$ obtained from the relation (21) using MVIM for the problem (1) converges when $0 < \alpha < 1$, $0 < \gamma < 1$.

Proof. The Proof is similar to the previous theorem.

Theorem 3.5. If the series solution (34) of problem (1) using HAM convergent then it converges to the exact solution of the problem (1).

Proof. We assume:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t),$$

$$\widehat{D}\big(u(x,t)\big) = \sum_{m=0}^{\infty} D(u_m(x,t)),$$

$$\widehat{H}\big(u(x,t)\big) = \sum_{m=0}^{\infty} H(u_m(x,t)).$$

where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)] = u_1+(u_2-u_1)+\cdots+(u_n-u_{n-1}) = u_n(x,t). \quad (38)$$

Hence, from (38),

$$\lim_{n\to\infty} u_n(x,t) = 0. \quad (39)$$

So, using (39) and the definition of the nonlinear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t) - \chi_m u_m(x,t)]] = 0.$$

therefore from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH_1(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}\left(u_{m-1}(x,t)\right) = 0. \quad (40)$$

\end{equation}

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (40) and simplifying it , we have

$$\sum_{m=1}^{\infty}\Re_{m-1}\left(u_{m-1}(x,t)\right) = \sum_{m=1}^{\infty}[u_{m-1}(x,t) + \int_0^t D\left(u_{m-1}(x,t)\right)dt + \int_0^t H(u_{m-1}(x,t))\,dt + (1-\chi_m)F(x,t)] = u(x,t)+F(x,t) + \int_0^t \widehat{D}\left(u(x,t)\right)dt + \int_0^t \widehat{H}\left(u(x,t)\right)dt. \quad (41)$$

From (40) and (41), we have

$$u(x,t) = -F(x,t) - \int_0^t \widehat{D}(u(x,t))\,dt - \int_0^t \widehat{H}(u(x,t))\,dt,$$

therefore, $u(x,t)$ must be the exact solution.

4. Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ϵ is a given positive value.

Algorithm: (ADM, MADM, and HAM)

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (10) for ADM , (13) for MADM and (34) for HAM.

Step 3. If $|u_{n+1}-u_n| < \epsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u(x,t)=\sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Algorithm: (VIM and MVIM)

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (20) for VIM and (21) for MVIM.

Step 3. If $|u_{n+1}-u_n| < \epsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1. Consider the BBM equation as follows:

$$u_t^6 - u_{xxx}^4 + 6u_x^4 = 0,$$

subject to the initial conditions:

$$u(x,0) = cos^{\frac{1}{2}}(x), \alpha = 0.467823, \beta = 0.563325.$$

Table 1
Numerical results for Example 4.1

(x,t)	Errors			
	ADM(n=26)	MADM(n=24)	VIM(n=19)	HAM(n=16)
(0.11, 0.16)	0.0073348	0.0062219	0.0044609	0.0031315
(0.21, 0.19)	0.0074605	0.0063314	0.0045781	0.0032457
(0.33, 0.27)	0.0075413	0.0064908	0.0046316	0.0033807
(0.42, 0.35)	0.0076715	0.0065516	0.0047865	0.0034922
(0.54, 0.41)	0.0077322	0.0066877	0.0048655	0.0035571
(0.65, 0.46)	0.0078288	0.0067228	0.0049567	0.0036108

(x,t)		Errors	
	HPM(n=17)	MHPM(n=15)	MVIM(n=10)
(0.11, 0.16)	0.0054876	0.0033609	0.0025926
(0.21, 0.19)	0.00552723	0.0034655	0.0026967
(0.33, 0.27)	0.0056437	0.0035926	0.0027217
(0.42, 0.35)	0.0057815	0.0036806	0.0027925
(0.54, 0.41)	0.0058122	0.0037356	0.0028715
(0.65, 0.46)	0.0059803	0.0038612	0.0029894

Table 1 shows that the approximate solution of the Benjamin-Bona-Mahony equation is convergence with ten iterations by using the MVIM . By comparing the results of table 1 , we can observe that the MVIM is more rapid convergence than the ADM, MADM, VIM, and HAM.

5. Conclusion

The MVIM has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with the approximations which convergent are rapid to exact solutions. In this work, the MVIM has been successfully employed to obtain the Benjamin-Bona-Mahony equation's approximate analytical solution.

References

[1]. S.Abbasbandy, A .Shirzadi, The first integral method for modified Benjamin-Bona-Mahony equation. Commun Nonlinear Sci Numer Simulat, In press, 2009.

[2]. S.Abbasbandy, Homptopy analysis method for generalized Benjamin-Bona-Mahony equation. Zeitschriff fur angewandte Mathematik und Physik (ZAMP), 59(2008) 51-62.

[3]. T.A.Abassy, El-Tawil,H.El.Zoheiry, Toward a modified variational iteration method (MVIM), J.Comput.Apll.Math, 207(2007) 137-147.

[4]. T.A.Abassy, El-Tawil,H.El.Zoheiry, Modified variational iteration method for Boussinesq equation, Comput.Math.Appl, 54(2007) 955-956.

[5]. T.B.Benjamin, J.L.Bona, J.J.Mahony, Model equations for long waves in nonlinear dispersive system. Philos Trans R Soc London, Sera, 27(1972) 47-78.

[6]. S.H.Behriy,H.Hashish, I.L.E-Kalla, A.Elsaid, A new algorithm for the decomposition solution of nonlinear differential equations.54(2007) 459-466.

[7]. A.O.Celebi, V.H.Kalantarov, M.Polat, Attractors for the generalized Benjamin-Bona-Mahony equation. Journal of Differential Equations, 157(1999) 439-451.

[8]. I.L.El-Kalla,Convergence of the Adomian method applied to a class of nonlinear integral equations. Appl.Math.Comput, 21(2008) 372-376.

[9]. Sh.Fang, B.Guo, The decay rates of solutions of generalized Benjamin-Bona-Mahony equations in multi-dimensions. Nonlinear Analysis, 69(2008) 2230-2235.

[10]. M.A.Fariborzi Araghi, Sh. Sadigh Behzadi, Solving nonlinear Volterra-Fredholm integral differential equations using the modified Adomian decomposition method, Comput. Methods in Appl. Math, 9(2009) 1-11.

[11]. D.D.Ganji, H.Babazadeh, M.H.Jalaei, H.Tashakkorian, Application of He's variational iteration method for solving nonlinear Benjamin-Bona-Mahony-Burgers equations and free vibration of systems. Acta Applicanda Mathematica: An International Survey Journal on

Applying Mathematics and Mathematical Applications, 106(2008) 359-367.

[12]. J.H.He, X.H.Wu, Exp-function method for nonlinear wave equations. Chaos, Solitons and Fractals, 30(2006) 700-708.

[13]. J.H.He, Variational principle for some nonlinear partial differential equations with variable cofficients, Chaos, Solitons, Fractals, 19 (2004) 847-851.

[14]. J.H. He, Variational iteration method for autonomous ordinary differential system, Appl. Math. Comput., 114 (2000) 115-123.

[15]. J.H. He, Wang.Shu-Qiang, Variational iteration method for solving integro-differential equations, Physics Letters A. 367 (2007) 188-191.

[16]. J.H.He, Variational iteration method some recent results and new interpretations, J. Comp. and Appl. Math., 207 (2007) 3-17.

[17]. N.Khiari, K.Omrani, On the convergence of difference scheme for the Benjamin-Bona-Mahony equation. Appl.Math.Comput, 182(2006) 999-1005.

[18]. Sh.Lai, X.LV, M.Shuai, The Jacobi elliptic function solutions to a generalized Benjamin-Bona-Mahony equation. Mathematical and computer Modelling, 49(2009) 369-378.

[19]. J.Limaco, H.R.Clark, L.A.Medeiros, Remarks on equations of Benjamin-Bona-Mahony type. J.Math.Anal.Appl, 328(2007) 1117-1140.

[20]. J.Limaco, H.R.Clark, L.A.Medeiros, On equations of Benjamin-Bona-Mahony type. Nonlinear Analysis, 59(2004) 1243-1265.

[21]. S.J.Liao , Beyond Perturbation: Introduction to the Homotopy Analysis Method.Chapman and Hall/CRC Press,Boca Raton,2003.

[22]. S.J.Liao ,Notes on the homotopy analysis method: some definitions and theorems, Communication in Nonlinear Science and Numerical Simulation 14(2009)983-997.

[23]. S.T.Mohyud-Din,M.A. Noor, Modified variational iteration method for solving Fisher's equations, J.Comput.Apll.Math, 2008.

[24]. S.Mica, On the controllability of the linearized Benjamin-Bona-Mahony equation. SIAM.Control Optim, 39(2001) 1667-1696.

[25]. K.Omrani, The convergence of fully discrete Galerkin approximations for the Benjamin-Bona-Mahony equation. Appl.Math.Comput, 180(2006) 614-621.

[26]. M.A.Raupp, Galerkin methods applied to the Benjamin-Bona-Mahony equation. Bol.Soc.Brasil.Mat, 6(1975) 65-77.

[27]. M.Song, Ch.Yang, B.Zhang, Exact solitary wave solutions of the Kudomtsov-Petviashvili-Benjamin-Bona-Mahony equation. Appl.Math.Comput, In press, 2009.

[28]. H.Tari,D.D.Ganji, Approximate explicit solutions of nonlinear Benjamin-Bona-Mahony-Burgers equations by He's methods and cpmparison with the exact solution.Phys.Lett.A, 367(2007) 95-101.

[29]. L.Wahlbin, Error estimates for a Galerkin method for a class of model equations for long waves. Numer.Math, 23(1975) 289-303.

[30]. A.M.Wazwaz, Exact solutions with compact and noncompact structures for the one-dimensional generalized Benjamin-Bona-Mahony equation. Communications in Nonlinear Science and Numerical Simulation, 10 (2005) 855-867.

[31]. B.Wang, Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains. J.Diffrential Equations, 246(2009) 2506-2537.

[32]. A.M.Wazwaz, Construction of solitary wave solution and rational solutions for the KdV equation by ADM.Chaos,Solution and fractals. 12(2001) 2283-2293.

[33]. A.M.Wazwaz , A first course in integral equations, WSPC, New Jersey; 1997.

[34]. E.Yusufoglu, A.Bekri, the variational iteration method for solitary patterns solutions of generalized Benjamin-Bona-Mahony equation.Phys.Lett.A, 367(2007) 461-464.

[35]. E.Yusufoglu, New solitonary solutions for the modified Benjamin-Bona-Mahony equations using Exp-function method. Phys.Lett.A, 372(2008) 442-464.

[36]. Y.H.Ye, L.F.Mo, He's variational method for the Benjamin-Bona-Mahony equation and Kawahara equation. Comput and Math with Appl, In press, 2009.

[37]. H.Yin, H.Zhao, J.Kim, Convergence rates of solutions forward boundary layer solutions of generalized Benjamin-Bona-Mahony equations in the half-space. J: Differential Equations, 245(2008) 3144-3216.

[38]. X.Zhao, W.Xu, Travelling wave solutions for a class of the generalized Benjamin-Bona-Mahony equations. Appl.Math.Comput, 192(2007) 507-519.

[39]. X.Zhao, W.Xu, Sh.Li, J.Shen, Bifurcations of traveling wave solutions for a class of the generalized Benjamin-Bona-Mahony equation. Appl.Math.Comput, 175(2006) 1760-1774.

[40]. L.Zeng, Existence and stability of solitary-wave solutions of quations of Benjamin-Bona-Mahony type, Differential Equations, 188(2003) 1-32.

[41]. M.Ikram, A.Muhammad, Muhammmad, Analytic solution to Benjam-Bona-Mahony by using laplace Adomian decomposition method ,Matrix, Science Mathematic ,3 (2019) 1-4.

[42]. A. Fakharian, M.T. Hamidi Beheshti and A. Davari, “Solving the Hamilton-Jacobi-Bellman equation Using Adomian Decomposition Method”, International Journal of Comp. Mathematics, 87(2010), 2769-2785.