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Generalization of Invariability of Block Matrices in Electrical Energy

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Abstract

Considering the essential role of inverse matrices in engineering sciences and considering the class of matrices inverses. Now, We introduce a new class of generalized inverse which is called n−strongly Drazin inverse and some elementary properties of the n−strongly Drazin inverse are obtained. Certain multiplicative and additive results for the n−strongly Drazin inverse in a Banach algebra are presented. We then apply some conditions under which a 2×2 block operator matrix has n−strongly Drazin inverse over Banach spaces.

Introduction

Matrix analysis is one of the appropriate mathematical methods used in engineering sciences. In methods based on matrix analysis special inverse matrix , properties of matrices are used in linear algebra problems. For the first time, the structure of matrix converters was presented by gyagy and pully under the purely theoretical concept with the motivation of optimizing the fundamental performance of cyclo-converters, in order to obtain output voltage with unlimited frequency and direct conversion using controllable two-way switches.

Considering the essential role of matrix inversion in problems related to electrical energy systems, in this article, a new method for the invariability of individual block matrices, using the n−strongly Drazin inverse definition, has been investigated and studied.

Let $\mathcal{A}$ be a Banach algebra with an identity. We first recall the definitions of some generalized inverses. As is well known, in 1958, Drazin [7] defined, an element $a \in \mathcal{A}$ has Drazin inverse if there is the element $x \in \mathcal{A}$ which satisfies

$$x \in Comm(a), xax = x, a - a^2x \in \mathcal{N}(\mathcal{A}).$$

Here $\mathcal{N}(\mathcal{A})$ is the set of all nilpotent elements in $\mathcal{A}$. The element b above is uniqe if it exists and is denoted by a^d and called the Drazin inverse of a. The Drazin inverse has many applications in singular differential equations and singular difference equations [1, 2], Markov chains [10, 11] and iterative methods [6]. Recently, several subclasses of the Drazin inverse have been studied. In 2017, Wang [14] gave the notion of the strongly Drazin inverse in a ring. An element $a \in \mathcal{A}$ has strongly Drazin inverse if there is a uniqe common solution to the equations

$$x \in Comm(a), xax = x, a - ax \in \mathcal{N}(\mathcal{A})$$

and we denoted by a^{sd} . We know that in a Banach algebra $\mathcal{A}$, $a \in \mathcal{A}^{sD}$ if and only if it is the sum of an idempotent and a nilpotent that commute, if and only if $a - a^2 \in \mathcal{N}(\mathcal{A})$ [4, Theorem 2-1]. Here, $q \in \mathcal{A}$ is an idempotent if $q^2 = q$. In same year, Chen

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and Sheibani [5] definded, the Hirano inverse of $a \in \mathcal{A}$ is the uniqe element $a \in \mathcal{A}$ satisfying

$$x \in Comm(a), xax = x, a^2 - ax \in \mathcal{N}(\mathcal{A})$$

and we denoted by a^h . They characterized the Hirano inverse by tripotents. Here, $q \in \mathcal{A}$ is a tripotent if $q^3 = q$. Also, they showed that, $a \in \mathcal{A}^H$ if and only if $a - a^3 \in \mathcal{A}$ [5, Theorem 3-1], if and only if it is the sum of a tripotenet and a nilpotent that commute [3, Theorem 2-8].

In 2019, Mosic [13] gave the notion of the n−strongly Drazin inverse (or ns−Drazin inverse), which is a new class of Drazin inverse and the ns−Drazin inverse $a \in \mathcal{A}$ is the uniqe element $x \in \mathcal{A}$ if such element exists, and if it satisfies

$$x \in Comm(a), xax = x, \ a^n - ax \in \mathcal{N}(\mathcal{A})$$

for some $n \in \mathbb{N}$ and we denoted by a^{nsd}. In [16], Zou and Mosic et al. investigate the structure of a ring in which every element satisfies the condition $a - a^{n+1} \in \mathcal{A}$ is nilpotent for a fixed n. We observed these inverses form a subclass of Drazin inverses which is related to periodic elements in a Banach algebra $\mathcal{A}$. We denote the set of all ns−Drazin invertible elements in $\mathcal{A}$ by $\mathcal{A}^{nsD}$.

We present some ns−Drazin inverses for a 2×2 operator matrix M under a number of different conditions, which generalize [16, Theorem 5-11]. If $a \in \mathcal{A}$ has ns−Drazin inverse a^{nsd}. The element $a^{\pi} = 1 - aa^{nsd}$ is called the spectral idempotent of a. In this Section, we consider the ns−Drazin inverse of a 2×2 operator matrix M under the perturbations on spectral idempotents. These also extends [8, Theorem 4-1] to wider cases.

1- Additive results

In this section we are concern on additive property of the n− strongly Drazin inverse of the sum in a Banach algebra $\mathcal{A}$. We introduce two known Lemmas which are related to the nilpotency.

Lemma 1-1. [17] Let $a, b \in \mathcal{A}$ with $ab = ba$.

(1) If $a \in \mathcal{N}(\mathcal{A})$ or $b \in \mathcal{N}(\mathcal{A})$, then $ab \in \mathcal{N}(\mathcal{A})$.

(2) If $a, b \in \mathcal{N}(\mathcal{A})$, then $a + b \in \mathcal{N}(\mathcal{A})$.

Lemma 2-1. [17] Let $a, b \in \mathcal{A}$ be such that $ab = 0$. Then

$$a, b \in \mathcal{N}(\mathcal{A}) \Leftrightarrow a + b \in \mathcal{N}(\mathcal{A}).$$

Proposition 3-1. [17] Every ns−Drazin invertible element in a Banach algebra is Drazin invertible element.

Theorem 4-1. [17] Let $n \in \mathbb{N}$. Then $a \in \mathcal{A}^{nsD}$ if and only if $a - a^{n+1} \in \mathcal{N}(\mathcal{A})$.

Theorem 5-1 [17] Let $a, b, c \in \mathcal{A}$. If $aba = aca$, then ac has ns−Drazin inverse if and only if ba has ns−Drazin inverse.

Corollary 6-1. Let $a, b \in \mathcal{A}$. If ab has ns−Drazin inverse, then so has ba.

Corollary 7-1. [17] Let $a \in \mathcal{A}$ and $m \in \mathbb{N}$. Then $a \in \mathcal{A}^{nsD}$ if and only if $a^m \in \mathcal{A}^{nsD}$.

Lemma 8-1. [17] Let $a, b \in \mathcal{A}$ and $ab = 0$. Then

$$a, b \in \mathcal{A}^{nsD} \Leftrightarrow a + b \in \mathcal{A}^{nsD}.$$

Theorem 9-1. Let $a, b \in \mathcal{A}^{nsD}$. If $a^3 b = 0$, $ba^2 b = 0$, $abab = 0$ and $b^2 ab = 0$, then $a + b \in \mathcal{A}^{nsD}$.

Proof. Let

$$M = \begin{bmatrix} ab^2 + a^2 b + bab + b^3 & a^2 + ab + ba + b^2 \\ ab^3 + a^2 b^2 & a^3 + ab^2 + a^2 b + aba \end{bmatrix}$$

$= F + G$

Where

$$F = \begin{bmatrix} b^3 & a^2 + ab + ba + b^2 \\ 0 & a^3 \end{bmatrix}$$

and

$$G = \begin{bmatrix} ab^2 + a^2 b + bab & 0 \\ ab^3 + a^2 b^2 & ab^2 + a^2 b + aba \end{bmatrix}$$

Obviously, we have $FG = 0$ and $G^2 = 0$. Further, we see that

$$F = \begin{bmatrix} b^3 & a^2 + ab + ba + b^2 \\ 0 & a^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a^2 + ba \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} b^3 & b^2 + ab \\ 0 & 0 \end{bmatrix} = K + L$$

It is easy to verify that

$$K = \begin{bmatrix} 0 & a^2 + ba \\ 0 & a^3 \end{bmatrix} = \begin{bmatrix} a + b \\ a^2 \end{bmatrix} [0 \quad a] .$$

Since

$$[0 \quad a] \begin{bmatrix} a + b \\ a^2 \end{bmatrix} = a^3 \in \mathcal{A}^{nsD}$$

by Corollary 6-1, we have K has ns–Drazin inverse. Similarly, L has ns–Drazin inverse. Obviously, $KL = 0$. In light of Lemma 2-1, F has ns–Drazin inverse and so M is ns–Drazin invertible. Also we compute that

$$M = \left(\begin{bmatrix} 1 \\ a \end{bmatrix} [b \quad 1] \right)^3 .$$

By using Corollary 6-1,

$$[b \quad 1] \begin{bmatrix} 1 \\ a \end{bmatrix}$$

has ns–Drazin inverse, which implies that a+b has ns–Drazin inverse, as required.

Corollary 10-1. [17] Let $a + b \in \mathcal{A}^{nsD}$. If $a^2 b = 0$ and $bab = 0$, then $a + b \in \mathcal{A}^{nsD}$.

Proof. This is immediate by Theorem 2-9. □

We are now ready to prove:

Theorem 11-1. Let $a, b \in \mathcal{A}^{nsD}$. If $a^2 b = 0$, $bab^2 = 0$ and $(ab)^2 = 0$, then $a + b \in \mathcal{A}^{nsD}$.

Proof. Let $p = a^2 + ab$ and $q = b^2 + ba$. Since $(ab)^2 = 0$ and we have $ab - (ab)^2 \in \mathcal{A}$ is nilpotent. Hence, by using Theorem 4-1, we see that $ab \in \mathcal{A}^{nsD}$. By using Corollary 6-1, $ba \in \mathcal{A}^{nsD}$. In view of Corollary 7-1, $a^2, b^2 \in \mathcal{A}^{nsD}$. Since $a^2(ab) = 0$, it follows by Lemma 8-1, that $p \in \mathcal{A}^{nsD}$. As $(ba)b^2 = 0$, we see that $q \in \mathcal{A}^{nsD}$. Clearly,

$$p^2 q = (a^2 + ab)(ab^3 + ab^2 a) = 0 ,$$
$$qpq = (b^2 + ba)(ab^3 + ab^2 a) = 0 .$$

According to Corollary 9-1,

$$(a + b)^2 = p + q \in \mathcal{A}^{nsD}.$$

Therefore $a + b \in A^{nsD}$, by Corollary 10-1. □

2- Operator Matrices

To illustrate the preceding results, we are concerned with the ns–Drazin inverse for an operator matrix. Throughout this section, the operator matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in L(X)^{nsD}$, $B \in L(X, Y)$, $C \in L(Y, X)$ and $D \in L(Y)^{nsD}$. Using different splitting approach and Theorem 11-1, we will obtain various conditions for the ns–Drazin inverse of M.

Lemma 1-2. Let $A \in L(X)^{nsD}$ and $D \in L(Y)^{nsD}$. Then

$$K = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \text{ and } H = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

have ns–Drazin inverse.

Proof. Consider the splitting of,

$$L = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad D] = PQ.$$

Since $QP = D$ has ns–Drazin inverse, by using Corollary 6-1, $L = PQ$ has ns–Drazin inverse. Simillary, K has ns–Drazin inverse.

For H, write

$$H = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} = p + q.$$

Since $pq = 0$, by Lemma 8-1, $H = p + q$ has ns–Drazin inverse. □

Lemma 2-2. [4] Let $B \in L(X, Y)$ and $C \in L(Y, X)$. If $CB \in L(Y)^{nsD}$, then

$$K = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

has ns–Drazin inverse.

Lemma 3-2. Let $D, CB \in L(Y)^{nsD}$. If $CBD = 0$, then

$$N = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$$

has ns–Drazin inverse.

Proof. Consider the splitting of,

$$N = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} = P + Q$$

By Lemmas 1-2 and 2-2, P and Q have ns–Drazin inverse. According to the assumptions, we have,

$$P^2Q = \begin{bmatrix} 0 & 0 \\ 0 & CBD \end{bmatrix}, QPQ^2 = O$$

and

$$(PQ)^2 = \begin{bmatrix} 0 & BD \\ 0 & 0 \end{bmatrix}^2$$

Therefore $P^2Q = 0$, $QPQ^2 = 0$ and $(PQ)^2 = 0$. Applying Theorem 11-1, $M = P + Q \in L(X \oplus Y)^{nsD}$, as asserted. □

Theorem 4-2. Let $A \in L(X)^{nsD}$ and $D, CB \in L(Y)^{nsD}$.
If $CBD = 0$, $A^2B = 0$ and $CAB = 0$, then

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has ns−Drazin inverse.

Proof. Write

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = P + Q$$

By Lemmas 1-2 and 3-2, P and Q have ns−Drazin inverse. We check that

$$P^2Q = \begin{bmatrix} 0 & A^2B \\ 0 & 0 \end{bmatrix}, QPQ^2 = \begin{bmatrix} 0 & 0 \\ CABC & CABD \end{bmatrix}$$

And

$$(PQ)^2 = \begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}^2.$$

Hence, we have

$$P^2Q = 0, QPQ^2 = 0 \text{ and } (PQ)^2 = 0.$$

Then by Theorem 11-1, we complete the proof and $M = P + Q \in L(X \oplus Y)^{nsD}$.
□

Corollary 5-2. Let $A \in L(X)^{nsD}$ and , $CB \in L(Y)^{nsD}$. If $CBD = 0$ and $AB = 0$, then $M \in L(X \oplus Y)^{nsD}$.

Proof. If $AB = 0$ then $A^2B = 0$ and $CAB = 0$. So we get the result by Theorem 4-2.
□

Proposition 6-2. Let $A \in L(X)^{nsD}$ and $D, CB \in L(Y)^{nsD}$. If $CBD = 0$, $BCA = 0$ and $DCA = 0$, then $M \in L(X \oplus Y)^{nsD}$.

Proof. Write

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = P + Q.$$

Similary Theorem 4-2, we have $P^2Q = 0$, $QPQ^2 = 0$ and $(PQ)^2 = 0$. By Theorem 11-1, we complete the proof and $M = P + Q \in L(X \oplus Y)^{nsD}$.
□

Corollary 7-2. Let $A \in L(X)^{nsD}$ and $D, CB \in L(Y)^{nsD}$. If $CBD = 0$ and $CA = 0$, then $M \in L(X \oplus Y)^{nsD}$.

Proof. We get the result by Peropsition6-2. □

3- Perturbation

Now let M be an operator matrix that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

It is of interest to consider the n−strongly Drazin inverse of M under generalized Schur condition $D = CA^{nsD}B$ (see [11, Theorem 2-1]). Let $W = AA^{nsd} + A^{nsd}BCA^{nsd}$. We now derive

Theorem 1-3. Let $A \in L(X)^{nsD}$ and $D \in L(Y)^{nsD}$. If $AA^{\pi}BC = 0$, $A^{\pi}BCA^{\pi} = 0$, $ABCA^{\pi} = 0$, $D = CA^{nsd}B$ and AW has ns−Drazin inverse, then $M \in L(X \oplus Y)^{nsD}$.

Proof. We easily see that

$$M = \begin{bmatrix} A & B \\ C & CA^{nsd}B \end{bmatrix} = P + Q$$

Where

$$P = \begin{bmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} A & AA^{nsd}B \\ C & CA^{nsd}B \end{bmatrix}.$$

It is not hard to see that

$P^3Q = 0$, $QP^2Q = 0$, $PQPQ = 0$ and $Q^2PQ = 0$.

In view of Theorem 2-9, P is nilpotent and it has ns−Drazin inverse. Moreover, we have

$$Q = Q_1 + Q_2$$

Where

$$Q_1 = \begin{bmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} A^2A^{nsd} & AA^{nsd}B \\ CAA^{nsd} & CA^{nsd}B \end{bmatrix}$$

that $Q_2Q_1 = 0$ and Q_1 is nilpotent. We easily check that

$$Q_2 = \begin{bmatrix} AA^{nsd} \\ CA^{nsd} \end{bmatrix} \begin{bmatrix} A & AA^{nsd}B \end{bmatrix}.$$

By hypothesis, we see that

$$[A \quad AA^{nsd}B]\begin{bmatrix} AA^{nsd} \\ CA^{nsd} \end{bmatrix} = A^2A^{nsd} + AA^{nsd}BCA^{nsd} = AW$$

has ns–Drazin inverse. Obviously, Q_2 has ns–Drazin inverse. Therefore Q has ns–Drazin inverse. According to Theorem 9-1, $M \in L(X \oplus Y)^{nsD}$. □

Corollary 2-3. Let $A \in L(X)^{nsD}$ and $D \in L(Y)^{nsD}$. If $ABC = 0$, $A^{\pi}BC = 0$, $D = CA^{nsd}B$ and AW has ns–Drazin inverse, then $M \in L(X \oplus Y)^{nsD}$.

Proof. This is obvious by Theorem 1-3. □

Regarding a complex matrix as the operator matrix on $\mathbb{C} \times \cdots \times \mathbb{C}$, we now present a numerical example to demonstrate Theorem 1-3.

Example 3-3. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

be complex matrices and set

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then

$$A^{nsd} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^{\pi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We easily check that

$$AA^{\pi}BC = 0, A^{\pi}BCA^{\pi} = 0,$$
$$ABCA^{\pi} = 0, D = CA^{nsd}B.$$

In this case, A and D have ns–Drazin inverses.

Theorem 4-3. Let $A \in L(X)^{nsD}$, $D \in L(Y)^{nsD}$. If $BCAA^{\pi} = 0$, $A^{\pi}BCA = 0$ and $D = CA^{nsd}B$, then $M \in L(X \oplus Y)^{nsD}$.

Proof. Clearly, we have

$$M = \begin{bmatrix} A & B \\ C & CA^{nsd}B \end{bmatrix} = P + Q$$

Where

$$P = \begin{bmatrix} A^2A^{nsd} & B \\ C & CA^{nsd}B \end{bmatrix}, Q = \begin{bmatrix} AA^{\pi} & 0 \\ 0 & 0 \end{bmatrix}.$$

By assumption, we verify that $P^2Q = 0$, $QPQ^2 = 0$ and $(PQ)^2 = 0$. In view of in Theorem 11-1, Q is nilpotent, and then it has ns–Drazin inverse. Moreover, we have

$$P = P_1 + P_2 = \begin{bmatrix} 0 & A^{\pi}B \\ CA^{\pi} & 0 \end{bmatrix} + \begin{bmatrix} A^2A^{nsd} & AA^{nsd}B \\ CAA^{nsd} & CA^{nsd}B \end{bmatrix}$$

and $P_1P_2 = 0$. Since ${P_1}^3 = 0$, therefore P_1 has ns–Drazin inverse. Moreover, we have

$$P_2 = \begin{bmatrix} AA^{nsd} \\ CA^{nsd} \end{bmatrix} [A \quad AA^{nsd}B].$$

By hypothesis, we see that

$$[A \quad AA^{nsd}B]\begin{bmatrix} AA^{nsd} \\ CA^{nsd} \end{bmatrix} = A^2A^{nsd} + AA^{nsd}BCA^{nsd} = AW$$

is ns–Drazin invertible. Therefore P_2 has ns–Drazin inverse. By virtue of Theorem 11-1, $M \in L(X \oplus Y)^{nsD}$, as required. □

Corollary 5-3. Let $A \in L(X)^{nsD}$, $D \in L(Y)^{nsD}$. If $BCA = 0$ and $D = CA^{nsd}B$, then $M \in L(X \oplus Y)^{nsD}$.

Example 6-3. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$$

be complex matrices and set

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then

$$A^{nsd} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^{\pi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We easily check that

$$BCAA^{\pi} = 0,\ A^{\pi}BCA = 0 \text{ and } D = CA^{nsd}B.$$

In this case, A and D have ns–Drazin inverses.

References

[1] S. L. Campbell, "The Drazin inverse and systems of second order linear differential Equations", Linear Multilinear Algebra, 14, 195–198, 1983.

[2] S. L. Campbell, C. D. Meyer, Generalized inverse of linear transformations. Dover, New York, 1991.

[3] H. Chen, M. Sheibani, "Strongly 2-nil-clean rings", J. Algebra Appl., 16, 175-178, 2017. DOI: 10.1142/ S021949881750178X.

[4] H. Chen, S. S. Nabavi, M. Sheibani Abdolyousefi, "Strongly Drazin inverse in rings", Operator and Matrices, 13, 495–505, 2019.

[5] H. Chen, M. Sheibani, "On Hirano inverse in rings", Turk. J. Math., 43, 2049–2057, 2019.

[6] J. J. Climent, M. Neumann, A. Sidi, "A semi-iterative method for real spectrum singular linear systems with an arbitrary index", J. Comput. Appl. Math., 87, 21–38, 1997.

[7] M. P. Drazin, "Pseudo-inverses in associative rings and semigroups", Amer. Math. Monthly , 65(7), 506-514, 1958.

[8] A. Ghaffari, T. Haddadi, M. Sheibani, "Extension of Hirano inverse in Banach algebra", Filomat,36(9), 3197-3206, 2022. https://doi.org/10.2298/ FIL2209 197G.

[9] L. Gyagy, B. R. Pully, Statics power frequency changers, Newyork, Willey, 1976.

[10] J. J. Hunter, "Generalized inverses of Markovian kernels in terms of properties of the Markov chain", Linear Algebra Appl., 447, 38–55, 2014.

[11] C. D. Meyer, J. M. Shoaf, "Upadating finite Markov chains by using techniques of group inversion", Statist. Comput. Simulation, 11, 163–181, 1980.

[12] D. Mosic, D. S. Djordjevic, "Several expressions for the generalized Drazin inverse of a block matrix in a Banach algebra", Appl. Math. Comput., 220, 374–381, 2013.

[13] D. Mosic, "The generalized and pseudo n-strong Drazin inverses in rings", Linear Multilinear Algebra, 2019. DOI: 10.1080/03081087.2019 .15 99806.

[14] Z. Wang, "A class of Drazin inverses in rings", Filomat, 31, 1781–1789, 2017.

[15] H. Zou, J. Chen, H. Zhu, "Characterizations for the n-strong Drazin invertibility in a ring", J. Algebra Appl., July 2020. DOI: 10.1142/ S02194 98821501413.

[16] H. Zou, D. Mosic, K. Zuo, Y. Chen, "On the n-strong Drazin invertibility in rings", Turk. J. Math., 43, 2659–2679, 2019.