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Stability of the Classification of Returns to Scale FDH Models in the Presence of Undesirable Data

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Abstract. This paper deals with the estimation of returns to scale (RTS) in free disposal hull (FDH) models for undesirable data and provides some stability intervals for preserving the RTS classification. It has been shown that the proposed stability intervals can be obtained via a polynomial-time algorithm based on the calculation of certain ratios of inputs and outputs, without solving any mathematical programming problem. The results of the study have been proved via some lemmas and theorems for undesirable data and have been illustrated by numerical example.

Keywords: DEA; Undesirable Data; FDH; Returns-to-Scale; Polynomial-Time Algorithms; Stability.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for evaluating decision making units (DMUs) based on the production possibility set. Free disposal hull (FDH) models, which were first formulated by Deprins et al. [5], rely on the sole assumption that production possibilities satisfy free disposability, and ensure that efficiency evaluations are effected from only actually observed performances. Estimation of returns to scale (RTS) is one of the vital topics in DEA and FDH models. Although there are many papers for estimating RTS in DEA models (see, e.g., [2–4, 7, 10, 12, 13, 28–30]), there are only few papers which concern the estimation of RTS in FDH models: Kerstens and Vanden Eeckaut [14], Podinovski [19], and Soleimani-damaneh et al[25, 26].The sensitivity and stability analysis of RTS classification in DEA was first studied by Seiford and Zhu [21] and improved by Jahanshahloo et al. [11]. These authors develop several linear programming formulations for investigating the stability of RTS classification. Soleimani-damaneh et al. [25, 26] provide two different enumeration algorithms for estimating the RTS in FDH models. In [25] an envelopment model has been considered which is an input oriented and contracts only inputs. In [26] an envelopment model has been considered which contracts the inputs and expands the outputs, simultaneously. The latter can provide the idea to generalize the concept of the right-RTS and the left-RTS [10, 29] to FDH models. Also, the latter shows that an extension of Banker and Thrall's method (see [4]) can be utilized to determine the RTS in FDH models. In this paper we seek some stability regions for the RTS classification in FDH models. To this end, the envelopment form of the FDH-CRS model1 and the enumeration technique to determine the RTS classification in FDH models are considered, and some stability intervals for radial variations in outputs while preserving the RTS classification are obtained. Also it can be shown that the proposed stability intervals can be obtained via a polynomial-time algorithm. In addition to obtaining the stability intervals, it is shown that the RTS classification of FDH-inefficient units can be determined by techniques provided in [25, 26] without obtaining the projection point all DEA models are formulated for desirable inputs and outputs. However, it was mentioned already by Koopmans[15] that the production process may also generate undesirable outputs like smoke pollution or waste. However, both desirable (good) and undesirable (bad) output and input factors may be present. Consider a paper mill production where paper is produced with undesirable outputs of pollutants such as biochemical oxygen demand, suspended solids, particulate and sulfur oxides. If inefficiency exists in the production, the undesirable pollutants should be reduced to improve the inefficiency, i.e, the undesirable and desirable outputs should be treated differently when we evaluate the production performance of paper mills. However, in the standard DEA model, decreases in outputs are not allowed and only inputs are allowed to decrease(Similarly, increases in inputs are not allowed and only outputs are allowed to increase). Faare et al[6] developed a non-linear DEA program to model the paper production

system where the desirable outputs are increased and the undesirable outputs are decreased. In DEA literature, there already existed much research concerning applications with undesirable inputs and/or outputs. Some of the existing approaches are brie summarized as follows:

First acceptable approach suggested by Koopmans. Main idea of his approach is to apply some transformations on data. Then the undesirable inputs or outputs will become desirable after this transformation. For other approaches based on data transformation one can see Scheel (1998, 2001), Ali and Seiford[1], Pastor[20], Seiford and Zhu[22], Golany and Roll [9], Lovell et al. [18]. There also exist many approaches that can avoid data transformation. For example, one may regard undesirable inputs as desirable outputs, or undesirable outputs as desirable inputs, see Liu and Sharp[17] for an initial attempt to formulate this method. For more information one can see Grosskopf [8], Silva Portela et al[23], Yu[31]. Recently Liu et al[16] present a systematic investigation on model building of DEA without transferring undesirable data. The remainder of this paper is organized as follows: In section 2, using[24] the determination of RTS in FDH models for undesirable data is reviewed and a new theorem for inefficient DMUs is proved. In Section 3, the required stability regions are provided and a polynomial-time algorithm to obtain these intervals is provided. Section 4, contains one examples with undesirable data.

2. RTS Classification in FDH Models for Undesirable Data

Now, we present a FDH model for undesirable data (see Liu et al[16]). Assume that $\{DO\},\{UI\},\{DI\}$ and $\{UO\}$ indicate fixed index sets independent of j, such that $x_{ij}(i \in \{DI\})$ and $y_{rj}(r \in \{DO\})$ are desirable inputs and outputs and $x_{ij}(i \in \{UI\})$, $y_{rj}(r \in \{UO\})$ are undesirable inputs and outputs.

Then FDH model can be readily extended into the following modified form:

$$\begin{aligned}
&\theta_o^{FDH} = \min \theta \\
&St. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \quad i \in \{DI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, \quad i \in \{UI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta y_{ro}, \quad r \in \{UO\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r \in \{DO\} \\
&\qquad \sum_{j \in J} \lambda_j = 1 \ , \ \lambda_j \in \{0,1\}; \ j \in J
\end{aligned} \tag{1}$$

DMU_o is considered as an FDH-efficient point if $\theta_o^{FDH} = 1$ or, equivalently DMU_o is an FDH-efficient point if $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$ is a boundary point of P, where

$P = \{(x, y) \mid \sum_{j \in J} \lambda_j x_j \le x \ , \sum_{j \in J} \lambda_j y_j \ge y \ , \ \sum_{j \in J} \lambda_j = 1 \ , \lambda_j \in \{0,1\}; j \in J \}$ Is FDH-PPS, under variable RTS(VRS) assumption. Now consider the following general input-oriented FDH model in undesirable data to evaluate $DMU_o \ (x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$:

$$\begin{aligned}
&\theta_o = \min \theta \\
&St. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \quad i \in \{DI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, \quad i \in \{UI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta y_{ro}, \quad r \in \{UO\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r \in \{DO\} \\
&\qquad \lambda_j = \delta w_j, \ w_j \in \{0,1\}; \quad j \in J \\
&\qquad \delta \in \Gamma, \ \sum_{j \in J} w_j = 1
\end{aligned} \tag{2}$$

Where Γ depending on the RTS assumption of the reference technology, is

$(i) \ \Gamma(s) = \{\delta \mid 0 < \delta\}$ for $s = CRS$ (3)

$(ii) \ \Gamma(s) = \{\delta \mid 0 < \delta \le 1\}$ for $s = NIRS$ (4)

$(iii) \ \Gamma(s) = \{\delta \mid \delta \ge 1\}$ for $s = NDRS$ (5)

The above model and the related technologies were proposed by Kerstens and Vanden Eeckaut[14].This model results in the $\theta_o^{CRS}, \theta_o^{NIRS}, \theta_o^{NDRS}$ input-oriented efficiency scores using the above mentioned Γ sets, respectively.

we express the convenience of Banker and Thrall's method to determine the RTS in FDH models:

Theorem 1. Considering DMU_o as an FDH-efficient DMU, we have

(i) If $\sum_{j\in J}\lambda_j^* = 1$ in any optimal solution of the FDH-CRS model, then CRS prevail.

(ii) If $\sum_{j\in J}\lambda_j^* < 1$ in all optimal solutions of the FDH-CRS model, then IRS prevail.

(iii) If $\sum_{j\in J}\lambda_j^* > 1$ in all optimal solutions of the FDH-CRS model, then DRS prevail.

(iv) If there exist two optimal solutions, (λ^*, θ^*) and (λ^{**}, θ^*), for the FDH-CRS model such that $\sum_{j\in J}\lambda_j^* > 1$ and $\sum_{j\in J}\lambda_j^{**} < 1$, then CRS prevail.

Proof. Omitted

Algorithm 1.

Step1.compute θ_o^{FDH}.

Step2. compute θ_o^{CRS} the FDH-CRS efficiency score corresponding to $(\theta_o^{FDH} x_{DI_o}, x_{UIo}, y_{DO_o}, \theta_o^{FDH} y_{U_{O_o}})$

Step 3. Compute $\overline{\lambda}_o$ and $\underline{\lambda}_o$.

$$\begin{aligned}
&\overline{\lambda_o} = \max \sum_{j\in J} \lambda_j \\
&St. \quad \sum_{j\in J} \lambda_j x_j \le \widehat{\theta}_{io}^{CRS}\, \widehat{\theta}_{io}^{FDH}\, xi_0, && i \in \{DI\} \\
&\quad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \le \widehat{\theta}_{ro}^{CRS}\, \widehat{\theta}_{ro}^{FDH}\, y_{ro}, && r \in \{UO\} \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\quad \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\quad \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned}$$

$$\begin{aligned}
&\underline{\lambda_o} = \min \sum_{j\in J} \lambda_j \\
&St. \quad \sum_{j\in J} \lambda_j x_j \le \widehat{\theta}_{io}^{CRS}\, \widehat{\theta}_{io}^{FDH}\, xi_0, && i \in \{DI\} \\
&\quad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \le \widehat{\theta}_{ro}^{CRS}\, \widehat{\theta}_{ro}^{FDH}\, y_{ro}, && r \in \{UO\} \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\quad \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\quad \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned} \tag{6}$$

Step4. if $\overline{\lambda_o} < 1$ then IRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else

If $\underline{\lambda_o} > 1$ then DRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else

CRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$

The following theorem helps to determine the RTS classification of $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$ in both efficient and inefficient cases, without needing to perform Step1 of Algorithm1.This reduces the computational requirements.

Theorem 2. Let θ_o^{CRS} be the FDH-CRS efficiency score corresponding to $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$ and consider the following models:

$$
\begin{aligned}
&\lambda_o^{+} = \max \sum_{j\in J} \lambda_j \\
&St. \quad \sum_{j\in J} \lambda_j x_j \le \theta_o^{CRS} x_{i0}, && i \in \{DI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta_o^{CRS} y_{ro}, && r \in \{UO\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\qquad \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\qquad \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&\underline{\lambda_o} = \min \sum_{j\in J} \lambda_j \\
&St. \quad \sum_{j\in J} \lambda_j x_j \le \theta_o^{CRS} x_{i0}, && i \in \{DI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta_o^{CRS} y_{ro}, && r \in \{UO\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\qquad \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\qquad \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned} \tag{7}
$$

Then we have $\overline{\lambda_o} = \lambda_o^{+}$ and $\underline{\lambda_o} = \lambda_o^{-}$.

Proof. Comparing Models (5) and (6) shows that to establish the theorem it is sufficient to show that

$$\theta_o^{CRS} = \widehat{\theta}_o^{CRS} \, \theta_o^{FDH}$$

Where $\widehat{\theta}_o^{CRS}$ is the FDH-CRS efficiency score corresponding to $(\theta_o^{FDH} x_{DI_o}, x_{UIo}, y_{DO_o}, \theta_o^{FDH} y_{Uo_o})$ and is obtained by using the following model:

$$\begin{aligned}
&\widehat{\theta}_o^{CRS} = \min \theta \\
&St. \quad \sum_{j\in J} \lambda_j x_j \le \theta(\theta_o^{FDH} x_{io}) x_{i0}, && i \in \{DI\} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \le \theta(\theta_o^{FDH} x_{io}) y_{ro}, && r \in \{UO\} \qquad (8) \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned}$$

Let $(\lambda^*, \theta_o^{CRS}, \delta^*, \omega^*)$ be an optimal solution to Model (CRS).

From the constraints of Model CRS we have

$$\begin{aligned}
&\sum_{j\in J} \lambda_j x_j \le \theta_o^{CRS} x_o = \frac{\theta_o^{CRS}}{\theta_o^{FDH}} \theta_o^{FDH} x_{io}, && i \in \{DI\} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \le \theta_o^{CRS} x_o = \frac{\theta_o^{CRS}}{\theta_o^{FDH}} \theta_o^{FDH} y_{ro}, && r \in \{UO\} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r \in \{DO\} \\
&\lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned}$$

These imply that $(\lambda = \lambda^*, \theta = \theta_o^{CRS} / \theta_o^{FDH}, \delta^*, w^*)$ is a feasible solution to Model (7) Thus $\hat{\theta}_o^{CRS} \le \theta^{CRS} / \theta_o^{FDH}$, which leads to

$$\theta_o^{CRS} \ge \hat{\theta}_o^{CRS} . \theta_o^{FDH} \qquad (9)$$

Now let $(\hat{\lambda}, \hat{\theta}_o^{CRS}, \hat{\delta}, \hat{w})$ be an optimal solution to Model(7). From the constraints of Model (7) we have

These imply that $(\lambda = \hat{\lambda}, \theta = \hat{\theta}_o^{CRS} \theta_o^{FDH}, \widehat{\delta}, \widehat{w})$ is a feasible solution to Model CRS. Thus

$$\theta_o^{CRS} \le \hat{\theta}_o^{CRS} \theta_o^{FDH}$$

By(8)and(9),

$$\theta_o^{CRS} = \hat{\theta}_o^{CRS} \theta_o^{FDH}$$

And the proof is complete.

Regarding Theorem2, the RTS classification of $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$ in both efficient and inefficient cases, can be obtained using the following Algorithm.

Algorithm 2.

Step1.compute θ_o^{CRS}.

Step 2.compute λ_o^+, λ_o^- by Models(6).

Step3.if $\lambda_o^+ < 1$ then IRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else

if $\lambda_o^- > 1$ then DRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else

CRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$.

It is evident that Algorithm2 is computationally economical compared to Algorithm1.Algorithm1 is performed through solving four models, while Algorithm2 is performed through solving three models. Using all three above-mentioned method requires solving at least three mixed integer linear(non-linear) programming problems for each DMU (see[4, 26]), and this can be expensive. So, two polynomial-time algorithms to do this have be improvised by Soleimani-damaneh et al.[25, 26]. The following theorem utilizes such polynomial-time tests to obtain the values of λ_o^+, λ_o^- and hence to determine the RTS classification. The proof of this theorem comes from the discussion provided in [25, 26, 27] and is hence omitted.

Theorem 3. Considering $DMU_o(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$.

$$\lambda^{jo} = \max_{1\le r\le s}\left\{\frac{y_{ro}}{y_{rj}}\right\}, j \in J \qquad (10)$$

$$\theta^{jo} = \max_{1\le i\le m}\left\{\frac{x_{ij}\lambda^{jo}}{x_{io}}\right\}, j \in J \qquad (11)$$

$$\theta_o^{CRS} = \min\{\theta^{jo}\}, \qquad (12)$$

and

$$A_o = \{k \in J : \theta^{ko} = \theta_o^{CRS}\} \quad (13)$$

We have

$$(i)\ \lambda_o^+ = \max\{\lambda^{ko} : k \in A_o\},$$
$$(ii)\ \lambda_o^- = \min\{\lambda^{ko} : k \in A_o\}.$$

Now regarding Theorem3, the RTS classification of $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, in both efficient and in efficient cases, can be obtained using the following algorithm

Note that this algorithm is a polynomial-time algorithm regarding Theorem3 in [26] and has many computational advantages compared to Algorithms. 1and 2. This algorithm is an enhanced version of the technique provided in [25].

Algorithm 3.

Step1.compute λ^{jo} for all $j \in J$ by (10).
Step 1.compute θ^{jo} for all $j \in J$ by (11).
Step 3. Compute θ_o^{CRS}, A_o by (12) and (13), respectively.
Step 4. Compute λ_o^+, λ_o^- by parts (i) and (ii) of Theorem 3, respectively.
Step 5. If $\lambda_o^+ < 1$, then IRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else
If $\lambda_o^- > 1$, then DRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$, else
CRS prevail at $(x_{Di_o}, x_{Ui_o}, y_{Do_o}, y_{Uo_o})$.

3. Stability

This section contains three theorems which provide stability intervals for preserving the RTS classification of DMUs in FDH models for undesirable data. The stability interval for preserving the RTS classification of a considered unit, DMU_o, is undesirable data with defined as follows.

Definition 1. The interval $I \subseteq R$ is a stability interval for preserving the RTS classification of a considered unit $DMU_o(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$ if

(x_o, y_o), $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ are in the same class in RTS classification, for each $\alpha \in I$.

In the following results φ_o^{FDH} In the following results φ_o^{FDH} denotes the output-oriented FDH efficiency score of DMU_o and is obtained via solving the model below :

$$
\begin{aligned}
\varphi_o^{FDH} &= \max \varphi \\
S.t \quad & \sum_{j \in J} \lambda_j x_j \le x_o, \\
& \sum_{j \in J} \lambda_j y_j \ge \varphi y_o \\
& \sum_{j \in J} \lambda_j = 1,\ \lambda_j \in \{1, 0\};\ j \in J
\end{aligned}
$$

Before providing the main theorems, we establish the following useful lemma

Lemma 4. Consider θ_o^{α} (for constant $\alpha > 0$) as the input-oriented FDH CRS efficiency score of $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$, which is obtained by solving the following model:

$$
\begin{aligned}
\theta_\alpha^{CRS} &= \min \theta \\
St. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, & i \in \{DI\} \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, & i \in \{UI\} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta y_{ro}, & r \in \{UO\} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge \alpha y_{ro}, & r \in \{DO\} \\
& \lambda_j = \delta w_j,\ w_j \in \{0,1\}; & j \in J \\
& \delta \in \Gamma,\ \sum_{j \in J} w_j = 1
\end{aligned}
$$

Then we have $\theta_\alpha^{CRS} = \alpha \theta_o^{CRS}$

Proof. Let $(\lambda^*, \theta_o^{CRS}, \delta^*, w^*)$ be an optimal solution to model CRS. Then, it is clear that $(\alpha\lambda^*, \alpha\theta_o^{CRS}, \alpha\delta^*, w^*)$ is a feasible solution to Model (15). Hence $\theta_\alpha^{CRS} \leq \alpha\theta_o^{CRS}$

If $\theta_\alpha^{CRS} < \alpha\theta_o^{CRS}$, then there exists a feasible solution to Model(15), say, $(\tilde{\lambda}, \tilde{\theta}, \tilde{\delta}, \tilde{w})$, such that $\tilde{\theta} < \alpha\theta_o^{CRS}$.This is evident that $(\tilde{\lambda} / \alpha, \tilde{\theta} / \alpha, \tilde{\delta} / \alpha, \tilde{w})$ is feasible to Model(2-CRS). Hence $\theta_o^{CRS} \leq \tilde{\theta} / \alpha$, while $\tilde{\theta} < \alpha\theta_o^{CRS}$.This is a clear contradiction. There fore $\theta_\alpha^{CRS} = \alpha\theta_o^{CRS}$ and the proof is complete.

Theorem 5. Suppose that CRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$. Then CRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ for

$$\alpha \in \left[1 / \lambda_o^+, \min\left\{1 / \lambda_o^-, \varphi_o^{FDH}\right\}\right]$$

Proof. The condition $\alpha \leq \varphi_o^{FDH}$ guarantees that $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o}) \in P$, because Model(14) shows that

$$\varphi_o^{FDH} = \max\{\alpha : (x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o}) \in P\}$$

(we recall that P is $FDH - PPS$, as defined in Section2). Also since CRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$, we have $1 / \lambda_o^+ \leq 1$ while $\varphi_o^{FDH} \geq 1$. Thus $1 / \lambda_o^+ \leq \varphi_o^{FDH}$ and the interval considered in the theorem for α is non empty (note that $1 / \lambda_o^+ \leq 1 / \lambda_o^-$).

Regarding Algorithm2, to establish the theorem it is sufficient to show that $\lambda_\alpha^+ \geq 1$ and $\lambda_\alpha^- \leq 1$, where

$$
\begin{aligned}
\lambda_\alpha^+ = \max & \sum_{j\in J} \lambda_j \\
St. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_\alpha^{CRS} x_{io}, && i \in \{DI\} \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \geq x_{io}, && i \in \{UI\} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \leq \theta y_{ro}, && r \in \{UO\} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \alpha y_{ro}, && r \in \{DO\} \\
& \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
& \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_\alpha^- = \min \sum_{j\in J} \lambda_j \\
&St. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_\alpha^{CRS} x_{io}, && i \in \{DI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{io}, && i \in \{UI\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \le \theta y_{ro}, && r \in \{UO\} \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge \alpha y_{ro}, && r \in \{DO\} \\
&\qquad \lambda_j = \delta w_j,\ w_j \in \{0,1\}; && j \in J \\
&\qquad \delta \in \Gamma,\ \sum_{j\in J} w_j = 1
\end{aligned}
$$

By Lemma4, $(\lambda^*, \theta^*, \delta^*, w^*)$ is an optimal solution to Models(16), if and only if $(\frac{\lambda^*}{\alpha}, \frac{\theta^*}{\alpha}, \frac{\delta^*}{\alpha}, \frac{w^*}{\alpha})$ is an optimal solution to Models(6).Hence $\lambda_\alpha^+ = \alpha\lambda_o^+$ and $\lambda_\alpha^- = \alpha\lambda_o^-$. By the assumption of the theorem we have

$$1/\lambda_o^+ \le \alpha \le 1/\lambda_o^-$$

Which implies that $\alpha\lambda_o^+ \ge 1$ and $\alpha\lambda_o^- \le 1$.Thus $\lambda_\alpha^+ \ge 1$ and $\lambda_\alpha^- \le 1$and hence CRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ by part(iii) of Theorem1.

Theorem6. Suppose that IRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$.Then IRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ fo

$$\alpha \in (0, \min\{1/\lambda_o^+, \varphi_o^{FDH}\})$$

Proof. See theorem (6) in M. Soleimani-damaneh, A. Mostafaee(2008)

Theorem7. Suppose that DRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$.Then DRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ for

$$\alpha \in (1/\lambda_o^-, \varphi_o^{FDH}]$$

Proof. The proof of this theorem is similar to that of Theorem6 and hence omitted.

Now all main discussions of the paper are summarized in the following. This algorithm determines the RTS classification of units in FDH models and provides the respective stability intervals. Note that this algorithm is polynomial-time (see Theorem3 in[26]).

Algorithm 4.

Step1. compute λ^{jo} for all $j \in J$ by(10).

Step2. compute θ^{jo} for all $j \in J$ by(11).

Step3. compute θ_o^{CRS} and A_o for all $j \in J$ by(12) and(13), respectively.

Step4. compute λ_o^+, λ_o^- by parts(i) and (ii) of Theorem3, respectively.

Step5. compute φ_o^{FDH} the output-oriented FDH efficiency score of $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$ as follows(see Model(14)).

$$\varphi_o^{FDH} = \max_{j \in J: x_j \le x_o} \left\{ \min_{1 \le r \le s} \{ y_{rj} / y_{ro} \} \right\}$$

Step6. If $\lambda_o^- > 1$ then DRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o})$. Furthermore DRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ for each $\alpha \in (1/\lambda_o^-, \varphi_o^{FDH}]$; else

If $\lambda_o^+ < 1$, then IRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o}))$. Furthermore IRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ for each $\alpha \in (0, \min\{1/\lambda_o^+, \varphi_o^{FDH}\})$; else CRS prevail at $(x_{DI_o}, x_{UI_o}, y_{DOo}, y_{Uo_o}))$.Further more CRS prevail at $(x_{Di_o}, \alpha x_{Ui_o}, \alpha y_{Do_o}, y_{Uo_o})$ for each $\alpha \in [1/\lambda_o^+, \min\{1/\lambda_o^-, \varphi_o^{FDH}\}]$.

4. Application

In this section, we show the ability of the provided approach using a numerical example. We apply the proposed method for evaluating 12 units, which each unit uses four inputs to produce three outputs with 3 input, and 2 output are desirable, one input and one output are undesirable. The data set for this example are shown in Table 1.

The RTS classification of all units as well as the related stability intervals have been obtained using Algorithm 4, and have been listed in Table 2.So, Algorithm 4 gives the RTS classification of inefficient DMUs directly without obtaining their projection. Also it should be noted that the RTS status of an inefficient DMU for undesirable data depends on the manner in which we move the DMU to the FDH frontier. Indeed, RTS is a property of the frontier at a specific point, not a property of the DMU that sites a that point.

Table 1

DMU	x_1	x_2	x_3	x_4	y_1	y_2	y_3
DMU_1	20	151	1560	874	60	110	**712**
DMU_2	19	131	1800	652	150	150	**294**
DMU_3	25	160	2120	680	160	55	**213**
DMU_4	16	168	1450	1043	180	87	**215**
DMU_5	22	158	1790	782	94	266	**349**
DMU_6	55	255	2100	550	230	90	**361**
DMU_7	33	235	1320	966	220	210	**338**
DMU_8	31	206	1900	782	152	80	**297**
DMU_9	30	244	2050	834	190	100	**401**
DMU_{10}	50	268	1630	716	250	50	**308**
DMU_{11}	53	306	1670	1112	260	147	**282**
DMU_{12}	38	284	2170	679	150	120	**286**

Table 2

DMU	λ_o^-	λ_o^+	φ_o^{FDH}	**RTS**	**Stability Interval**
1	0.88	0.88	1.03	IRS	[0, 1.03]
2	1	1	1	CRS	[1, 1]
3	0.9	0.9	1.08	IRS	[0, 1.08]
4	1	1	1	CRS	[1, 1]
5	1	1	1	CRS	[1, 1]
6	1.23	1.26	1.17	DRS	[0.81, 1.17]
7	1	1	1	CRS	[1, 1]
8	0.86	0.86	1.46	IRS	[0, 1.16]
9	1.03	1.03	1.36	DRS	[0.97, 1.36]
10	1	1	1	CRS	[1, 1]
11	1	1	1	CRS	[1, 1]
12	0.67	0.67	1.4	IRS	[0, 1.4]

5. Conclusion

Economic notion of returns to scale and calculation method which has been widely studied in the context of DEA, This, in turn, extends the application of the DEA. This paper seeks to complete a polynomial time algorithm to determine the returns to scale for both the efficient and inefficient units is undesirable for the data based on the ratio of input and outputs, without solving any mathematical programming problem. Also, the a major part of article, to preserve some stability classification of returns to scale is proposed. Talk presented can be useful in the development of performance analysis of a practical project that can be done theoretically and computationally.

References

[1] Liu, W.B., W. Meng, X.X. Li and D.Q., DEA models with undesirable inputs and outputs, Annals of Operations Research Zhang (2010), 173, 177-194.

[2] R.D. Banker, A. Charnes, W.W. Cooper, Some models for estimating technicall and scale efficiencies in data envelopment analysis, Management Science 30 (1984) 1078–1092.

[4] W.W. Cooper, L.M. Sieford, K. Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA Solver Software, Kluwer Academic Publishers, (2000).

[5] D. Deprins, L. Simar, H. Tulkens, Measuring labor-efficiency in post offices, in: M. Marchand, P. Pestieau, H. Tulkens (Eds.), The Performance of Public.

[6] R. Fare, S. Grosskopf, C.A.K. Lovell and S. Yaisawarang, Derivation of shadow prices for undesirable outputs: a distance function approach, Rev. Econ. Statist, (1993), 75, (2) 374-380.

[7] R. Fare, S. Grosskopf, Estimating of returns to scale using data envelopment analysis: A comment, European Journal of Operational Research 79 (3) (1995) 379–382.

[8] R. Fare. and S. Grosskopf Modeling undesirable factors in e_ciency evaluation: Comment, European Journal of Operational Research, (2004), 157, 242-245.

[9] B. Golany. and Y. Roll An application procedure for DEA, Omega: The International Journal of Management Science, (1989), 17, 237-250

[10] B. Golany, G. Yu, Estimating returns to scale in DEA, European Journal of Operational Research 103 (1) (1997) 28–37.

[11] G.R. Jahanshahloo, F. Hosseinzadeh Lotfi, M. Zohrehbandian, Notes on sensitivity and stability of the classification of returns to scale in data envelopment analysis, Journal of Productivity Analysis 23 (2005) 309–313.

[12] G.R. Jahanshahloo, M. Soleimani-damaneh, Estimating returns-to-scale in data envelopment analysis: A new procedure, Applied Mathematics and Computation 150 (1) (2004) 89–98.

[13] G.R. Jahanshahloo, M. Soleimani-damaneh, M. Rostamy-Malkhalifeh, An enhanced procedure for estimating returns-to-scale in DEA, Applied Mathematics and Computation 171 (2005) 1226–1238.

[14] K. Kerstens, P. Vanden Eeckaut, Estimating returns-to-scale using nonparametric deterministic technologies: A new method based on goodnessof- fit, European Journal of Operational Research 113 (1999) 206–214.

[15] Koopmans, T.C. Analysis of production as an e_cient combination of activities, in: T.C. Koopmans (Ed.), Activity Analysis of Production and Allocation, Cowles Commission, Wiley, New York (1951), 33-97

[16] Liu, W.B., W. Meng, X.X. Li and D.Q. Zhang (2010), DEA models with undesirable inputs and outputs, Annals of Operations Research(2010), 173, 177-194.

[17] Liu, W.B. and J. Sharp, DEA models via goal programming, In G. Westerman(Ed.), Data envelopment analysis in the public and private sector. Deutscher Universitats-Verlag. (1999).

[18] Lovell, C.A.K., J.T. Pastor and J.A Turner, Measuring macroeconomic performance in the OECD: a comparison of European and non-European countries, European Journal of Operational Research, (1995), 87, 507-518.

[19] V.V. Podinovski, On the linearisation of reference technologies for testing returns to scale in FDH models, European Journal of Operational Research 152 (3) (2004) 800–802.

[20] Pastor, J.T., Translation invariance in data envelopment analysis, Annals of Operations Research, (1996), 66, 93-102.

[21] L.M. Seiford, J. Zhu, Sensitivity and stability of the classifications of returns to scale in data envelopment analysis, Journal of Productivity Analysis 12 (1999) 55–75.

[22] Seiford, L.M. and J. Zhu Modeling undesirable factors in e_ciency evaluation, European Journal of Operational research, (2002), 142, 16-20.

[23] Silva Portela, M.C.A., E. Thanassoulis and G. Simpson Negative data in DEA: a directional distance approach applied to bank branches, Journal of the Operational Research Society, (2004), 55, 1111-1121.

[24]M.Soleimani-damaneh, A.Mostafaee Stability of the classification of return to scale in FDH models Original Research Article European Journal of Operation Research, Volume 196, Issue 3, (2009), pages 1223-1228

[25] M. Soleimani-damaneh, G.R. Jahanshahloo, M. Reshadi, On the estimation of returns to scale in FDH models, European Journal of Operational Research 174 (2) (2006) 1055–1059.

[26] M. Soleimani-damaneh, M. Reshadi, A polynomial-time algorithm to estimate returns to scale in FDH models, Computers and Operations Research 34 (7) (2007) 2168–2176.

[27] Tulkens H. On FDH efficiency analysis: Some methodological issues and applications to retail banking, courts and urban transit. Journal of Productivity Analysis 1993;4:183-210.

[28] M. Zarepisheh, M. Soleimani-damaneh, Global Variation of Outputs with Respect to the Variation of Inputs in Performance Analysis; Generalized RTS, European Journal of Operational Research 186 (2008) 786–800.

[29] M. Zarepisheh, M. Soleimani-damaneh, A Dual Simplex-based Method for Determination of the Right and Left Returns to Scale in DEA. European Journal of Operational Research, (in press).

[30] M. Zarepisheh, M. Soleimani-damaneh, L. Pourkarimi, Determination of returns to scale by CCR formulation without chasing down alternative optimal solutions, Applied Mathematics Letters 19 (9) (2006) 964–967.

[31] Yu, M.M. Measuring physical e_ciency of domestic airports in Taiwan with undesirable outputs and environment factors, Journal of Air Transport Man-agement, (2004), 10, 295-303.