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Original Research

An Uncertain Renewal Stock Model for Barrier Options Pricing with Floating Interest Rate

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ABSTRACT

Option pricing is a main topic in contemporary financial theories, captivating the attention of numerous financial analysts and economists. Barrier option, classified as an exotic option, derives its value from the behavior of an underlying asset. The outcome of this option is based on whether or not the price of the underlying asset has reached a predetermined barrier level. Over the years, the stock price has been represented through continuous stochastic processes, with the prominent one being the Brownian motion process. Correspondingly, the widely used Black-Scholes model has been employed. Nevertheless, it has become evident that utilizing stochastic differential equations to characterize the stock price process is unsuitable and leads to a perplexing paradox. As a result, many researchers have turned to incorporating fuzzy or uncertain environments in such situations. This study presents a methodology for pricing barrier options on stocks in an uncertain environment, in which the interarrival times are uncertain variables. The approach employs the Liu process and renewal uncertain process, considering the interest rate as dynamic and floating. The pricing formulas for knock-in barrier options are derived using α-paths of uncertain differential equations with jumps.

1 Introduction

The pricing of options holds great importance in the financial markets, and it is a subject of considerable interest in mathematical finance. Nevertheless, barrier options and vanilla options share similarities, with the exception that barrier options are either activated or deactivated when the underlying asset price touches the barrier price before the maturity time. Barrier options have been traded in the over-the-counter (OTC) market since 1967 and have become the preferred choice among exotic options. Different pricing methods have been widely utilized in option pricing, including the Black-Scholes [1] and Merton's [2] option pricing theory, in which the price process for underlying assets follows the

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stochastic differential equations (SDEs). Merton [2] was the first to propose a theory for pricing rational options, focusing on down and out options. Rich, on the other hand, contributed to the pricing of barrier options. Subsequently, numerous researchers have explored various approaches for pricing such options. For example, Nouri, Abbasi, et al. [3, 4] introduced an enhanced Monte Carlo algorithm for pricing different types of barrier options. Additionally, [5] employed a Lie-algebraic method to determine the value of moving barrier options, and [6] conducted a study on the analytical valuation of American double barrier options. In 2013, Liu [7] argued that the application of stochastic differential equations to characterize the stock price process is unsuitable and leads to a perplexing paradox. This perspective is substantiated by empirical observations, which reveal that the peak of the distribution of underlying assets exceeds that of a normal probability distribution, accompanied by heavier tails. Numerous empirical studies have shown that the behavior of underlying asset prices does not conform to the principles of probability and randomness. So many researchers have applied fuzzy and uncertain environments to compute option pricing formulas [8-10]. Considering the influence of both randomness and human uncertainty on financial markets, it is evident that an investor's belief holds great importance in shaping market dynamics. As investors tend to base their decisions on their beliefs rather than solely on probabilities. In support of this, Kahneman [11] demonstrated that the degrees of beliefs exhibit a much wider range of variation compared to frequency. In 2004 Cont and Tankov [12] employed jump-diffusion models as an uncertain source and demonstrated the extensive structure these models possess for asset pricing. In 2007 Liu [13] established a theory of uncertainty within the framework of uncertain measure, focusing on the degree of belief. In 2008, Liu [14] introduced the concept of uncertain process to enhance the modelling of uncertain phenomena. Researchers in [15-17] have developed various methods for solving uncertain differential equations (UDEs) based on this work. Additionally, Yao [16] has proposed several numerical techniques for computing integration and differentiation, which can be applied to renewal uncertain processes. Furthermore, Chen and Liu [15] have demonstrated the existence and uniqueness theorem for the solutions of UDEs, and besides Liu [18] has proven the stability of UDEs. In 2009, Liu [18] developed several formulas for option pricing based on an uncertain stock model. Following that, researchers in [19- 23] extensively explored uncertain stock pricing models. Furthermore, Chen [24] introduced a formula to price American options in 2011. Meanwhile, Liu [13] highlighted the importance of uncertain renewal processes, specifically focusing on cases where the interarrival times are uncertain variables. Later on, Liu [25] proposed a renewal reward process that accounted for the uncertainty of interarrival times and rewards. In 2012, Yao [26] established a theory on uncertainty calculus specifically for renewal processes. Jia and Chen [27] conducted a study in 2020, uncovering noteworthy findings on pricing formulas for Knock-in barrier options within an uncertain stock pricing model featuring a floating interest rate. Additionally, Gao, et al. [28] investigated pricing American barrier option of currency model in uncertain environment. Section 2 of the paper begins with the necessary preliminaries. Subsequently, Section 3 presents the stock pricing model in uncertain space, which specifically focuses on real decision problems and incorporates a floating interest rate. Section 4, offers the proof for European knock-in options pricing formulas within the framework of the uncertain stock model. Finally, Section 5 concludes the paper by presenting a summary of the findings.

2 Preliminaries

Consider Γ denote a non-empty set, and define the σ-algebra L be a collection of all the events $\theta \epsilon L$ over Γ. We can define it as a function that assigns to each event θ the belief degree$\mathcal{M}\{\theta\}$, which represents our confidence in the occurrence ofθ. Liu [14] proposed five axioms to provide an axiomatic definition of uncertain measure to ensure that the number $\mathcal{M}\{\theta\}$ is not arbitrary and has special mathematical properties;

1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;

2: (Monotonicity axiom) $\mathcal{M}(\theta_1) \leq \mathcal{M}(\theta_2)$ whenever $\theta_1 \subseteq \theta_2$;

3: (Duality axiom) $\mathcal{M}(\theta) + \mathcal{M}(\theta^c) = 1$ for every event θ;

4: (subadditivity axiom) For each sequence of events $\{\theta_i\}$, that can be counted, we have

$$\mathcal{M}(\bigcup_{i=1}^{\infty} \Theta_i) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Theta_i) \tag{1}$$

Definition 1. [18]. The set function $\mathcal{M}$ which satisfies the above axioms, is called an uncertain measure.

Definition 2. [18]. Consider Γ be a non-empty set, the σ -algebra L, be a collection of all the events over Γ and $\mathcal{M}$ be an uncertain measure according to the above definition. Then the triple $(\Gamma, L, \mathcal{M})$ is called an uncertain space.

5: (Product Measure Axiom) [18]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$ where $\Gamma = \Gamma_1 \times \Gamma_2 \times$ ... and $L = L_1 \times L_2 \times$... be uncertainty space for $k = 1,2,\ldots,$n , then product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ -algebra satisfying the product uncertain measure $\mathcal{M}$ is uncertain measure satisfying,

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Theta_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Theta_k\} \tag{2}$$

Where Θ_k, are arbitrary chosen event from L_k for $k = 1,2,\ldots,n$, respectively.

Definition 3. [18]. The uncertainty distribution for an uncertain variable such as ξ is defined by function $\Phi : R \to [0,1]$ that $\Phi(x) = \mathcal{M}\{\xi \leq x\}$.

Definition 4. Following uncertainty distribution is called normal

$$\Phi(x) = (1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right))^{-1}, \quad x \in \mathbb{R}, \tag{3}$$

If ξ be an uncertain variable, in this case $\sigma > 0$ and e are real numbers and it is shown by$N(e, \sigma)$. The normal uncertainty distribution can be called standard, if $e = 0$ and$\sigma = 1$. So $\Phi^{-1}(\alpha)$، $\alpha \in (0،1)$ is the inverse uncertainty distribution of ξ, if it exists. The expected value of an uncertain variable ξ is defined as

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha, \tag{4}$$

Definition 5. [14] following UDE (uncertain differential equation),

$$dX_t = h(t, X_t)dt + k(t, X_t)dC_t, \quad (5)$$

Has an α-path X_t^α $(0 < \alpha < 1)$, if it solves the bellow corresponding ODE

$$dX_t^\alpha = h(t, X_t^\alpha)dt + |k(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt, \quad (6)$$

Where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha} \quad (7)$$

Definition 6. [18]. Liu process is an uncertain process C_t which have bellow properties

1- $C_0 = 0$;
2- C_t has independent and stationary increments;
3- Almost all sample paths are Lipschitz continuous;
4- All increments C_{t+s} - C_s are normal uncertain variables with expected value 0 and variance t^2.

Theorem 1. Let X_t be the solution of the UDE eq. (5) and α -path X_t^α be the solution of ODE eq. (6). Then

$$\begin{aligned} &\mathcal{M}\{X_t \le X_t^\alpha, \quad \forall t \in [0, T]\} = \alpha, \\ &\mathcal{M}\{X_t > X_t^\alpha, \quad \forall t \in [0, T]\} = 1 - \alpha, \end{aligned} \quad (8)$$

Definition 7. [14] The uncertain process

$$N_t = \max_{n \ge 0}\{n \mid Sn \le t\} \quad (9)$$

is called an uncertain renewal process, if $\xi_1, \xi_2, \xi_3, \ldots$ be iid positive uncertain variables. Also $S_0 = 0$ and $S_n = \sum_{n\ge1} \xi_n$.

The uncertain renewal process N_t has an expected value

$$E[N_t] = \sum_{k=1}^{\infty} \Phi(\frac{t}{n}) \quad (10)$$

Where Φ denote the uncertainty distribution of ξ_is.

Definition 8. [29] Consider that $\xi_1, \xi_2, \xi_3, \ldots$ indicate the interarrival times of sequential events. Hence, N_t is the number of renewals in $(0, T\,]$ and S_n is the total waiting time before the nth event occurs. The relation between the fundamental formulas of an uncertain renewal process are as below:

$$\begin{aligned} &N_t \ge n \Leftrightarrow S_n \le t, \\ &N_t \le n \Leftrightarrow S_{n+1} > t, \end{aligned} \quad (11)$$

Theorem 2. [29] Consider N_t be an uncertain renewal process, if interarrival times $\xi_1, \xi_2, \xi_3, \ldots$ have an uncertainty distribution Φ, then N_t has an uncertainty distribution

$$\Upsilon_t(\alpha) = 1 - \Phi(\frac{t}{\lfloor\alpha\rfloor + 1}), \quad (12)$$

for all $\alpha \ge 0$, where $\lfloor\alpha\rfloor$ denotes the largest integer that is less than or equal to α.

3. Uncertain Model for Stock Pricing with Floating Interest Rate

Assume that the stock price S_t and interest rates r_t follows:

$$\begin{cases} r_t = \mu_1 + \delta \dfrac{dC_{1t}}{dt}, \\ dS_t = \mu_2 S_t dt + \sigma_2 S_t dC_{2t} + \nu_2 S_t dN_t. \end{cases} \tag{13}$$

where C_{1t} and C_{2t} are independent Liu process, δ is a positive real number, μ_1 is the riskless interest rate, μ_2 is log-drift, σ_2 is log-diffusion, ν_2 is the stock price jump size and N_t is an uncertain renewal process. Based on these assumptions, the discount rate is

$$e^{-\int_0^s r_t dt} = e^{-\int_0^s \left(\mu_1 + \delta \frac{dC_{1t}}{dt}\right) dt} = e^{-\mu_1 s - \delta C_{1s}} \tag{14}$$

By solving the differential equation

$$dS_t^\alpha = \mu_2 S_t^\alpha dt + \frac{\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha} \sigma_2 |S_t^\alpha| dt + \nu_2 S_t^\alpha dN_t \tag{15}$$

have an α-path for S_t as

$$S_t^\alpha = S_0 e^{\mu_2 t + \frac{\sqrt{3}\sigma_2 t}{\pi} ln \frac{\alpha}{1-\alpha}} (1+\nu_2)^{N_t} \tag{16}$$

We have

$$\frac{dS_t{}^\alpha}{S_t{}^\alpha} = \mu_2 dt + \sigma_2 dC_t + \nu_2 dN_t \tag{17}$$

Integrating both sides, we get

$$\int_0^t \frac{dS_t{}^\alpha}{S_t{}^\alpha} = \mu_2 t + \sigma_2 C_t + \sum_{i=1}^{N_t} \ln(1+\nu_{s_i}) \tag{18}$$

This means

$$\ln S_t{}^\alpha - \ln S_0 = \mu_2 t + \sigma_2 C_t + \sum_{i=1}^{N_t} \ln(1+\nu_{s_i}),$$

$$\ln S_t{}^\alpha = \ln S_0 + \mu_2 t + \sigma_2 C_t + \textstyle\sum_{i=1}^{N_t} \ln(1+\nu_{s_i}) \ ,$$

$$S_t{}^\alpha = S_0 \exp[\mu_2 t + \sigma_2 C_t + \sum_{i=1}^{N_t} \ln(1+\nu_{s_i})] \tag{19}$$

$$= S_0 \exp(\mu_2 t + \sigma_2 C_t) \prod_{i=1}^{N_t} (1+\nu_{s_i})$$

$$= S_0 \exp(\mu_2 t + \sigma_2 C_t)(1+\nu_2)^{N_t}$$

$$= S_0 \exp(\mu_2 t + \frac{\sqrt{3}\sigma_2 t}{\pi} \ln \frac{\alpha}{1-\alpha})(1+\upsilon_2)^{N_t}$$

Which is a solution to "Eq. (13)".

4 European Knock- In Options

One kind of barrier options is knock-in option which contract that only comes in existence when underlying asset crosses a certain price level. This means that traders can buy or sell this type of options only at the moment and after that the price reaches a particular prespecified level. If the knock-in price level has touched at any time during the lifetime of the options contract, the payoff of the option is converted into a vanilla option and the knock-in barrier option expires worthless. In this section we have presented formula of pricing European UIC (up-and-in call) option which asset price follows Eq. (13) the renewal uncertain model with floating interest rate.

4.1 Pricing Formula for Call Options

Consider an UIC option which in that barrier level is L,The exercise price is K, and the maturity time is T . This call option is invalid and has payoff equal to 0, if before the maturity T , the spot price S_t always be under the barrier level L, i.e.,

$$\sup_{0\le t\le T} S_t < L. \tag{20}$$

If the price of underlying asset S_t hits the designated barrier L and goes above that before the maturity, i.e.,

$$\sup_{0\le t\le T} S_t \ge L. \tag{21}$$

Then, this UIC option will become into existence, and its payoff will be $max(S_t - K, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$B_L(\eta) = \begin{cases} 1, & \eta \ge L, \\ 0, & \eta < L. \end{cases} \tag{22}$$

Hence, the payoff on the maturity time is written as;

$$\text{payoff} = (S_t - K)^+ B_L(\sup_{0\le t\le T} S_t) \tag{23}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$U_{ic} = e^{\mu_1 t + \delta C_{1t}} (S_t - K)^+ B_L(\sup_{0\le t\le T} S_t) \tag{24}$$

and a fair price of this kind of barrier options (UIC option) is

$$f_{ui}^c = E[U_{ic}] = E[e^{\mu_1 T + \delta C_{1t}} (S_t - K)^+ B_L(\sup_{0\le t\le T} S_t)] \tag{25}$$

Theorem 3. Consider an up-and-in call (UIC) option for stock pricing model that underlying uncertain Eq. (13) has a barrier level L, exercise price K, and the maturity data T. Then the fair price of the option is defined by

$$f_{ui}^c = \int_{\alpha_0}^{1} e^{-\mu_1 T - \frac{\sqrt{3}\delta T}{\pi} \ln\frac{1-\alpha}{\alpha}} . \Upsilon(\alpha) d\alpha \tag{26}$$

Where

$$\alpha_0 = (1 + e^{\frac{\pi(ln(S_0) - lnL + \mu_2 T + n ln(1+\nu_2))}{\sqrt{3}\sigma_2 T}})^{-1} \tag{27}$$

and

$$\Upsilon(\alpha) = \sup_{n\geq 0}(\Psi(\frac{T}{n})) \wedge (1 - \Phi(\frac{ln(K + \frac{\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha}) - ln(S_0) - \mu_2 T - nln(1+\nu_2)}{\sigma_2 T})) \tag{28}$$

Here Φ represents the uncertain standard normal distribution of uncertain variables, and Ψ represents the distribution of the interarrival times in uncertain environment for the uncertain renewal process.

Proof. For each $x \geq 0$, we arrive

$$\begin{aligned} M\{(S_T - K)^+ \geq x\} &= M\{S_0 e^{\mu_2 T + \sigma_2 C_{2t}}(1+\nu_2)^{N_t} \geq K + x\} \\ &= M\{\sigma_2 C_{2t} + N_t ln(1+\nu_2) \geq ln(K+x) - ln(S_0) - \mu_2 T\} \\ &= \sup_{n\geq 0} M\{N_T \geq n\} \wedge M\{C_{2t} \geq \frac{ln(K+x) - ln(S_0) - \mu_2 T - nln(1+\nu_2)}{\sigma_2}\} \\ &= \sup_{n\geq 0} M\{S_n \leq T\} \wedge M\{C_{2t} \geq \frac{ln(K+x) - ln(S_0) - \mu_2 T - nln(1+\nu_2)}{\sigma_2}\} \\ &= \sup_{n\geq 0}(\Psi(\frac{T}{n})) \wedge (1 - \Phi(\frac{ln(K+x) - ln(S_0) - \mu_2 T - nln(1+\nu_2)}{\sigma_2 T})) = \Upsilon(x) \end{aligned} \tag{29}$$

Now with Substitute $x = \Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$

$$\Upsilon(\alpha) = \sup_{n\geq 0}(\Psi(\frac{T}{n})) \wedge (1 - \Phi(\frac{\ln(K + \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}) - \ln(S_0) - \mu_2 T - n\ln(1+\nu_2)}{\sigma_2 T})) \tag{30}$$

and so

$$f_{ui}^{c} = \int_0^1 e^{-\mu_1 T - \frac{\sqrt{3}\delta T}{\pi}\ln\frac{1-\alpha}{\alpha}}.\Upsilon(\alpha)d\alpha \tag{31}$$

note that

$$B_L(\sup_{0\leq t\leq T} S_t^{\alpha}) = 1 \tag{32}$$

if and only if

$$\sup_{0\leq t\leq T} S_t^{\alpha} \geq L. \tag{33}$$

and

$$B_L(\sup_{0\leq t\leq T} S_t^{\alpha}) = 0 \tag{34}$$

if and only if

$$\sup_{0\leq t\leq T} S_t^{\alpha} < L. \tag{35}$$

in addition

$$
\begin{aligned}
& S_t^{\alpha} = S_0 e^{\mu_2 T + \frac{\sqrt{3}\sigma_2 T}{\pi} ln\frac{\alpha}{1-\alpha}} (1+\nu_2)^{N_t} \geq L \\
\Rightarrow\quad & ln(S_0) + (\mu_2 T + \frac{\sqrt{3}\sigma_2 T}{\pi} ln\frac{\alpha}{1-\alpha}) + nln(1+\nu_2) \geq lnL \\
\Rightarrow\quad & ln(S_0) - lnL + \mu_2 T + nln(1+\nu_2) \geq -ln\frac{\alpha}{1-\alpha}\frac{\sqrt{3}\sigma_2 T}{\pi} \\
\Rightarrow\quad & \frac{\pi}{\sqrt{3}\sigma_2 T}(ln(S_0) - lnL + \mu_2 T + nln(1+\nu_2)) \geq ln\frac{1-\alpha}{\alpha} \\
\Rightarrow\quad & e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(ln(S_0) - lnL + \mu_2 T + nln(1+\nu_2))} \geq \frac{1-\alpha}{\alpha} \\
\Rightarrow\quad & 1 + e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(ln(S_0) - lnL + \mu_2 T + nln(1+\nu_2))} \geq \frac{1}{\alpha} \\
\Rightarrow\quad & \alpha \geq (1 + e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(ln(S_0) - lnL + \mu_2 T + nln(1+\nu_2))})^{-1} = \alpha_0
\end{aligned}
\tag{36}
$$

so

$$
f_{ui}^{c} = \int_{\alpha_0}^{1} e^{-\mu_1 T - \frac{\sqrt{3}\delta T}{\pi} ln\frac{1-\alpha}{\alpha}} .\Upsilon(\alpha) d\alpha. \tag{37}
$$

Example 1. Assume the initial stock price $S_0 = 15$, risk-less interest rate $\mu_1 = 0.02$, log-drift $\mu_2 = 0.06$, log-diffusion $\sigma_2 = 0.3$, jump size of stock price $\nu_2 = 0.2$, barrier level $L = 26$, strike price $K = 20$, time to maturity $T = 1$, parameter $\delta = 0.005$ and $n = 1$. Then the price of an up-and-in call option is 0.1039.

4.2 Pricing Formula for Put Option

Consider a DIP (down-and-input) option which, in that barrier level is L, exercise price is K, and the maturity time is T . This put option is invalid, and has payoff equal to 0, if before the maturity T , the spot price S_t consistently remains above the barrier level L, i.e.,

$$
\inf_{0\leq t\leq T} S_t > L. \tag{38}
$$

If the price of underlying asset S_t hits the designated level L and goes below of that before the maturity , i.e.,

$$
\inf_{0\leq t\leq T} S_t \leq L. \tag{39}
$$

Then, this DIP option will come into existence, and its payoff will be $\max(K - S_t, 0)$ on the maturity date. Now we assign $\eta^+ = \max(\eta, 0)$ and apply an indicator function

$$
B_L(\eta) = \begin{cases} 1, & \eta > L, \\ 0, & \eta \leq L. \end{cases} \tag{40}
$$

Hence, the payoff on the maturity time is written as

$$
payoff = (K - S_t)^+ (1 - B_L(\inf_{0\leq t\leq T} S_t)) \tag{41}
$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$
D_{ip} = e^{\mu_1 t + \delta C_{1t}} (K - S_t)^+ (1 - B_L(\inf_{0\leq t\leq T} S_t)) \tag{42}
$$

and a fair price of this kind of barrier options (DIP option) is

$$
f_{di}^{p} = E[D_{ip}] = E[e^{\mu_1 T + \delta C_{1t}} (K - S_t)^+ (1 - B_L(\inf_{0\leq t\leq T} S_t))] \tag{43}
$$

Theorem 4. Consider a down-and-input (DIP) option for stock pricing model which follows the uncertain Eq. (13) has a lower barrier$L$, exercise price$K$, and the maturity data$T$. Then the fair price of the option is

$$f_{di}^{p}=\int_{0}^{\alpha_0} e^{-\mu_1 T-\frac{\sqrt{3}\delta T}{\pi}ln\frac{\alpha}{1-\alpha}}.\Upsilon(\alpha)d\alpha \tag{44}$$

Where

$$\alpha_0=(1+e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(ln(S_0)-lnL+\mu_2 T+nln(1+v_2))})^{-1} \tag{45}$$

And

$$\Upsilon(\alpha)=\sup_{n\geq 0}(1-\Psi(\frac{T}{n+1}))\wedge\Phi(\frac{ln(K-\frac{\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha})-ln(S_0)-\mu_2 T-nln(1+v_2)}{\sigma_2 T}) \tag{46}$$

Here Φ represents the uncertain standard normal distribution of uncertain variables, and Ψ represents the distribution of the interarrival times in uncertain environment for the uncertain renewal process.

Proof. For each $x\in[0,K\,]$, we arrive

$$\begin{aligned}M\{(K-S_T)^{+}\geq x\} &= M\{S_0 e^{\mu_2 T+\sigma_2 C_{2t}}(1+v_2)^{N_t}\geq K-x\}\\ &= M\{\sigma_2 C_{2t}+N_t ln(1+v_2)\geq ln(K-x)-ln(S_0)-\mu_2 T\}\\ &= \sup_{n\geq 0} M\{N_T\leq n\}\wedge M\{C_{2t}\leq\frac{ln(K-x)-ln(S_0)-\mu_2 T-nln(1+v_2)}{\sigma_2}\}\\ &= \sup_{n\geq 0} M\{S_n\geq T\}\wedge M\{C_{2t}\leq\frac{ln(K-x)-ln(S_0)-\mu_2 T-nln(1+v_2)}{\sigma_2}\}\\ &= \sup_{n\geq 0}(1-\Psi(\frac{T}{n+1}))\wedge(\Phi(\frac{ln(K-x)-ln(S_0)-\mu_2 T-nln(1+v_2)}{\sigma_2 T}))\\ &= \Upsilon(x)\end{aligned} \tag{47}$$

Now with Substitute $x=\Phi^{-1}(\alpha)=\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$

$$\Upsilon(\alpha)=\sup_{n\geq 0}(1-\Psi(\frac{T}{n+1}))\wedge\Phi(\frac{ln(K-\frac{\sqrt{3}}{\pi}ln\frac{\alpha}{1-\alpha})-ln(S_0)-\mu_2 T-nln(1+v_2)}{\sigma_2 T}) \tag{48}$$

and so

$$f_{di}^{p}=\int_{0}^{1} e^{-\mu_1 t-\frac{\sqrt{3}\delta T}{\pi}ln\frac{1-\alpha}{\alpha}}.\Upsilon(\alpha)d\alpha \tag{49}$$

note that

$$1-B_L(\inf_{0\leq t\leq T} S_t^{1-\alpha})=1 \tag{50}$$

if and only if

$$\inf_{0\leq t\leq T} S_t^{1-\alpha}\leq L. \tag{51}$$

and

$$1-B_L(\inf_{0\leq t\leq T} S_t^{1-\alpha})=0 \tag{52}$$

if and only if

$$\inf_{0\le t\le T} S_t^{1-\alpha} > L. \tag{53}$$

Moreover

$$\begin{aligned}
& S_t^{1-\alpha} = S_0 e^{\mu_2 T + \frac{\sqrt{3}\sigma_2 T}{\pi} ln\frac{1-\alpha}{\alpha}}(1+\nu_2)^{N_t} \le L \\
\Rightarrow\ & ln(S_0) + (\mu_2 T + \frac{\sqrt{3}\sigma_2 T}{\pi} ln\frac{1-\alpha}{\alpha}) + nln(1+\nu_2) \le lnL \\
\Rightarrow\ & lnL - ln(S_0) - nln(1+\nu_2) - \mu_2 T \ge \frac{\sqrt{3}\sigma_2 T}{\pi} ln\frac{1-\alpha}{\alpha} \\
\Rightarrow\ & \frac{\pi}{\sqrt{3}\sigma_2 T}(lnL - ln(S_0) - nln(1+\nu_2) - \mu_2 T) \ge ln\frac{1-\alpha}{\alpha} \\
\Rightarrow\ & e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(lnL - ln(S_0) - \mu_2 T - nln(1+\nu_2))} \ge \frac{1-\alpha}{\alpha} \\
\Rightarrow\ & 1 + e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(lnL - ln(S_0) - \mu_2 T - nln(1+\nu_2))} \ge \frac{1}{\alpha} \\
\Rightarrow\ & \alpha \ge (1 + e^{\frac{\pi}{\sqrt{3}\sigma_2 T}(lnL - \ln(S_0) - \mu_2 T - nln(1+\nu_2))})^{-1} = 1 - \alpha_0
\end{aligned} \tag{54}$$

so

$$f_{di}^{p} = \int_{1-\alpha_0}^{1} e^{-\mu_1 T - \frac{\sqrt{3}\delta T}{\pi} ln\frac{1-\alpha}{\alpha}}.\Upsilon(\alpha)d\alpha = \int_{0}^{\alpha_0} e^{-\mu_1 T - \frac{\sqrt{3}\delta T}{\pi} ln\frac{\alpha}{1-\alpha}}.\Upsilon(\alpha)d\alpha \tag{55}$$

Example 2. Assume the initial stock price $S_0 = 20$, risk-less interest rate $\mu_1 = 0.02$, log-drift $\mu_2 = 0.06$, log-diffusion $\sigma_2 = 0.3$, jump size of stock price $\nu_2 = 0.2$, barrier level $L = 8$, strike price $K = 15$, time to maturity $T = 1$, parameter $\delta = 0.005$ and $n = 1$. Then the price of a down-and-in put option is 0.0028.

5 Conclusion

Since the probability space and randomness aren't sufficient space for simulation of investor decisions, many researchers propose Liu uncertain space for using in such cases. In this paper, we have presented an uncertain renewal process, in which interarrival times are uncertain variables to compute the prices of barrier options on stocks, using the Liu process and renewal uncertain process which in the interest rate is floating and dynamic. Formulas for pricing two kinds of knocked-in options (up-and-in call options and down-and-in put options) are arrived by α-paths of UDEs with jumps.

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