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Research Paper

Generalized Krasnoselskii–Mann Type Iterations for Two Non-expansive Mappings in Real Hilbert Spaces

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ABSTRACT

In this paper, we propose a novel Mann iterative algorithm for discovering a shared fixed point of two nonexpansive mappings in real Hilbert spaces. We establish the weak convergence of this fixed-point approach under new conditions, and additionally demonstrate its strong convergence with the inclusion of an additional requirement. Our findings expand upon previous results presented by Kanzow and Shehu, as well as by Cho et al. Furthermore, we showcase the versatility of our main results through the presentation of various applications in the last section, accompanied by illustrative examples.

1 Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. An operator $T : C \longrightarrow C$ (C is a nonempty, closed and convex subset of H) is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C.$ For the set of all fixed points of $T,$ we use the notation $F(T)$. One of the important fixed point algorithm is the Krasnoselskii-Mann iteration [15.17]. This algorithm, is given by the following iterative sequence

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \qquad \forall n \in \mathbb{N},$$

where $x_0 \in H$ and $\alpha_n \in]0,1[$ for all $n \in \mathbb{N}$. This algorithm was developed by a number of authors; see for example [1-3,5-8,10,11,14,16,19,20] and the references therein. Xu and Ori [22] proposed the

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implicit Mann-like iterative method for a finite family of nonexpansive mappings $\{T_1, T_2, ..., T_N\}$ with a real sequence $\{\alpha_n\}$ in $]0,1[$ and an initial point $x_0 \in C$;

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n \qquad \forall n \in \mathbb{N},$$

where $T_n \equiv T\ (n \bmod N)$. They obtained some weak convergence results, by using this sequence.
In 2017 Kanzow and Shehu [13] considered the following iteration and generalized the Krasnoselskii-Mann algorithm:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n \qquad \forall n \geq 0,$$

where $\alpha_n, \beta_n \in]0,1[$ are satisfy $\alpha_n + \beta_n \leq 1$, and the residual vector r_n. By considering this algorithm, they obtained the following theorem.

Theorem 1.1. Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : H \longrightarrow K$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_1 \in H$ and using the recursion

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n \qquad \forall n \geq 1, \tag{1}$$

where r_n denotes the residual vector. Here we assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$ and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

(b) $\sum_{n=1}^{\infty} \| r_n \| < \infty$,

(c) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{x_n\}$ generated by (1) converges weakly to a fixed point of T.

The following useful lemmas are required for the main results of this article.
Lemma 1.2. [12] Let E be a uniformly convex Banach space. Let $s > 0$ be a positive number and let $B_s(0)$ be a closed ball of E. There exits a continuous, strictly increasing and convex function $g : [0,\infty[\longrightarrow [0,\infty[$ with $g(0) = 0$ such that

$$\|ax + by + cz + dw\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|)$$

for all $x, y, z, w \in B_s(0) = \{ x \in E ;\ \|x\| \leq s \}$ and $a, b, c, d \in [0,1]$ such that $a + b + c + d = 1$.

Lemma 1.3. [18] If in a Hilbert space H the sequence $\{x_n\}$ is weakly convergent to x_0, then for any $x \neq x_0$,

$$\liminf_{n\to\infty}\|x_n - x_0\| < \liminf_{n\to\infty}\|x_n - x\| \qquad \forall x_0 \in H.$$

Lemma 1.4. [21] Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying the following condition:

$$a_{n+1} \leq a_n + b_n, \qquad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.5. [4] Let E be a real uniformly convex Banach space, C be a nonempty, closed and convex subset of E and let $T: C \longrightarrow C$ be a nonexpansive mapping. Then $I-T$ is demiclosed at zero, that is, $x_n \xrightarrow{w} x$ and $x_n - Tx_n \xrightarrow{s} 0$ imply that $Tx = x$.

In Section 2, by using lemma 1.2, we give a new proof and develop a novel Mann-type approach for two nonexpansive mappings and show that it is weakly convergent by considering new requirements. In Section 3, we show that the algorithm proposed in Section 2 has strong convergence to a common fixed point of two nonexpansive mappings by considering an extra criterion.

2 Weak Convergence

In this section, we first describe a new generalized Krasnoselskii-Mann algorithm for identifying a common fixed point of two nonexpansive mappings, as well as explore its weak convergence. The following theorem, is the main results of this section. In particular, the following theorem is a fairly stright forward generalization of Theorem 4 of [13], which considers two functions.

Theorem 2.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T_1, T_2 : C \longrightarrow C$ are two nonexpansive mappings with $F(T_1) \cap F(T_2) \neq \phi$. Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \lambda_n r_n, \qquad \forall n \geq 1 \tag{2}$$

where $x_0 \in H$, $\{r_n\}$ denote the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n + \lambda_n = 1$ for all $n \geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \lambda_n < \infty;$

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \alpha_n \gamma_n = \infty;$

(c) $\{r_n\}$ is bounded.

Then the sequence $\{x_n\}$ generated by (2) converges weakly to some $\hat{x} \in F(T_1) \cap F(T_2)$.

proof. Let us first observe that the sequence $\{x_n\}$ is bounded. For this purpose, suppose $x^* \in F(T_1) \cap F(T_2)$. For all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \lambda_n r_n - x^*\| \\ &= \|\alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \lambda_n r_n - (\alpha_n + \beta_n + \gamma_n + \lambda_n) x^*\| \\ &\le \alpha_n \|x_n - x^*\| + \beta_n \|T_1 x_n - x^*\| + \gamma_n \|T_2 x_n - x^*\| + \lambda_n \|r_n - x^*\|.\end{aligned}$$

According to the above inequality and the nonexpansiveness of T_1 and T_2, we get:

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \alpha_n \|x_n - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| + \lambda_n \|r_n - x^*\| \\ &\le (\alpha_n + \beta_n + \gamma_n) \|x_n - x^*\| + \lambda_n \|r_n - x^*\| \\ &\le \|x_n - x^*\| + \lambda_n \|r_n - x^*\|.\end{aligned}$$

By using Lemma 1.4, and from the condition (a), we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This shows that the sequence $\{x_n\}$ is bounded. Now, there exists a function g_1 that satisfies the conditions in Lemma 1.2 such that, for all $n \in \mathbb{N}$:

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \lambda_n r_n - x^*\|^2 \\ &= \|\alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \lambda_n r_n - (\alpha_n + \beta_n + \gamma_n + \lambda_n) x^*\|^2 \\ &= \|\alpha_n (x_n - x^*) + \beta_n (T_1 x_n - T_1 x^*) + \gamma_n (T_2 x_n - T_2 x^*) + \lambda_n (r_n - x^*)\|^2 \\ &\le \alpha_n \|x_n - x^*\|^2 + \beta_n \|T_1 x_n - T_1 x^*\|^2 + \gamma_n \|T_2 x_n - T_2 x^*\|^2 + \lambda_n \|r_n - x^*\|^2 \\ &\quad - \alpha_n \beta_n g_1(\|x_n - T_1 x_n\|) \\ &\le \alpha_n \|x_n - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 + \lambda_n \|r_n - x^*\|^2 \\ &\quad - \alpha_n \beta_n g_1(\|x_n - T_1 x_n\|).\end{aligned}$$

It follows that

$$\begin{aligned}\alpha_n \beta_n g_1(\|x_n - T_1 x_n\|) &\le (\alpha_n + \beta_n + \gamma_n) \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \lambda_n \|r_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \lambda_n \|r_n - x^*\|^2\end{aligned}$$

and hence:

$$\sum_{n=1}^{k}\alpha_n\beta_n g_1(\|x_n-T_1x_n\|)\le\|x_0-x^*\|^2-\|x_k-x^*\|^2+\sum_{n=1}^{k}\lambda_n\|r_n-x^*\|^2$$
$$\le\|x_0-x^*\|^2+\sum_{n=1}^{k}\lambda_n\|r_n-x^*\|^2.$$

From above inequality, and by using the condition (a), we conclude that:

$$\sum_{n=1}^{\infty}\alpha_n\beta_n g_1(\|x_n-T_1x_n\|)\le\|x_0-x^*\|^2+\sum_{n=1}^{\infty}\lambda_n\|r_n-x^*\|^2<\infty$$

and hence we get from the condition (b) that

$\lim_{n\to\infty} g_1(\|x_n-T_1x_n\|)=0.$

Since g_1 is strictly increasing and continuous, and from $g_1(0)=0,$ we obtain that

$\lim_{n\to\infty}\|x_n-T_1x_n\|=0.$ (3)

Again, there exists a function g_2 that satisfies the conditions in Lemma 1.2 such that

$$\|x_{n+1}-x^*\|^2=\|\alpha_nx_n+\beta_n T_1x_n+\gamma_nT_2x_n+\lambda_nr_n-x^*\|^2$$
$$=\|\alpha_n(x_n-x^*)+\beta_n(T_1x_n-T_1x^*)+\gamma_n(T_2x_n-T_2x^*)+\lambda_n(r_n-x^*)\|^2$$
$$\le\alpha_n\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2+\lambda_n\|r_n-x^*\|^2$$
$$-\alpha_n\gamma_n g_2(\|x_n-T_2x_n\|).$$

Using the same technique as in the previous case, the same result may be derived for T_2. Hence

$\lim_{n\to\infty}\|x_n-T_2x_n\|=0.$ (4)

Then we show that the sequence $\{x_n\}$ converges weakly. Since $\{x_n\}$ is bounded, then t exists a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $\hat{x}\in C$. Note from Lemma 1.5 and the relations (3) and (4) that $\hat{x}\in F(T_1)\cap F(T_2)$.

Next we show $\{x_n\}$ converges weakly to some $\hat{x}$. Suppose the contrary, then there exists some subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $\{x_{m_k}\}$ converges weakly to some $\bar{x}\in C$, where $\bar{x}\neq\hat{x}$. Similarly, we can show $\bar{x}\in F(T_1)\cap F(T_2)$. Notice that we have proved that $\lim_{n\to\infty}\|x_n-x^*\|$ exists for each $x^*\in F(T_1)\cap F(T_2)$. Assume that $\lim_{n\to\infty}\left\|x_n-\hat{x}\right\|=d$. By Lemma 1.3; we see that

$$d=\lim_{n\to\infty}\left\|x_n-\hat{x}\right\|=\lim_{k\to\infty}\left\|x_{n_k}-\hat{x}\right\|<\liminf_{k\to\infty}\left\|x_{n_k}-\bar{x}\right\|=\lim_{n\to\infty}\left\|x_n-\bar{x}\right\|$$
$$=\lim_{k\to\infty}\left\|x_{m_k}-\bar{x}\right\|=\liminf_{k\to\infty}\left\|x_{m_k}-\bar{x}\right\|<\liminf_{k\to\infty}\left\|x_{m_k}-\hat{x}\right\|=\lim_{n\to\infty}\left\|x_n-\hat{x}\right\|=d.$$

This is a contradiction. Hence $\bar{x} = \hat{x}$ and this completes the proof. □

By taking $T_1 = T$ and $T_2 = I$ and replacing β_n and γ_n in Theorem 2.1 by $\frac{\beta_n}{2}$, we conclude the following corollary.

Corollary 2.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : C \longrightarrow C$ is a nonexpansive mapping with $F(T) \neq \phi$.
Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \lambda_n r_n, \qquad \forall n \geq 1 \tag{5}$$

where $x_0 \in H$, $\{r_n\}$ denote the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are real sequences in [0,1] such that $\alpha_n + \beta_n + \lambda_n = 1$ for all $n \geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(c) $\{r_n\}$ is bounded.

Then the sequence $\{x_n\}$ generated by (5) converges weakly to some $\hat{x} \in F(T)$.

We can consider the general case of Theorem 2.1 as follows, that extends the previous result given by Cho at et. al. [9].

Theorem 2.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T_1, T_2, ..., T_N : C \longrightarrow C$ are nonexpansive mappings with $F(T_1) \cap ... \cap F(T_N) \neq \phi$.
Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \sum_{i=1}^{N} \beta_{i_n} T_i x_n + \lambda_n r_n, \qquad \forall n \geq 1$$

where $x_0 \in H$, $\{r_n\}$ denote the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are real sequences in [0,1] such that $\alpha_n + \sum_{i=1}^{N} \beta_{i_n} + \lambda_n = 1$ for all $n \geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_{i_n} = \infty$; $1 \leq i \leq N$;

(c) $\{r_n\}$ is bounded.

Then the sequence $\{x_n\}$ converges weakly to some $\hat{x} \in \bigcap_{i=1}^{N} F(T_i)$.

proof. The proof is similar to the proof of Theorem 2.1. $\square$

Example 2.4. Suppose $H = \mathbb{R}$, $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be defined by $T_1 x = 0$, $T_2 x = x$ for all $x \in \mathbb{R}$. Then it is clear that T_1, T_2 are nonexpansive and $0 \in F(T_1) \cap F(T_2)$. Furthermore, let us take $\alpha_n = \frac{1}{3n}$, $\beta_n = \gamma_n = \frac{1}{2}(1 - \frac{1}{3n})$, $\lambda_n = 0$, $r_n = 0$ for all $n \geq 1$, Then it is easy to see that, the sequence $\{r_n\}$ is bounded and the following conditions hold.

$$\sum_{n=1}^{\infty} \lambda_n = 0 < \infty;$$

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \frac{1}{3n} \times \frac{1}{2}(1 - \frac{1}{3n}) = \infty;$$

$$\sum_{n=1}^{\infty} \alpha_n \gamma_n = \sum_{n=1}^{\infty} \frac{1}{3n} \times \frac{1}{2}(1 - \frac{1}{3n}) = \infty.$$

Now, for any initial point $x_1 \in \mathbb{R}$, our iterative scheme (2) becomes

$$x_{n+1} = \frac{1}{3n} x_n + \frac{1}{2}(1 - \frac{1}{3n})(0) + \frac{1}{2}(1 - \frac{1}{3n}) x_n + 0 = (\frac{1}{2} + \frac{1}{3n} - \frac{1}{6n}) x_n.$$

It is then clear that, the sequence $\{x_n\}$ converges to $0 \in F(T_1) \cap F(T_2)$.

3 Strong Convergence

In this section, by considering an additional condition in Theorem 2.1, we prove the strong convergence of the sequence $\{x_n\}$ that introduced in Theorem 2.1.

Theorem 3.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T_1, T_2 : C \longrightarrow C$ are two nonexpansive mappings with $F = F(T_1) \cap F(T_2) \neq \phi$. Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n + \delta_n u_n, \qquad \forall n \geq 1 \tag{6}$$

where $x_0 \in H$, $\{u_n\}$ is a bounded sequence, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 1$, and such that the following conditions hold:

(a) $\sum_{n=1}^{\infty} \delta_n < \infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \alpha_n \gamma_n = \infty$;

(c) there exists a nondecreasing function $f:[0,\infty[\longrightarrow[0,\infty[$ with $f^{-1}(0)=\{0\}$ such that $f(d(x,F))\le\|x-T_1x\|+\|x-T_2x\|$ for all $x\in X$.

Then the sequence $\{x_n\}$ generated by (6) converges strongly to some $x^*\in F$.

proof. First we show that the sequence $\{x_n\}$ is bounded. Suppose $x^*\in F$. From (6), for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n x_n+\beta_n T_1x_n+\gamma_n T_2x_n+\delta_n u_n-x^*\|\\ &= \|\alpha_n x_n+\beta_n T_1x_n+\gamma_n T_2x_n+\delta_n u_n-(\alpha_n+\beta_n+\gamma_n+\delta_n)x^*\|\\ &\le \alpha_n\|x_n-x^*\|+\beta_n\|T_1x_n-x^*\|+\gamma_n\|T_2x_n-x^*\|+\delta_n\|u_n-x^*\|.\end{aligned}$$

Then we obtain from $x^*\in F$, recent relation and the nonexpansiveness of T_1,T_2 that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \alpha_n\|x_n-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|x_n-x^*\|+\delta_n\|u_n-x^*\|\\ &\le(\alpha_n+\beta_n+\gamma_n)\|x_n-x^*\|+\delta_n\|u_n-x^*\|.\end{aligned}$$

Thus:

$$\|x_{n+1}-x^*\|\le\|x_n-x^*\|+\delta_n\|u_n-x^*\|. \tag{7}$$

From Lemma 1.4, we see by using the restriction (a) that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. It follows that the sequence $\{x_n\}$ is bounded.

Now by using Lemma 1.2 there exists a mapping g_1(that satisfies the conditions in Lemma 1.2) such that for all $n\in\mathbb{N}$

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n x_n+\beta_n T_1x_n+\gamma_n T_2x_n+\delta_n u_n-x^*\|^2\\ &= \|\alpha_n(x_n-x^*)+\beta_n(T_1x_n-T_1x^*)+\gamma_n(T_2x_n-T_2x^*)+\delta_n(u_n-x^*)\|^2\\ &\le \alpha_n\|x_n-x^*\|^2+\beta_n\|T_1x_n-T_1x^*\|^2+\gamma_n\|T_2x_n-T_2x^*\|^2+\delta_n\|u_n-x^*\|^2\\ &\quad-\alpha_n\beta_n g_1(\|x_n-T_1x_n\|)\\ &\le \alpha_n\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2+\delta_n\|u_n-x^*\|^2\\ &\quad-\alpha_n\beta_n g_1(\|x_n-T_1x_n\|).\end{aligned}$$

The above inequality shows that

$$\alpha_n\beta_n g_1(\|x_n-T_1x_n\|)\le(\alpha_n+\beta_n+\gamma_n)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\delta_n\|u_n-x^*\|^2$$

$$\leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + \delta_n \left\|u_n - x^*\right\|^2.$$

Therefore,

$$\sum_{n=1}^{k} \alpha_n \beta_n g_1(\|x_n - T_1 x_n\|) \leq \left\|x_0 - x^*\right\|^2 + \sum_{n=1}^{k} \delta_n \left\|u_n - x^*\right\|^2.$$

By the condition (a), we have

$$\sum_{n=1}^{\infty} \alpha_n \beta_n g_1(\|x_n - T_1 x_n\|) \leq \left\|x_0 - x^*\right\|^2 + \sum_{n=1}^{\infty} \delta_n \left\|u_n - x^*\right\|^2 < \infty$$

Now we conclude from the condition (b) that,

$\lim_{n\to\infty} g_1(\|x_n - T_1 x_n\|) = 0.$

By using the properties of g_1, we obtain from the above inequality that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{8}$$

By a similar method used for T_1, we conclude that

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0. \tag{9}$$

Now we obtain from (8), (9) and by taking the limsup as $n \to \infty$ of the inequality in the condition (c) that

$\lim_{n\to\infty} f(d(x_n, F)) = 0$

And hence

$\lim_{n\to\infty} d(x_n, F) = 0.$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$ then for any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, F) < \frac{\varepsilon}{2}$ for all $n \geq N$. Putting $\theta_n = \delta_n \left\|u_n - x^*\right\|$, we see from (7) that

$\left\|x_{n+1} - x^*\right\| \leq \left\|x_n - x^*\right\| + \theta_n.$

For all $n \in \mathbb{N}$ there exists $x^* \in F$ such that

$\left\|x_n - x^*\right\| \leq d(x_n, F) + \frac{\varepsilon}{2}.$

Thus for any positive integers m, n, with $m > n$, we have

$$\begin{aligned}
\left\|x_m - x^*\right\| &\leq \left\|x_n - x^*\right\| + \sum_{j=n+1}^{m} \theta_j \\
&\leq d(x_n, F) + \frac{\varepsilon}{2} + \sum_{j=n+1}^{m} \theta_j \\
&\leq \varepsilon + \sum_{j=n+1}^{m} \theta_j
\end{aligned}$$

and therefore

$$\|x_n - x_m\| = \|x_n - x_m - x^* + x^*\| \le \|x_n - x^*\| + \|x_m - x^*\| \le 2\|x_n - x^*\| + \sum_{j=n+1}^{m} \theta_j.$$

This implies that

$$\|x_n - x_m\| \le 2\varepsilon + \sum_{j=n+1}^{m} \theta_j.$$

It follows from the restriction (a) that $\{x_n\}$ is a Cauchy sequence in C and so $\{x_n\}$ converges strongly to some $\hat{x} \in C.$ Since $F = F(T_1) \cap F(T_2)$ is closed, we obtain that $\hat{x} \in F.$ This completes the proof. $\square$

By taking $T_1 = T$ and $T_2 = I$ and replacing β_n and γ_n in Theorem 3.1 by $\frac{\beta_n}{2},$ we conclude the following corollary.

Corollary 3.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : C \longrightarrow C$ is a nonexpansive mapping with $F(T) \neq \phi$. Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \delta_n u_n, \qquad \forall n \ge 1 \tag{10}$$

where $x_0 \in H,$ $\{u_n\}$ denotes the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] such that $\alpha_n + \beta_n + \delta_n = 1$ for all $n \ge 1,$ and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \delta_n < \infty;$

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty;$

(c) there exists a nondecreasing function $f : [0,\infty[\longrightarrow [0,\infty[$ with $f^{-1}(0) = \{0\}$ such that $f(d(x,F)) \le \|x - Tx\|$ for all $x \in X$.

Then the sequence $\{x_n\}$ generated by (10) converges strongly to some $\hat{x} \in F(T).$

The following theorem, is a general case of Theorem 3.1 for finite family of nonexpansive mappings.

Theorem 3.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T_1, T_2, ..., T_N : C \longrightarrow C$ are nonexpansive mappings with $F(T_1) \cap ... \cap F(T_N) \neq \phi.$ Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1} = \alpha_n x_n + \sum_{i=1}^{N} \beta_{i_n} T_i x_n + \delta_n u_n, \qquad \forall n \geq 1$$

where $x_0 \in H$, $\{u_n\}$ is a bounded sequence, and where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] such that $\alpha_n + \sum_{i=1}^{N} \beta_{i_n} + \delta_n = 1$ for all $n \geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \delta_n < \infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n \beta_{i_n} = \infty;\ 1 \leq i \leq N$;

(c) there exists a nondecreasing function $f : [0,\infty[\longrightarrow [0,\infty[$ with $f^{-1}(0) = \{0\}$ such that $f(d(x,F)) \leq \sum_{i=1}^{N} \|x - T_i x\|$ for all $x \in X$.

Then the sequence $\{x_n\}$ converges strongly to some $\hat{x} \in \bigcap_{i=1}^{N} F(T_i)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Example 3.4. Suppose $H = \mathbb{R}$, $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be defined by $T_1 x = -x + 2$, $T_2 x = \frac{x}{2} + \frac{1}{2}$ for all $x \in \mathbb{R}$. Then it is clear that T_1, T_2 are nonexpansive and $F(T_1) \cap F(T_2) = \{1\}$. Furthermore, let us take $\alpha_n = 1 - \frac{1}{2n} - \frac{1}{2n^2}$, $\beta_n = \gamma_n = \frac{1}{4n}$, $\delta_n = \frac{1}{2n^2}$, $u_n = 0$ for all $n \geq 1$, and $f(x) = \frac{x}{5}$. Then it is easy to see that

$$\sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} \frac{1}{2n^2} < \infty;$$

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \alpha_n \gamma_n = \sum_{n=1}^{\infty} (1 - \frac{1}{2n} - \frac{1}{2n^2}) \times \frac{1}{4n} = \infty;$$

and for every $x \in \mathbb{R}$

$$\frac{d(x,1)}{5} = f(d(x,1)) \leq \|x - (-x+2)\| + \left\|x - (\frac{x}{2} + \frac{1}{2})\right\| \leq \|2x - 2\| + \left\|\frac{x}{2} + \frac{1}{2}\right\| \leq \frac{5}{2}\|x - 1\|.$$

Therefor all conditions in Theorem 3.1 are hold. Now, for any initial point $x_1 \in \mathbb{R}$, our iterative scheme (6) becomes

$$x_{n+1} = (1 - \frac{1}{2n} - \frac{1}{2n^2}) x_n + \frac{1}{4n}(-x_n + 2) + \frac{1}{4n}(\frac{x_n}{2} + \frac{1}{2}) + \frac{1}{2n^2} \times 0$$

$$= (1 - \frac{5}{8n} - \frac{1}{2n^2}) x_n + \frac{5}{8n}$$

Then it is clear that the sequence $\{x_n\}$ convergence to $x=1$.

4 Applications

In this section, some applications of the main results are shown. To begin, we demonstrate how our results may be used to the Douglas-Rachford splitting method for obtaining the zeros of an operator T that is the sum of two maximal monotone operators, i.e. $T=A+B$ where $A,B:H\to 2^H$ are maximal monotone multi-functions on a real Hilbert space H. The method was originally introduced in [12] in a finite-dimensional setting, its extension to maximal monotone mappings in Hilbert spaces can be found in [21].

Theorem 4.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $A,B:H\to 2^H$ are two maximal monotone operators with $A^{-1}(0)\cap B^{-1}(0)\neq\phi.$ Let $J_\gamma A, J_\gamma B:H\to C$ be resolvant operators that induced by A and B, respectively. Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1}=\alpha_n x_n+\beta_n J_\gamma A x_n+\gamma_n J_\gamma B x_n+\lambda_n r_n, \qquad \forall n\geq 1 \tag{11}$$

where $x_0\in H,$ the bounded sequence $\{r_n\}$ denote the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n+\lambda_n=1$ for all $n\geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty}\lambda_n<\infty;$

(b) $\sum_{n=1}^{\infty}\alpha_n\beta_n=\sum_{n=1}^{\infty}\alpha_n\gamma_n=\infty;$

(c) $\{r_n\}$ is bounded.

Then the sequence $\{x_n\}$ generated by (11) converges weakly to some $x^*\in A^{-1}(0)\cap B^{-1}(0).$

Proof. We know the corresponding resolvant operators $J_\gamma A, J_\gamma B$ are (firmly) nonexpansive then by using Theorem 2.1, the result is obtained. □

Theorem 4.2. Let $A,B\subseteq H$ be two nonempty, closed, and convex subsets of a real Hilbert space H, and suppose that $A\cap B\neq\phi.$ Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1}=\alpha_n x_n+\beta_n P_A\, x_n+\gamma_n P_B x_n+\lambda_n r_n, \qquad \forall n\geq 1 \tag{12}$$

where $x_0\in H,$ the bounded sequence $\{r_n\}$ denote the residual vector, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n+\lambda_n=1$ for all $n\geq 1$, and the following conditions hold:

(a) $\sum_{n=1}^{\infty}\lambda_n<\infty;$

(b) $\sum_{n=1}^{\infty}\alpha_n\beta_n=\sum_{n=1}^{\infty}\alpha_n\gamma_n=\infty$;

(c) $\{r_n\}$ is bounded.

Then the sequence $\{x_n\}$ generated by (12) converges weakly to some $x^*\in A\cap B$.

Proof. We know the corresponding projection operators P_A, P_B are (firmly) nonexpansive then by using Theorem 2.1, the result is obtained. □

Considering an additional condition, we show in the following theorems that the algorithm introduced in the above theorems has a strong convergence.

Theorem 4.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $A,B:H\to 2^H$ are two maximal monotone operators with $A^{-1}(0)\cap B^{-1}(0)\neq\phi$. Let $J_\gamma A, J_\gamma B:H\to C$ be resolvant operators that induced by A and B, respectively.
Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1}=\alpha_n x_n+\beta_n J_\gamma Ax_n+\gamma_n J_\gamma Bx_n+\delta_n u_n,\qquad \forall n\geq 1 \tag{13}$$

where $x_0\in H$, $\{u_n\}$ is a bounded sequence, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\geq 1$, and such that the following conditions hold:

(a) $\sum_{n=1}^{\infty}\delta_n<\infty$;

(b) $\sum_{n=1}^{\infty}\alpha_n\beta_n=\sum_{n=1}^{\infty}\alpha_n\gamma_n=\infty$;

(c) there exists a nondecreasing function $f:[0,\infty[\longrightarrow[0,\infty[$ with $f^{-1}(0)=\{0\}$ such that $f(d(x,F))\leq\|x-J_\gamma Ax\|+\|x-J_\gamma Bx\|$ for all $x\in X$.

Then the sequence $\{x_n\}$ generated by (13) converges strongly to some $x^*\in A^{-1}(0)\cap B^{-1}(0)$.

Proof. We know the corresponding resolvant operators $J_\gamma A, J_\gamma B$ are (firmly) nonexpansive then by using Theorem 3.1, the result is obtained. □

Theorem 4.4. Let $A,B\subseteq H$ be two nonempty, closed, and convex subsets of a real Hilbert space H, and suppose that $A\cap B\neq\phi$. Let the sequence $\{x_n\}$ in H be generated as follows:

$$x_{n+1}=\alpha_n x_n+\beta_n P_A\, x_n+\gamma_n P_B x_n+\delta_n u_n,\qquad \forall n\geq 1 \tag{14}$$

where $x_0\in H$, $\{u_n\}$ is a bounded sequence, and where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n+\delta_n=1$ for all $n\geq 1$, and such that the following conditions hold:

(a) $\sum_{n=1}^{\infty}\delta_n<\infty$;

(b) $\sum_{n=1}^{\infty}\alpha_n\beta_n=\sum_{n=1}^{\infty}\alpha_n\gamma_n=\infty$;

(c) there exists a nondecreasing function $f:[0,\infty[\longrightarrow[0,\infty[$ with $f^{-1}(0)=\{0\}$ such that $f(d(x,F))\le\|x-P_Ax\|+\|x-P_Bx\|$ for all $x\in X$.

Then the sequence $\{x_n\}$ generated by (14) converges strongly to some $x^*\in A\cap B$.

Proof. We know the corresponding projection operators P_A, P_B are (firmly) nonexpansive then by using Theorem 3.1, the result is obtained. □

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