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Research Paper

Which Currency Will Evolve?

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ABSTRACT

In this paper, using Evolutionary Game Theory, we study the evolution of currency as an important and influential problem in the world economy by presenting three new models. In examining the evolution of two different currencies, we prove that if the parties chose either of the two currencies with a fixed probability, it will not have an impact on the evolutionary process of these currencies. Considering a new action based on either the possibility of using any two currencies or a preference between the two currencies, we illustrate their impact on the evolutionary process of currencies. In fact, we examine the impact of different actions and strategies on the evolutionary process between the two types of currencies.

1 Introduction

In many interactive situations, there is interdependence between the two parties' decision. Game theory as an effective tool tries to analyze such situations. Therefore, this theory has many applications in economic, social, political and biological issues and so on. (See [2], [3], [4], [5], [6], [7], [8], [15], [14], [17], and [21]). A field of game theory as Evolutionary Game Theory considers a population of decision makers. In this population, the frequency of a specific decision can alter in response to the decisions made by all individuals in the population. That is, in these situations, different strategies can have different fitness (expected payoffs) and so the profile of the population will change toward the strategy with higher fitness. Such strategies are called an Evolutionary Stable Strategy, ESS. In this regard, an interesting question that arises is: What are the endpoints of the evolution? (See [22], [23]).
According to the above explanation the concept of ESS has many applications, especially in biology, because it can examine fitness of a mutation and its prevalence in the population (See [1], [10], [12], and [14]). On the other hand, in today's society, individuals who play different games in different situations change their strategy to achieve greater payoffs and so such cases can be analyzed in form of population games (See [9], [11], [13], [18], [19], [20], [23], and [24]).
Population games in Evolutionary Game Theory have many applications, especially in topics whose endpoints are important in their evolutionary process. In population games, different strategies and dynamics can be examined in the evolutionary process and their results can be compared. It should be

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noted, however, that presenting a strategic game that reflects the conditions of the game is very important to provide a more accurate analysis because it makes the modeling closer to the actual situation. One of the issues that can be studied as a population game is the evolution of currency because in the human society, each of the common currencies is used in a frequency. Nowadays, because of the validity of some currencies, such as the dollar and the euro, most of the current transactions are made with these currencies. On the other hand, with the advancement of technology and the emergence of digital currencies such as Bitcoin this important question arises: What kind of currency will eventually evolve in the world? And what strategies will be effective and helpful in this evolutionary process?
In this study, we have tried to answer such questions. The example of the evolution of money in [23] examines the evolution of currency as a population game. Taking an idea of this example and considering the importance of the evolution of currency, we present new models with different actions. Also, the phase portraits of Replicator dynamics for each model are plotted in different cases.

2 Main Results

To examine the evolution of currency, we assume that two types of currency are used for trading in the community under discussion and trading is possible when both sides use the same currency for exchange. It should be noted that these different currencies can be real currencies such as dollars, euros, pounds and so on and can be digital currencies such as bitcoin and so on. Therefore, the results of this analysis can be used to examine the evolution of each of these currencies against another. We assume that society uses two types of currency for trading, C and D. If both parties agree on a single currency to exchange, that transaction will be made. A trader gets an outcome of 1 if the deal is successful and 0 if it fails. Therefore, the game of this financial transaction is

Player 2

Player 1

	C	D
C	1,1	0,0
D	0,0	1,1

This game has symmetric Nash equilibria $\sigma_1 = (1,0), \sigma_2 = (0,1)$. We find mixed strategy Nash equilibria using the Equality of Payoffs theorem.

$$\pi_1(D, \sigma^*) = \pi_1(C, \sigma^*)$$

$$q^* = 1 - q^*$$

$$q^* = \frac{1}{2}.$$

By symmetry $p^* = \frac{1}{2}$, so the mixed-strategy Nash equilibrium is $\sigma_{3=}\left(\frac{1}{2}, \frac{1}{2}\right)$. We investigate the conditions of ESS for symmetric Nash equilibria:

I) For $\sigma^* = (1,0)$ and $\sigma = (p, 1-p)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p$. Since $\sigma^* \neq \sigma$ then $p < 1$ obviously $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$, so $\sigma_1 = (1,0)$ is an ESS.

II) Similarly, $\sigma_2 = (0,1)$ is also an ESS.

III) For $\sigma^* = \sigma_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\sigma = (p, 1-p) \neq \sigma^*$, we have $\pi(\sigma^*, \sigma^*) = \frac{1}{2} = \pi(\sigma, \sigma^*)$.

It should be noted that, in general, the condition $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ for all mixed-strategy Nash equilibria is obviously established and the main condition for ESS is $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$.

On the other hand

$$\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) = -\frac{1}{2} - 2p(p-1) > 0$$

$$\Leftrightarrow -\frac{1}{2} - 2p^2 + 2p > 0$$

$$\Leftrightarrow (2p-1)^2 < 0.$$

Since $p \neq \frac{1}{2}$, this condition cannot be satisfied, so $\sigma_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$ is not an ESS. We consider the above game as a pairwise contest. Let x be the proportion of individuals using C then the population profile is $X = (x, 1-x)$ and

$$\pi(C, X) = x, \quad \pi(D, X) = 1 - x.$$

So the replicator dynamics for the proportion of individuals using C is

$$\begin{aligned} \dot{x} &= x(\dot{1} - x)\,)(\pi(C, X) - \pi(D, X)) \\ &= x(1-x)(2x-1) \end{aligned} \tag{2.1}$$

So the fixed points are $x^* = 0$, $x^* = 1$, and $x^* = \frac{1}{2}$. According to equation (2.1), if $x > \frac{1}{2}$ then $\dot{x} > 0$ and so the population will evolve towards that fixed point that corresponds to the Nash equilibrium of $\sigma_1 = (1,0)$. And if $x < \frac{1}{2}$ then $\dot{x} < 0$ and so the population will evolve towards that fixed point that corresponds to the Nash equilibrium of$\sigma_2 = (0,1)$. Putting together the above results we can conclude that the population of the community evolves only to use one of these two types of currency, and of course it depends only on the proportion of people who initially choose C or D. This result itself is highly controversial. Since governments have a major role in the world economy, they can politically encourage people to use their preferred currency, whether real or digital, then they will benefit greatly from this evolutionary process. Now we want to add other actions to the above model to provide a more detailed analysis of the evolution of currency.

2.1 Model 1

In this model we add a third action M to the basic model. M means that a person always chooses C with constant probability $0 < p < 1$ and D with probability constant $1 - p$. In this case, the game of this model is

Player 2

Player 1

	C	D	M
C	1,1	0,0	p, p
D	0,0	1,1	$1-p, 1-p$
M	p, p	$1-p, 1-p$	$p^2+(1-p)^2, p^2+(1-p)^2$

But the necessary condition for $\sigma = (0,0,1)$ to be a symmetric Nash equilibrium strategy is that

$$p^2 + (1-p)^2 \geq p \qquad \wedge \qquad p^2 + (1-p)^2 \geq 1-p.$$

But

$$p^2 + (1-p)^2 \geq p$$

$$\Leftrightarrow 2p^2 - 3p + 1 \geq 0$$

$$\Leftrightarrow p \leq \frac{1}{2}$$

And

$$p^2 + (1-p)^2 \geq 1-p$$

$$\Leftrightarrow 2p^2 - p \geq 0$$

$$\Leftrightarrow p \geq \frac{1}{2}.$$

So it follows from these two conditions that $\sigma = (0,0,1)$ is a symmetric Nash equilibrium strategy if $p = \frac{1}{2}$ and in that case, the game of this model becomes

Player 2

Player 1

	C	D	M
C	1,1	0,0	½,½
D	0,0	1,1	½,½
M	½,½	½,½	½,½

Clearly this game has three symmetric pure-strategy Nash equilibria $\sigma_1 = (1,0,0)$, $\sigma_2 = (0,1,0)$ and $\sigma_3 = (0,0,1)$. To find a mixed-strategy Nash equilibrium, we use the Equality of Payoffs theorem. Assuming $\sigma^* = (p_1, p_2, p_3)$, we have

$$\pi_1(D, \sigma^*) = \pi_1(C, \sigma^*) = \pi_1(M, \sigma^*).$$

$$p_1 + \frac{p_2}{2} = p_2 + \frac{p_3}{2} = \frac{1}{2}.$$

But

$$\begin{cases} p_1 + p_2 + p_3 = 1 \\ p_1 + \dfrac{p_3}{2} = \dfrac{1}{2} \\ p_2 + \dfrac{p_3}{2} = \dfrac{1}{2} \end{cases} \tag{2.2}$$

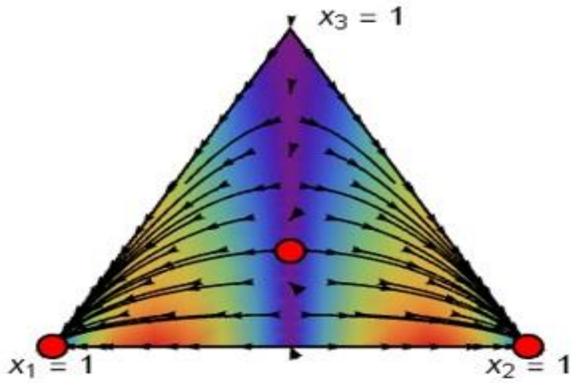
has an infinite number of solutions. In fact $\forall \;\; 0 \leq p < 1 \;\; \sigma^* = (p, p, 1 - 2p)$ is a solution of (2.2). Now we investigate the conditions of ESS for symmetric Nash equilibria:

I) For $\sigma^* = \sigma_1 = (1,0,0)$ and $\forall\, \sigma = (p_1, p_2, p_3)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p_1 + 0 + \frac{p_3}{2}$. Since $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$, so $\sigma_1 = (1,0,0)$ is an ESS.

II) Similarly, $\sigma_2 = (0,1,0)$ is also an ESS.

III) If $\sigma^* = \sigma_3 = (0,0,1)$, $\sigma = (p_1, p_2, p_3) \neq \sigma^*$, we have

$$\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*) = \frac{1}{2}.$$



critical points

x_1	x_2	x_3	λ_1	λ_2
0.0000400028	0.9999	0.0000600024	−0.99975	−0.49988
0.349993	0.350007	0.299999	0.349915	−0.0000500002
0.99991	0.0000300018	0.0000600012	−0.99979	−0.4999

Fig.1: The Phase Portrait of Replicator Dynamics of Model 1.

On the other hand

$$\pi(\sigma^*,\sigma)-\pi(\sigma,\sigma)=\frac{1}{2}-p_1^2-p_1p_3-\frac{p_3^2}{2}>0$$
$$\Leftrightarrow \tfrac{1}{2}-p_1^2-p_2^2-\tfrac{1}{2}(1-(p_1+p_2)^2)>0$$
$$\Leftrightarrow -\frac{1}{2}(p_1-p_2)^2>0.$$

But this condition cannot be satisfied, so $\sigma_3=(0,0,1)$ is not an ESS.

IV) Since $\forall\ o<p<1\quad \sigma^*=(p,p,1-2p)$ is a mixed strategy of this game and

$\pi(\sigma^*,\sigma^*)=\pi(\sigma,\sigma^*)=\frac{1}{2}$ then the ESS condition becomes $\pi(\sigma^*,\sigma)>\pi(\sigma,\sigma)$. Now $\pi(\sigma^*,\sigma)=\frac{1}{2}$ and $\pi(\sigma,\sigma)=p_1^2+p_2^2+\frac{1}{2}(1-(p_1+p_2))(1+(p_1+p_2))$. So

$$\pi(\sigma^*,\sigma)-\pi(\sigma,\sigma)>0$$
$$\Leftrightarrow \frac{1}{2}-p_1^2-p_2^2-\frac{1}{2}(1-(p_1+p_2)^2)>0$$
$$\Leftrightarrow -\frac{1}{2}(p_1-p_2)^2>0$$

But this condition cannot be satisfied, so $\forall\ o<p<1\ \ \sigma^*=(p,p,1-2p)$ is not an ESS. According to the above, this model only has two ESSs, $\sigma_1=(1,0,0)$, and $\sigma_2=(0,1,0)$. (see Figure 1). That is, adding M where individuals always use each of these two currencies with a fixed probability, has no effect on the evolutionary process of the currencies and the results of this model are the same as the main model. In fact, the strategy $\sigma_3=(0,0,1)$ in model 1 is the same as strategy $\sigma_3=(\frac{1}{2},\frac{1}{2})$ in the main model.

2.2 Model 2

In this model we add the third action N to the basic model. In this action one can trade with both C and D currencies but there is no preference between C and D. In fact, by introducing this action, we are considering individuals who invest in different things and have the potential for both types of trading. According to the above assumptions, the game of this model is

Player 2

Player 1		C	D	N
	C	1,1	0,0	1,1
	D	0,0	1,1	1,1
	N	1,1	1,1	1,1

About $u_1(N,N)=1$, we can say when both players select N, they choose either C or D. This game has three symmetric pure-strategy Nash equilibria $\sigma_1=(1,0,0)$, $\sigma_2=(0,1,0)$, and $\sigma_3=(0,0,1)$.To find a mixed-strategy Nash equilibrium, we assume $\sigma^* = (p_1, p_2, p_3)$ then

$$p_1 + p_3 = p_2 + p_3 = p_1 + p_2 + p_3 = 1.$$

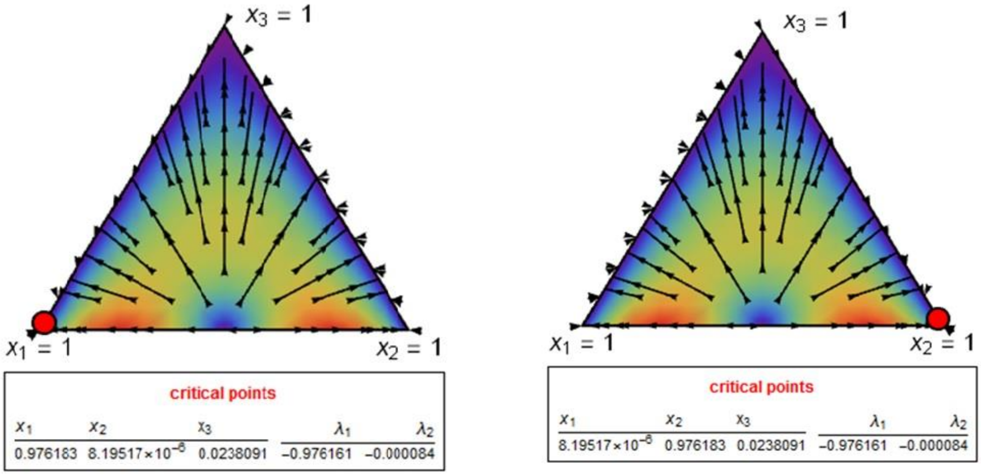


Fig. 2: The Phase Portraits of Replicator Dynamics of Model 2

But

$$\begin{cases} p_1 + p_2 + p_3 = 1 \\ p_1 + p_3 = 1 \\ p_2 + p_3 = 1 \end{cases} \tag{2.3}$$

has a unique solution, $\sigma^* = (0,0,1)$. In other words, this game only has three symmetric pure-strategy Nash equilibria σ_1, σ_2 and σ_3. So, to find the ESSs of this model, we examine the following three cases:

I) For $\sigma^* = \sigma_1 = (1,0,0)$ and $\forall\ \sigma = (p_1, p_2, p_3)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p_1 + p_3$. Since $\pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*)$, so we must examine the condition $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$ for $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$. If $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ then $p_2 = 0$ and $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) = (p_1 + p_3) - (p_1^2 + p_1p_3 + p_2^2 + p_2p_3 + p_3p_1 + p_3p_2 + p_3^2) = p_1 + p_3 - (p_1 + p_3)^2 = 0.$ *So* $\sigma_1 = (1,0,0)$ is not an ESS.

II) Similarly, $\sigma_2 = (0,1,0)$ is not an ESS.

III) For $\sigma^* = \sigma_3 = (0,0,1)$ and $\forall\ \sigma = (p_1, p_2, p_3)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p_1 + p_1 + p_3 = 1$. Since $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$, so we have to investigate the second condition of being ESS for $\sigma^* = \sigma_3 = (0,0,1)$, that is, $\forall\ \sigma^* \neq \sigma, \pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$. But $\pi(\sigma^*, \sigma) = 1$ *and* $\pi(\sigma, \sigma) = p_1^2 + p_1p_3 + p_2^2 + p_2p_3 + p_3p_1 + p_3p_2 + p_3^2$ then

$$\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$$

$$\Leftrightarrow 1-(p_1^2+p_1p_3+p_2^2+p_2p_3+p_3p_1+p_3p_2+p_3^2)>0$$
$$\Leftrightarrow 1-p_1^2-p_2^2-p_3^2-2p_3(p_1+p_2)>0$$
$$\Leftrightarrow 1-p_1^2-p_2^2-p_3^2-2p_3(1-p_3)>0$$
$$\Leftrightarrow 1-p_1^2-p_2^2+p_3^2-2p_3>0$$
$$\Leftrightarrow 1-p_1^2-p_2^2+(1-p_1-p_2)^2-2(1-p_1-p_2)>0$$

$$\Leftrightarrow 2p_1p_2>0 \tag{2.4}$$

But (2.4) cannot be satisfied $\forall\ \sigma=(p_1,p_2,p_3)\neq\sigma^*$. For example, consider $\sigma=(0.2,0,0.8)$, then $2p_1p_2=0$. Therefore $\sigma_3=(0,0,1)$ is not an ESS. (see Figure 2).

So this model has no ESS unlike previous models. Thus the proportion of individuals who initially choose C or D will not influence the evolution of these currencies. Therefore, choosing actions such as N will have much impact on changing the population profile. Also, according to this model, when there is no difference between the two currencies, the population profile will not evolve into a particular currency. This model shows that the use of new actions such as N can alter the evolutionary process. Therefore, the results of this model encourage economic powers to find new actions that lead to their desired population profile.

2.3 Model 3

In this model we consider R as the third action, which in some sources is referred to as retaliation. In R, the individual can trade with both types of currency but prefer one and here we assume D to be preferred. According to the above description, the game of this model is

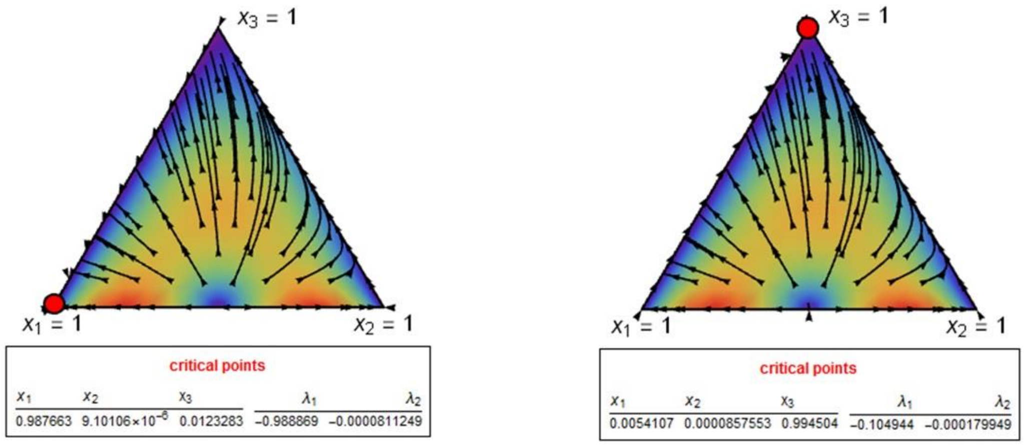


Fig. 3: The Phase Portraits of Replicator Dynamics of Model 3.

The symmetric pure-strategy Nash equilibria of this game are $\sigma_1=(1,0,0)$, and $\sigma_2=(0,0,1)$. Also using the Equality of Payoffs theorem proves that this game has no symmetric mixed-strategy Nash equilibrium.

Player 2

Player 1

	C	D	R
C	1,1	0,0	1,1
D	0,0	1,1	0.9,1.1
R	1,1	1.1,0.9	1,1

Now we discuss the ESS condition for symmetric Nash equilibria:

I) For $\sigma^* = \sigma_1 = (1,0,0)$ and $\forall\ \sigma = (p_1, p_2, p_3)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p_1 + p_3$. Since $\pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*)$, then the ESS condition becomes $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$ for $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$. If $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ then $p_2 = 0$ and $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) = 1 - (p_1^2 + p_1p_3 + p_2^2 + (0.9)p_2p_3 + p_3p_1 + (1.1)p_3p_2 + p_3^2) = 1 - (p_1 + p_3)^2 = 0.$ So $\sigma_1 = (1,0,0)$ is not an ESS.

II) For $\sigma^* = \sigma_2 = (0,0,1)$ and $\forall\ \sigma = (p_1, p_2, p_3)$ in which $\sigma^* \neq \sigma$, we have $\pi(\sigma^*, \sigma^*) = 1$ and $\pi(\sigma, \sigma^*) = p_1 + 0.9\, p_2 + p_3$. Since $\pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*)$, so we have to investigate the second condition of being ESS for $\sigma^* = \sigma_2 = (0,0,1)$, that is, $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$. If $\pi(\sigma^*, \sigma) = 1 = p_1 + 0.9\, p_2 + p_3$ then $p_2 = 0$. On the other hand $\pi(\sigma^*, \sigma) = p_1 + 1.1\, p_2 + p_3$ and $\pi(\sigma, \sigma) = p_1^2 + p_1p_3 + p_2^2 + (0.9)p_2p_3 + p_3p_1 + (1.1)p_3p_2 + p_3^2 = p_1^2 + p_2^2 + p_3^2 + 2p_1p_3 + 2p_2p_3$.

then

$$\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) = (p_1 + 1.1\, p_2 + p_3) - (p_1^2 + p_2^2 + p_3^2 + 2p_1p_3 + 2p_2p_3$$
$$= p_1 + p_3 - (p_1 + p_3)^2 = 0$$

which proves that σ_2 is not an ESS. (see Figure 3).

Therefore, the third model showed that adding a new action that indicates individuals' preference for a particular currency will lead to changes in the evolutionary process of currencies.

3 Conclusion

Considering the importance of the currency evolution for trading worldwide, we have examined three models in this study. It was proved that if the parties chose either of the two currencies with a fixed probability, it would not have an impact on the evolutionary process of these currencies. But if we consider a new action based on either the possibility of using any two currencies or a preference between two currencies, the evolutionary process of currencies will change. So if the economic powers decide to dominate their preferred currency or eliminate a particular currency in the world, the second and third models will show the success for which they will make a lot of profit. The models presented in this study suggest that for this purpose, new actions and strategies that lead to new ESSs or elimination of existing ESSs need to be found and explored.

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