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On a Generalized Subclass of p-Valent Meromorphic Functions by Defined q-Derivative Operator

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ABSTRACT

Financial Mathematics is the application of mathematical methods to financial problems. It is shown that p-valent functions play important roles in Financial Mathematics. In this paper, we define a new subclass of meromorphically p-valent functions by using q-derivative operator and fractional q-calculus operator. We obtain some geometric properties of coefficient estimates, extreme points, convex linear combination, radii of starlikeness and convexity. Finally, ε-neighborhood property will be investigated.

1 Introduction

The field of Mathematical Finance has undergone a remarkable development since the seminal papers by Black and Scholes [1] and Merton [2], in which the famous Meromorphic Functions was derived. Although, Louis Bachelier is considered the author of the first scholarly work on mathematical finance, published in 1900, mathematical finance emerged as a discipline in the 1970s, following the work of Fischer Black, Myron Scholes and Robert Merton [1,2] on option pricing theory [3]. Mathematical finance, also known as quantitative finance and financial mathematics, is a field of applied mathematics, concerned with mathematical modeling of financial markets. Generally, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to financial theory, taking observed market prices as input [4]. Meromorphic Functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Meromorphic Functions have been studied in many areas such as differential geometers [5-9]; mathematical finance [10-14]. Let $\sum_{p,n}$ denote the class of meromorphic functions of the type

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{+\infty} a_{n-p} z^{n-p}, \quad (a_{n-p} \geq 0, p \in \mathbb{N}) \tag{1}$$

which are analytic in the punctured open disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If $f \in \sum_{p,n}$ *is* given by Eq. (1) and $g \in \sum_{p,n}$ given by $g(z) = \frac{1}{z^p} + \sum_{n=1}^{+\infty} a_{n-p} z^{n-p}$ then the Hadamard product (or convolution) $f * g$ of f and g is defined by $(f * g)(z) = \frac{1}{z^p} + \sum_{n=1}^{+\infty} a_{n-p} b_{n-p} z^{n-p}$.

The q-shifted factorial is defined for $w, q \in \mathbb{C}$ as a product of n factors by:

$$(w;q)_n = \begin{cases} 1 & , n = 0 \\ (1-w)(1-wq)\cdots(1-wq^{n-1}) & , n \in \mathbb{N} \end{cases}$$

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and in terms of the basic analogue of the gamma function

$$(q^w;q)_n = \frac{\Gamma_q(w+n)(1-q)^n}{\Gamma_q(w)}, \tag{2}$$

Where the q-gamma function is defined by [15]; $\Gamma_q(z) = \frac{(q;q)_\infty(1-q)^{1-z}}{(q^z;q)_\infty}$, $(0 < q < 1)$.

In view of the relation (2), we get $\lim_{q\to 1} \frac{(q^w;q)_n}{(1-q)^n} = (w)_n$, Where $(w)_n$ denotes the Pochhammer symbol given by

$$(w)_n = \frac{\Gamma(w+n)}{\Gamma(w)} = \begin{cases} 1 & , n = 0 \\ w(w+1)\cdots(w+n-1) & , n \in \mathbb{N}. \end{cases}$$

Also, Jacksons q-derivative and q-integral of a function $f(z)$ defined on a subset of $\mathbb{C}$ are, respectively, given by

$$D_{q,z}f(z) = \frac{f(qz)-f(z)}{(q-1)z} \quad (z \neq 0\,, q \neq 1) \tag{3}$$

and

$$\int_0^z f(x)dx = f(t)d(t;q) = z(1-q)\sum_{k=0}^{+\infty} q^k f(zq^k). \tag{4}$$

see Gasper and Rahman [16, 17,18]. Purohit and Raina [16], used fractional q-calculus operator investigating certain classes of functions which are analytic in the open disc. For some recent investigations on the sub classes of analytic functions defined by using q −calculus operator and related topics [19-23].

2 Main Results

Here, we tend to investigate some important concepts of Harmonic Functions that are useful in theory of mathematical finance [24]. In order to put forward our methodology, we start with introducing the following important concepts that are used throughout the paper. In this section, we by using the definitions of the fractional q-calculus operator (fractional q-integral operator and fractional q-derivative operator) a new subclass of meromorphically p-valent functions. Following Gasper and Rahman [15], the fractional q- integral operator $I^{\varepsilon}_{q,z}f(z)$ of a function $f(z)$ of order δ is defined by

$$I^{\delta}_{q,z}f(z) = D^{-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-tq)_{\delta-1}f(t)d(t;q) \ , \ (\delta > 0) \tag{5}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin the q-binomial function $(z-tq)_{\delta-1}$ is given by

$$(z-tq)_{\delta-1} = z^{\delta-1}\ {}_1\Phi_0\left[q^{-\delta+1}; -; q, tq^{\delta}\,/\,z\right].$$

The series ${}_1\Phi_0$ [δ; −; q , z] is single valued when $|arg(z)| < \pi$ and $|z| < 1$ (see for details [15]). Therefore, the function $(z-tq)_{\delta-1}$ in (5) is single valued when $|arg(-tq^{\delta}/z)| < \pi$, $|tq^{\delta}\,/z| < 1$ and $|arg(z)| < \pi$. The fractional q-derivative operator $D^{\delta}_{q,z}f(z)$ of a function $f(z)$ of order δ is defined by

$$D_{q,z}^{\delta}f(z)=D_{q,z}I_{q,z}^{1-\delta}f(z)=\frac{1}{\Gamma_q(1-\delta)}D_{q,z}\int_0^z(z-tq)_{-\delta}f(t)d(t:q)\quad,\quad(0\leq\delta\leq1)\tag{6}$$

And $D_{q,z}^{\delta}f(z)=D_{q,z}^{k}I_{q,z}^{k-\delta}f(z)$,$(-1\leq\delta<k\ ,k\in\mathbb{N})$. By (5) and (6), we give the following image formulas for the function z^{n-p} under the fractional q- integral and fractional q-derivative operators defined.

$$I_{q,z}^{\delta}z^{n-p}=\frac{\Gamma_q(n-p+1)}{\Gamma_q(n-p+\delta+1)}z^{n-p+\delta},\qquad \alpha\geq0\,,\delta>0\,,n\in\mathbb{N})$$

and

$$I_{q,z}^{\delta}z^{n-p}=\frac{\Gamma_q(n-p+1)}{\Gamma_q(n-p-\delta+1)}z^{n-p-\delta},\qquad \alpha\geq0\,,\delta>0\,,n\in\mathbb{N})$$

Juma et al. [22] define linear multiplier fractional q-differintegral operator $\Omega_{q,\lambda}^{\varepsilon,m}$, as $\Omega_{q,\lambda}^{\varepsilon,m}:\Sigma_p\to\Sigma_p$ Such that:

$$\Omega_{q,\lambda}^{\varepsilon,m}f(z)=\frac{1}{z^p}+\sum_{n=1}^{+\infty}\left[\frac{\Gamma_q(k-p+2)\Gamma_q(1-p-\varepsilon)}{\Gamma_q(1-\mathrm{p})\Gamma_q(k-p+2-\varepsilon)}\left[1-\lambda+\frac{[k-p+1]_q\lambda}{p}\right]\right]^m a_{k-p+1}z^{k-p+1}.$$

For the purpose of this paper, by using fractional q-differintegral operators $D_{q,z}^{\delta}$ and $\Omega_{q,\lambda}^{\varepsilon,m}$, we define a fractional q-differintegral operator $\Psi_{p,q}^{\delta,m}(\alpha,\lambda):\Sigma\mathrm{p},\alpha\to\Sigma\mathrm{p},\alpha$, as follows Corresponding to the function

$$\Omega_{p,q}^{\delta,m}(\lambda;z)=\frac{1}{z^p}+\sum_{n=1}^{+\infty}\left[\frac{\Gamma_q(1-p-\delta)\Gamma_q(n-p+1)(p+p^{\lambda}+(n-p)q^{\lambda})}{p\Gamma_q(1-p)\Gamma_q(n-p-\delta+1)}\right]^m z^{n-p}.$$

$(z\in U,p\in\mathbb{N},0\leq q<1,m\in\mathbb{N}_0\,,\lambda\geq0,\delta>0)$.

Let us define the function $\Omega_{p,q}^{\delta,m,-1}(\alpha,\lambda;z)$, the generalized multiplicative inverse of $\Omega_{p,q}^{\delta,m}(\lambda;z)$ given by the relation

$$\Omega_{p,q}^{\delta,m}(\lambda;z)*\Omega_{p,q}^{\delta,m,-1}(\alpha,\lambda;z)=\frac{1}{z^p(1-z)^{\alpha+p}}\ ,\qquad(\alpha+p>0)$$

Note that if $\alpha+p=1$, then $\Omega_{p,q}^{\delta,m,-1}(\alpha,\lambda;z)$ is the inverse of $\Omega_{p,q}^{\delta,m}(\lambda;z)$ with respect to the Hadamard product $*$. Using this function, we define the following family of transforms $\Psi_{p,q}^{\delta,m}(\alpha,\lambda)$ defined by

$$\Psi_{p,q}^{\delta,m}(\alpha,\lambda)f(z)=\Omega_{p,q}^{\delta,m,-1}(\alpha,\lambda;z)*f(z)=\frac{1}{z^p}+\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)a_{n-p}z^{n-p}$$

with

$$\Phi_{p,q}^{\delta,m}(n,\lambda)=\left[\frac{p\Gamma_q(1-p)\Gamma_q(n-p-\delta+1)}{\Gamma_q(1-p-\delta)\Gamma_q(n-p+1)(p+p\lambda+(\mathrm{n}-\mathrm{p})q^{\lambda})}\right]^m$$

$(z\in U,p\in\mathbb{N},0\leq q<1,m\in\mathbb{N}_0\,,\alpha>-p\,,\lambda\geq0,\delta>0)$.

We now apply $\Psi_{p,q}^{\delta,m}(\alpha,\lambda)$ to define the subclass $\Sigma_{p,n}$ as follows. Let $-1 \le B < A \le 1$, and $0 < q < 1$. Then a function $f \in \Sigma_{p,n}$ given (1) is said to be in the class $\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$ if it satisfies the inequality

$$\left|\frac{\mu z^{p+2}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)f(z))''+z^{p+1}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)f(z))'-p[\mu(p+1)-1]}{B[\mu z^{p+2}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)f(z))''+z^{p+1}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)f(z))']-Ap[\mu(p+1)-1]}\right| < 1, \tag{7}$$

$(z \in U, p \in \mathbb{N}, -1 \le B \le 0, 0 \le \mu < \dfrac{1}{p+1}, m \in \mathbb{N}_0, \alpha > -p, \lambda \ge 0, \delta > 0)$.

We note that

(i) For $\alpha = 1-p, m = 0$ and $\mu = \lambda$ we get $\Sigma_{p,q}^{\delta,\mu,0}(1-p,\lambda,A,B) = \Sigma_{s,p}^{\lambda}(A,B)$, (see [13]).

(ii) For $\alpha = 1-p, m = 0$ and $\mu = \lambda = 0$ we get $\Sigma_{p,q}^{\delta,0,0}(1-p,\lambda,A,B) = \Sigma_{s,p}^{0}(A,B) = H(p;\ A,B)$, (see [25]).

We now obtain the following coefficient bounds for function of the form (1) to belong to the class $\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$ that defined above.

Theorem 1: *Let $f \in \Sigma_{p,n}$. The $f\ \Sigma \in_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$ if and only if*

$$\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]a_{n-p} \le p(B-A)[\mu(p+1)-1] \tag{8}$$

The integrality is sharp for $F(z)$ given by

$$F(z) = \frac{1}{z^p} + \frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}z^{n-p}, (n \ge 1)$$

Proof.

Assume that the inequality (8) holds true and let $0 < |z| = r < 1$. Then from (7) we have:

$$\mathrm{M}(f) = \left|\mu z^{p+2}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda))f(z))'' + z^{p+1}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda))f(z))' - p[\mu(p+1)-1]\right|$$

$$-\left|B[\mu z^{p+2}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda))f(z))'' + z^{p+1}(\Psi_{p,q}^{\delta,m}(\alpha,\lambda))f(z))'] - Ap[\mu(p+1)-1]\right|$$

$$= \left|\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n\right|$$

$$-\left|p(B-A)[\mu(p+1)-1] + B\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n\right|$$

$$\le \sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}r^n - p(B-A)[\mu(p+1)-1]$$

$$-B\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}r^n$$

$\leq \sum_{n=1}^{+\infty} \frac{(\alpha+p)_n}{(1)_n} \Phi_{p,q}^{\delta,m} (n,\lambda)(1 - B)(n - p)[\mu(n - p - 1) + 1]a_{n-p} - p(B - A)[\mu(p + 1) - 1] \leq 0.$

by virtue of (8). Hence, by the principle of maximum modulus, $f \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. Conversely, Let $f \in \Sigma_{p,q}^{\delta,\mu,m} (\alpha,\lambda,A,B)$. Then

$$\left|\frac{\mu z^{p+2}(\Phi_{p,q}^{\delta,m} (\alpha,\lambda))f(z))'' + z^{p+2} (\Phi_{p,q}^{\delta,m} (\alpha,\lambda))f(z))' - p[\mu(p + 1) - 1]}{B[\mu z^{p+2}(\Phi_{p,q}^{\delta,m} (\alpha,\lambda))f(z))'' + z^{p+2} (\Phi_{p,q}^{\delta,m} (\alpha,\lambda))f(z))'] - A\, p[\mu(p + 1) - 1]}\right|$$

$$= \left|\frac{\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n}{p(B-A)[\mu(p+1)-1]+B\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n}\right| < 1 .$$

Since $Re(z) \leq |z|$ for all z,

$$Re\left\{\frac{\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n}{p(B-A)[\mu(p+1)-1]+B\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(n-p)[\mu(n-p-1)+1]a_{n-p}z^n}\right\} < 1.$$

By letting $z \to \overline{1}$ through real values, we have $\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m} (n,\lambda)(1 - B)(n - p)[\mu(n - p - 1) + 1]a_{n-p} \leq p(B - A)[\mu(p + 1) - 1]$. Finally, sharpness follows if we take

$$F(z) = \frac{1}{z^p} + \frac{(1)_n(p(B-A)}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]} z^{n-p},$$

where $n \geq 1$. This completes the proof. □

Corollary 1: *Let* $f \in \Sigma_{p,q}^{\delta,\mu,m} (\alpha,\lambda,A,B)$. *Then* $a_{n-p} \leq \frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]} z^{n-p}$,

where $z \in U, p \in \mathbb{N}, -1 \leq B \leq 0, 0 \leq \mu < \frac{1}{p+1}, m \in \mathbb{N}, \alpha > -p, \lambda \geq 0$ *and* $\delta > 0$.

Corollary 2: *Let* $0 \leq \mu_1 < \mu_2 < \frac{1}{p+1}$. *Then* $\Sigma_{p,q}^{\delta,\mu_2,m}(\alpha,\lambda,A,B) \subset \Sigma_{p,q}^{\delta,\mu_1,m} (\alpha,\lambda,A,B)$.

Theorem ***2:*** *The function* $f(z)$ *of the form (1) belongs to* $\Sigma_{p,q}^{\delta,\mu,m} (\alpha,\lambda,A,B)$ *if and only if it can be expressed by*

$$f(z) = \sum_{n=0}^{\infty} d_{n-p}f_{n-p}(z) ,\ d_{n-p} \geq 0 ,\ \sum_{n=0}^{\infty} d_{n-p} = 1$$

Where

$$f_{-p}(z) = \frac{1}{z^p} ,$$

$$f_{-p}(z)=\frac{1}{z^p}+\frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}z^{n-p},(n\geq 1).$$

Proof.

Suppose that $f(z)=\sum_{n=0}^{\infty}d_{n-p}f_{n-p}(z)$, $d_{n-p}\geq 0$, $\sum_{n=0}^{\infty}d_{n-p}=1$ Then, $f(z)=d_{-p}f_{-p}(z)$ $+\sum_{n=1}^{\infty}d_{n-p}\left[\frac{1}{z^p}+\frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}z^{n-p}\right]$

$$+\sum_{n=1}^{\infty}d_{n-p}\left[\frac{1}{z^p}+\frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}z^{n-p}\right]$$

$$=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}d_{n-p}z^{n-p}.$$

Now by using Theorem 1 we conclude that $f\in\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. Conversely, if f given by (1), belongs to $f\in\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$, by letting $d_{-p}=1-\sum_{n=1}^{+\infty}d_{n-p}$, where

$d_n=\frac{(1)_n(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}a_{n-p}$, $(n\geq 1)$. This concludes the result. □

Theorem 3: *Let for* $k=1,2,\cdots,t$, $f_k(z)=\frac{1}{z^p}+\sum_{n=1}^{+\infty}a_{n-p,k}z^{n-p}$ *belongs to* $\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$, *then* $F(z)=\sum_{k=1}^{t}d_{k-p}f_k(z)$ *is also in the same class, where* $\sum_{k=1}^{t}d_{k-p}=1$.

Proof.

According to Theorem 1, for every $k=1,2,\cdots,t$, we have

$\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]a_{n-p,k}\leq p(B-A)[\mu(p+1)-1]$.

But

$F(z)=\sum_{k=1}^{t}d_{k-p}f_k(z)=\sum_{k=1}^{t}d_{k-p}\left(\frac{1}{z^p}+\sum_{n=1}^{\infty}a_{n-p,k}z^{n-p}\right)$

$=\frac{1}{z^p}\sum_{k=1}^{t}d_{k-p}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{t}d_{k-p}a_{n-p,k}\right)z^{k-p}=\frac{1}{z^p}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{t}d_{k-p}a_{n-p,k}\right)z^{k-p}$.

Since,

$\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]\left(\sum_{k=1}^{t}d_{k-p}a_{n-p,k}\right)$

$=\sum_{k=1}^{t}d_{k-p}\left(\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n}{(1)_n}\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]a_{n-p,k}\right)$

$\leq\sum_{k=1}^{t}d_{k-p}\ (\mathrm{p}(\mathrm{B}-\mathrm{A})[\mu(\mathrm{p}+1)-1])$

$=p(B-A)[\mu(p+1)-1]\ \sum_{k=1}^{t}d_{k-p}=p(B-A)[\mu(p+1)-1]$.

This issue completes the proof. □

Theorem 4: *Let* $f \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$ *and has the form (1). Then for* $0 < r = |z| < 1$ *and* $-1 \leq B \leq 0$, we have

$$\frac{1}{r^p}\left(1-\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right)$$

$$\leq |f(z)| \leq \frac{1}{r^p}\left(1-\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right).$$

Proof.

Since $f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p}\, z^{n-p}$ then

$$|f(z)| = \left|\frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p}\, z^{n-p}\right| \leq \frac{1}{|z|^p} + \sum_{n=1}^{\infty} a_{n-p}\,|z|^{n-p} \quad (a_{n-p} \geq 0) \leq \frac{1}{r^p}\left(1+\sum_{n=1}^{\infty} a_{n-p}\right) \tag{9}$$

By Corollary 1, we have $a_{n-p} \leq \frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}$. Thus from (9), we obtain

$$|f(z)| \leq \frac{1}{r^p}\left(1-\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right).$$

Similarly,

$$|f(z)| \geq \frac{1}{|z|^p} + \sum_{n=1}^{\infty} a_{n-p,k}\,|z|^{n-p} \qquad (a_{n-p} \geq 0)$$

$$\geq \frac{1}{r^p}\left(1+\sum_{n=1}^{\infty} a_{n-p}\right)$$

$$\geq \frac{1}{r^p}\left(1-\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right).$$

as desired. □

Theorem 5: *Let* $f \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$ *and has the form of (1). Then for* $0 < r = |z| < 1$ *and* $-1 \leq B \leq 0$,

$$\frac{p}{r^{p+1}}\left(1-r\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right)$$

$$\leq |f'(z)| \leq \frac{p}{r^{p+1}}\left(1+r\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right).$$

Proof.

Suppose that $f \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. Then,

$$|f'(z)| = \left|\frac{p}{z^{p+1}} + \sum_{n=1}^{\infty} a_{n-p,}\,(n-p)z^{n-p-1}\right| \leq \frac{p}{|z|^{p+1}} + \sum_{n=1}^{\infty} a_{n-p,}\,(n-p)|z|^{n-p-1} \quad (a_{n-p} \geq 0)$$

$$\leq \frac{p}{z^{p+1}}\left(1+r\sum_{n=1}^{\infty}(a_{n-p})\right) \leq \frac{p}{r^{p+1}}\left(1+r\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right)$$

On the other hand,

$$|f'(z)| = \left|\frac{p}{z^{p+1}}-\sum_{n=1}^{\infty}a_{n-p,}(n-p)z^{n-p-1}\right| \geq \frac{p}{|z|^{p+1}}-\sum_{n=1}^{\infty}a_{n-p,}(n-p)|z|^{n-p-1} \quad (a_{n-p}\geq 0)$$

$$\geq \frac{p}{z^{p+1}}\left(1-r\sum_{n=1}^{\infty}(a_{n-p})\right) \geq \frac{p}{r^{p+1}}\left(1-r\frac{(1)_n\,(p(B-A)[\mu(p+1)-1])}{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}\right)$$

which complete the proof. □

2.1 Radii Condition and Partial Sum Property

Mathematical modelling has gained popularity in financial modeling due to the dependence structure of their increments and the roughness of their results [4]. In this section we obtain radii condition of starlikeness and convexity and investigate about partial sum property.

Theorem 6: *If $f(z) \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$, then f is meromorphically univalent starlike of order γ in disk $|z| < R_1$, and it is meromerphically univalent convex of order γ* in disk $|z| < R_2$ *where*

$$R_1 = inf_n\left\{\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(p-\sigma)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,p(n+p-\sigma)(B-A)[\mu(p+1)-1]}\right\}^{\frac{1}{n}} \quad (10)$$

$$R_2 = inf_n\left\{\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n(n+p-\sigma)\,(B-A)[\mu(p+1)-1]}\right\}^{\frac{1}{n}} \quad (11)$$

Proof.

For starlikeness it is enough to show that $\left|\frac{zf'(z)}{f(z)}+p\right| < p-\sigma, \quad (0\leq\sigma\leq p)$ But $\left|\frac{zf'(z)}{f(z)}+p\right| = \left|\frac{\sum_{n=1}^{\infty}na_{n-p}z^n}{1+\sum_{n=1}^{\infty}a_{n-p}z^n}\right| \leq \frac{\sum_{n=1}^{\infty}na_{n-p}|z|^n}{1-\sum_{n=1}^{\infty}a_{n-p}|z|^n} \leq p-\sigma,$ or $\sum_{n=1}^{\infty}na_{n-p}|z|^n \leq (p-\sigma)-(p-\sigma)\sum_{n=1}^{\infty}a_{n-p}|z|^n$

Or $\sum_{n=1}^{+\infty}\frac{n+p-\sigma}{p-\sigma}a_{n-p}|z|^n \leq 1.$

By Corollary 1, $\frac{n+p-\sigma}{p-\sigma}\,|z|^n \leq \frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,p(B-A)[\mu(p+1)-1]}$. So, it is enough to suppose

$$|z|^n \leq \frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}(n,\lambda)(p-\sigma)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,p(n+p-\sigma)(B-A)[\mu(p+1)-1]}.$$

Hence, we get the required result (10). For convexity, by using Alexander's Theorem, if f be an analytic function in the unit disk and normalized by $f(0) = f'(0)-1 = 0$, then $f(z)$ is convex if and only if $zf'(z)$ is starlike and applying an easy calculation we conclude the required result (11) and the proof is completed. □

Theorem 7: *Let* $-1 \le B < A \le 1$ *and* $f \in \Sigma_{p,n}$ *be given by (6) and define* $S_1(z) = \frac{1}{z^p}$ $, S_t(z) = \frac{1}{z} + \sum_{n=1}^{t-1} a_{n-p} z^{n-p}$, $(t = 2, 3, \cdots)$. Also suppose that $\sum_{n=1}^{+\infty} x_{n-p} a_{n-p} \le 1$, where

$$x_{n-p} = \frac{(\alpha+p)_n \, \Phi_{p,q}^{\delta,m}(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n \, (p(B-A)[\mu(p+1)-1])} \tag{12}$$

Then

$$Re\left(\frac{f(z)}{S_t(z)}\right) > 1 - \frac{1}{x_t} \ , \quad Re\left(\frac{S_t(z)}{f(z)}\right) > \frac{x_t}{1+x_t} \tag{13}$$

Proof.

Since $\Sigma_{n=1}^{+\infty} x_{n-p} a_{n-p} \le 1$, by Theorem 1, $f \in \Sigma_{p,q}^{\delta,m}(\alpha, \lambda, A, B)$. We can see from

that $x_{n-p+1} > x_{n-p} > 1$, $n = 1, 2, \cdots,$. Therefore,

$$\sum_{n=1}^{t-1} a_{n-p} + x_t \sum_{n=t}^{+\infty} a_{n-p} \le \sum_{n=1}^{+\infty} x_{n-p} a_{n-p} \le 1 \tag{14}$$

By setting $X_1(z) = x_t \left[\frac{f(z)}{S_t(z)} - \left(1 - \frac{1}{x_t}\right)\right] = \frac{x_t \sum_{n=1}^{t-1} a_{n-p} z^{n-p}}{1 + \sum_{n=1}^{t-1} a_{n-p} z^{n-p}} + 1$, and applying (14), we find that

$$Re\left(\frac{X_1(z)-1}{X_1(z)+1}\right) = \left|\frac{X_1(z)-1}{X_1(z)+1}\right| = \left| \frac{x_t \sum_{n=1}^{+\infty} a_{n-p} z^{n-p}}{2 + x_t \sum_{n=1}^{+\infty} a_{n-p} z^{n-p} + 2x_t \sum_{n=1}^{t-1} a_{n-p} z^{n-p}} \right|$$

$$\le \frac{x_t \sum_{n=1}^{+\infty} a_{n-p} z^{n-p}}{2 - x_t \sum_{n=1}^{+\infty} a_{n-p} z^{n-p} - 2\sum_{n=1}^{t-1} a_{n-p} z^{n-p}} \le 1.$$

By a simple calculation we get $Re(X_1(z)) > 0$, and therefore$Re\left(\frac{X_1(z)}{x_t}\right) > 0$, or Equivalently $Re\left[\frac{f(z)}{S_t(z)} - (1 - \frac{1}{x_t})\right] > 0$. this gives the first inequality in (13). For the second inequality we consider

$$X_2(z) = (1 + x_t)\left[\frac{S_t(z)}{f(z)} - \frac{x_t}{1+x_t}\right] = 1 - \frac{(1+x_t)\sum_{n=1}^{+\infty} a_{n-p} z^{n-p}}{1 + \sum_{n=1}^{+\infty} a_{n-p} z^{n-p}}.$$

and by using (14), $\left|\frac{X_2(z)-1}{X_2(z)+1}\right| \le 1$. Hence $Re(X_2(z)) > 0$, and therefore $Re\left(\frac{X_2(z)}{1+x_t}\right) > 0$, or equivalenty $Re\left[\frac{S_t(z)}{f(z)} - \frac{x_t}{1+x_t}\right] > 0$. This shows that the second inequality in (13). □

2.2 Neighborhoods Results

In this sense, this section is devoted to show some properties of the developed subclass that are useful in the mathematical theory of finance and economics. Mainly, in this section, some concepts such as investigated neighborhoods for analytic univalent functions are going to be introduced. Altintas and Owa [26], Goodman [27], Lashin [28], Raina and Srivastava [16], and, Ruscheweyh [17], have investigated neighborhoods for analytic univalent functions. In this section, we start by introducing the ε-neighborhood of a function $f \in \Sigma_{p,n}$. To do this, we assume that $-1 \le B < A \le 1, p \in \mathbb{N}, 0 \le \mu < \frac{1}{p+1}, m \in \mathbb{N}_0, \alpha > -p, \lambda \ge 0, \delta > 0$ *and* $\varepsilon \ge 0$. Define ε-neighborhood of a function $f \in \Sigma_{p,n}$ of the from of (1) as:

$N_{\varepsilon}(z)=\left\{g(z):\ g(z)\ =\frac{1}{z^p}+\sum_{n=1}^{+\infty} b_{n-p}\, z^{n-p}\in\Sigma_{p,\alpha}\ and\ \theta\ \le\ \varepsilon\right\}$

Where

$$\theta=\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}\,(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,(\,p(B-A)[\mu(p+1)-1])}\left|a_{n-p}-b_{n-p}\right| \tag{15}$$

Theorem 8: *Let the function $f(z)$ defined by (1) be in the class* $\Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. *If $f(z)$ satisfies the following condition*:

$\frac{f(z)+vz^{-p}}{1+v}\in\ \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B),\qquad (v\ \in\ \mathbb{C}\,,|v|\ <\ \varepsilon,\varepsilon\ >\ 0)$

Then $N\varepsilon(f)\subset\ \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$.

Proof.

By using (7), we obtain $f\in\ \Sigma_{p,q}^{\delta,\mu,m}\ (\alpha,\lambda,A,B)$ if and only if for any $\beta\ \in\ C,|\beta|\ =\ 1,$

$$\frac{\mu z^{p+2}\left(\Psi_{p,q}^{\delta,m}(\alpha,\mu)\right)f(z))''+z^{p+1}\left(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)\right)f(z))'-p[\mu(p+1)-1]}{B\,[\mu z^{p+2}\left(\Psi_{p,q}^{\delta,m}(\alpha,\mu)\right)f(z))''+z^{p+1}\left(\Psi_{p,q}^{\delta,m}(\alpha,\lambda)\right)f(z))']-p[\mu(p+1)-1]}\neq\beta$$

which is equivalent to

$$\frac{(f*Q)(z)}{z^{-p}}\neq\ 0\,,(z\ \in\ U^*) \tag{16}$$

Where

$$Q(z)=\frac{1}{z^p}+\sum_{n=1}^{+\infty} e_{n-p}\,z^{n-p}\ ,\ \ (z\in U^*)$$

such that

$$e_{n-p}=\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}\,(n,\lambda)(1-\beta\,B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,(\beta\,p(B-A)[\mu(p+1)-1])} \tag{17}$$

It follows from (17) that

$$\left|e_{n-p}\right|=\left|\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}\,(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,(\,p(B-A)[\mu(p+1)-1])}\right|$$

$$\le\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}\,(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,(\,p(B-A)[\mu(p+1)-1])}$$

Since $\frac{f(z)+vz^{-p}}{1+v}\in\Sigma_{p,q}^{\delta,m}(\alpha,\lambda,A,B,)$ by (16) we get

$$\frac{\frac{f(z)+vz^{-p}}{1+v}*Q)(z)}{z^{-p}}\neq 0 \tag{18}$$

Now assume that $\left|\frac{(f*Q)(z)}{z^{-p}}\right|<\varepsilon$. Then, by (18) we get

$$\left|\frac{1}{1+v}\frac{(f\ *\ Q)(z)}{z^{-p}}+\frac{v}{1+v}\right|\geq\frac{1}{|1+v|}(|v|-1)\left|\frac{(f\ *\ Q)(z)}{z^{-p}}\right|>\frac{|v|-\varepsilon}{|1+v|}\geq 0.$$

This is a contradiction with $|v| < \varepsilon$. Therefore $\left|\frac{(f*Q)(z)}{z^{-p}}\right| < \varepsilon$. Now, if we suppose that

g(z) $=\frac{1}{z^{-p}}+\sum_{n=1}^{\infty} b_{n-p}\, z^{n-p} \in N_{\varepsilon}(f)$ then

$$\left|\frac{(f-g)(z)*Q)(z)}{z^{-p}}\right| = \left|\sum_{n=1}^{+\infty}\left(a_{n-p}-b_{n-p}\right)e_{n-p}z^{n-p}\right| \leq \sum_{n=1}^{+\infty}\left|a_{n-p}-b_{n-p}\right|\left|e_{n-p}\right|\left|z^{n-p}\right|$$

$$\leq\ |z^{n-p}|\sum_{n=1}^{+\infty}\frac{(\alpha+p)_n\,\Phi_{p,q}^{\delta,m}\,(n,\lambda)(1-B)(n-p)[\mu(n-p-1)+1]}{(1)_n\,(p(B-A)[\mu(p+1)-1])}\left|a_{n-p}-b_{n-p}\right| \leq\ \varepsilon.$$

Thus, for any complex number β such that $|\beta| = 1$, we have

$$\frac{(g*Q)(z)}{z^{-p}} \neq 0\,,(z\in U^{*})\,,$$

which implies that $g \in \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. So $N_{\varepsilon}f(z) \subset \Sigma_{p,q}^{\delta,\mu,m}(\alpha,\lambda,A,B)$. □

Our motivation came from mathematical finance, more precisely from establishing a subclass of harmonic univalent functions that have important role in finance.

3 Conclusion

In this paper, we define a new subclass of meromorphically p-valent functions by using q-derivative operator and fractional q-calculus operator that are useful in mathematical finance. As a result, we obtained some geometric properties of coefficient estimates, extreme points, convex linear combination, radii of starlikeness and convexity. Furthermore, we investigated the ε-neighborhood of the presented classes.

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