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A numerical solution of a Kawahara equation by using Multiquadric radial basis function

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Abstract

In this article, we apply the Multiquadric radial basis function (RBF) interpolation method for finding the numerical approximation of traveling wave solutions of the Kawahara equation. The scheme is based on the Crank-Nicolson formulation for space derivative. The performance of the method is shown in numerical examples.

Key words: Kawahara equation, Traveling wave solution, Radial basis function(RBFs).
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1 Introduction

The Kawahara equation occurs in the theory of magneto-acoustic waves in a plasmas [1] and in the theory of shallow water waves with surface tension [2]. This equation is one of the simplest one-dimensional PDE's which exhibits complex dynamical behavior. As an evolution equation it arises in a number of applications including concentration waves and plasma physics [3-4], flame propagation and reaction diffusion combustion dynamics [5-6], free surface film-flows [7-8] and two-phase flows in cylindrical or plane geometries [9].

The origin of our study is the equation

$$u_t + uu_x + u_{3x} - u_{5x} = 0, \tag{1.1}$$

which is a Kawahara-type equation with the initial condition

$$u(x, 0) = f(x). \tag{1.2}$$

Nonlinear evolution equation (1.1) has been studied by a number of authors from various viewpoints[10-16]. This equation has drown much attention not only because it is interesting as a simple one-dimensional nonlinear evolution equation including effects of instability and dissipation but also it is important for description of engineering and scientific problems.

In this article, we consider the interpolation with Multiquadric(MQ) radial basis function(RBF) for solving the Kawahara equation. In this method we use the finite difference formula for time derivative and Crank-Nicolson scheme for simplifying the space derivative. Also we have linearized the nonlinear term in the equation by using the form introduced by the Rubin and Graves [10].

Although applications of radial basis functions(RBFs) have bloomed in recent years, using RBFs to solve evolutionary partial differential equations(PDEs) is a young research field. The strength of the method is in its ability to achieve spectral or high-order accuracy for scattered node layouts while being able to node refine in areas where increased resolution is needed. The multiquadric radial basis functions(MQ) method is a recent mesh-free collocation method

with global basis functions. The MQ method for the solution of partial differential equations was first introduced by Kansa in the early 1990s and showed exponential convergence for interpolation problems.
It was originally proposed by Hardy in 1970 [14-15] for interpolation of scattered data. Madych and co-worker [11-12] and Wu and Schaback [13] showed that the MQ method has exponential convergence for approximation of functions.

The construction of this paper is as follows. In the next section, we straight go to explanation of the method. Section 3, studies some numerical examples to show the applicability of the scheme. Finally in Section 4, a brief conclusion is given.

2 Construction of the method

In this section, for the purpose of completeness, we present the radial basis function method for the numerical solution of the Equation (1.1). Consider the Kawahara Eq. (1.1)

$$\frac{\partial u(x,t)}{\partial t}+u\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3 u(x,t)}{\partial x^3}-\frac{\partial^5 u(x,t)}{\partial x^5}=0, \qquad a\le x\le b, \qquad t>0, \tag{2.1}$$

subject to the initial condition

$$u(x,0)=f(x). \tag{2.2}$$

We discrete the time derivative of the equation using a finite difference formula and applying Crank-Nicolson scheme to the space derivative at two successive time level n and $n+1$. So we get

$$[\frac{u^{n+1}-u^n}{\delta t}]+[\frac{(uu_x)^{n+1}+(uu_x)^n}{2}]+[\frac{(u_{3x})^{n+1}+(u_{3x})^n}{2}]-[\frac{(u_{5x})^{n+1}+(u_{5x})^n}{2}]=0, \tag{2.3}$$

where $u^{n+1} = u(x, t^{n+1})$, $t^{n+1} = t^n + \delta t$. To linearized the nonlinear term $(uu_x)^{n+1}$ we use the following linearization form given by Rubin and Graves [10]:

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u^n u_x^{n+1} - (uu_x)^n. \tag{2.4}$$

Substituting values from Eq.(2.4) in Eq.(2.3) we get

$$[\frac{u^{n+1} - u^n}{\delta t}] + [\frac{u^{n+1}u_x^n + u^n u_x^{n+1}}{2}] + [\frac{(u_{3x})^{n+1} + (u_{3x})^n}{2}] - [\frac{(u_{5x})^{n+1} + (u_{5x})^n}{2}] = 0. \tag{2.5}$$

Rearranging the terms and simplifying we have

$$u^{n+1} + \frac{\delta t}{2}[u^{n+1}u_x^n + u^n u_x^n + (u_{3x})^{n+1} - (u_{5x})^{n+1}] = u^n - \frac{\delta t}{2}[(u_{3x})^n - (u_{5x})^n]. \tag{2.6}$$

Let $x_i, i = 1, 2, ..., N$ be the collocation points in the interval $[a, b]$ such that $x_1 = a$ and $x_N = b$. We use the

$$u^n(x) = \sum_{j=1}^{N} \lambda_j^n \varphi(r_j), \tag{2.7}$$

to approximate the Eq.(2.1). Where φ is a radial basis function and $r_j(x) = ||x - x_j||$ represents the Euclidean norm between x and x_j, where x_j's are known as centers. The unknown parameters λ_j in Eq.(2.7) should be determined. Substituting the points $x = x_i$ for $i = 1, 2, ..., N$ we have

$$u^n(x_i) = \sum_{j=1}^{N} \lambda_j^n \varphi(r_{ij}), \qquad i = 1, 2, ..., N. \tag{2.8}$$

Eq.(2.8) can be expressed in a matrix form as

$$u^n = A\lambda^n, \tag{2.9}$$

where

$$A = \begin{pmatrix} \varphi(r_{11}) & \varphi(r_{12}) & \cdots & \varphi(r_{1N}) \\ \varphi(r_{21}) & \varphi(r_{22}) & \cdots & \varphi(r_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(r_{N1}) & \varphi(r_{N2}) & \cdots & \varphi(r_{NN}) \end{pmatrix},$$

and $\lambda^n = [\lambda_1^n, \lambda_2^n, \cdots, \lambda_N^n]^T$. Now by using Eq.(2.6) and Eq.(2.8), we get the following equation for the points in set $[a, b]$:

$$\sum_{j=1}^{N} \lambda_j^{n+1}\varphi(r_{ij}) + \frac{\delta t}{2}[(\sum_{j=1}^{N} \lambda_j^{n}\varphi(r_{ij}) \sum_{j=1}^{N} \lambda_j^{n+1}\varphi'(r_{ij})$$
$$+ \sum_{j=1}^{N} \lambda_j^{n+1}\varphi(r_{ij}) \sum_{j=1}^{N} \lambda_j^{n}\varphi'(r_{ij})) + \sum_{j=1}^{N} \lambda_j^{n+1}\varphi'''(r_{ij}) - \sum_{j=1}^{N} \lambda_j^{n+1}\varphi^{(5)}(r_{ij})] =$$
$$\sum_{j=1}^{N} \lambda_j^{n}\varphi(r_{ij}) - \frac{\delta t}{2}[\sum_{j=1}^{N} \lambda_j^{n}\varphi'''(r_{ij}) - \sum_{j=1}^{N} \lambda_j^{n}\varphi^{(5)}(r_{ij})], \tag{2.10}$$

where $\varphi'(r_{ij}) = \frac{d}{dx}\varphi(||x - x_j||)|_{x=x_i}$, $\varphi'''(r_{ij}) = \frac{d^3}{dx^3}\varphi(||x - x_j||)|_{x=x_i}$ and $\varphi^{(5)}(r_{ij}) = \frac{d^5}{dx^5}\varphi(||x - x_j||)|_{x=x_i}, i = 1, 2, ..., N$.

We introduce the $N \times N$ matrices D_1, D_2 and D_3 such that

$$D_1 = [\varphi'(r_{ij}) : \quad 1 \le i \le N, \quad 1 \le j \le N], \tag{2.11}$$
$$D_2 = [\varphi'''(r_{ij}) : \quad 1 \le i \le N, \quad 1 \le j \le N], \tag{2.12}$$
$$D_3 = [\varphi^{(5)}(r_{ij}) : \quad 1 \le i \le N, \quad 1 \le j \le N], \tag{2.13}$$

also we have

$$u_x^n = D_1\lambda^n, \quad F = u_x^n * A, \quad and \quad E = u^n * D_1,$$

which the symbol " $*$ " is used for component by component multiplication. At this time for the sake of simplification Eq.(2.12), with the help of above relationships and by using the Eqs.(2.11), (2.12) and (2.13), we get the Eq.(2.10) in the matrix form

$$[A+\frac{\delta t}{2}(E+F+D_3-D_5)]\lambda^{n+1}=[A-\frac{\delta t}{2}(D_3-D_5)]\lambda^n \quad (2.14)$$

Equation (2.14) can be rewritten as

$$\lambda^{n+1}=R^{-1}L\lambda^n, \quad (2.15)$$

where $R=[A+\frac{\delta t}{2}(E+F+D_3-D_5)]$ and $L=[A-\frac{\delta t}{2}(D_3-D_5)]$. Equation (2.14) represents a system of N linear equations with N unknown parameters λ_j. This system can be solved by the Gaussian elimination method. Then from Eqs. (2.9) and (2.15) we can write

$$u^{n+1}=AR^{-1}LA^{-1}u^n.$$

In this literature we use the following radial basis function:

Multiquadric (MQ) $\varphi(r_j)=\sqrt{j^2+c^2}$.

3 Numerical tests and results

In this section we present the results of the numerical tests of our scheme for the solution of the Kawahara equations (2.1) and (2.2). The value of the parameters K and c used in the examples is taken as

$$C=\frac{36}{169}, \quad K=\frac{1}{2\sqrt{13}}. \quad (3.1)$$

All programs are run in Mathematica(7).

Example 3.1 *Consider the Kawahara equation which has the traveling wave solution of which is the exact solution*

$$u(x,t)=\frac{-72}{169}+\frac{105}{169}\sec h^4[K(x+Ct)], \quad (3.2)$$

where the parameters c and K are given as (3.1). The initial condition of Eq. (2.1) is given as

$$u(x,0) = \frac{-72}{169} + \frac{105}{169}\sec h^4(Kx). \tag{3.3}$$

The boundary conditions are extracted from the exact solution (3.2). For this example, we consider $\delta t = 0.001$ and $N = 120$. The values of u_{exact} and u_{app} are computed where u_{exact} and u_{app} represent the exact and approximate solutions respectively. Also the difference between these values are calculated. The numerical results are given in Table 1 for MQ radial basis function in the space interval $0 \le x \le 60$ when the time interval is $0 \le t \le 0.5$. Value of the shape parameter MQ radial basis function is $c = 0.5$. The L_∞ error distribution at $t = 0.5$ is shown in Figure 2.

Table 1: The values of u_{ex}, u_{app} and $|u_{ex} - u_{app}|$ for Example 3.1.

x	u_{ex}	u_{app}	$\|u_{ex} - u_{app}\|$
0	-4.26036×10^{-1}	1.83567×10^{-1}	2.42469×10^{-1}
10	1.67004×10^{-1}	1.45296×10^{-1}	2.17079×10^{-2}
20	-4.07712×10^{-1}	-3.18642×10^{-1}	8.90698×10^{-2}
30	-4.25947×10^{-1}	-4.20649×10^{-1}	5.29755×10^{-3}
40	-4.26035×10^{-1}	-4.2568×10^{-1}	3.54789×10^{-4}
50	-4.26036×10^{-1}	-4.26066×10^{-1}	3.07329×10^{-5}
60	-4.26036×10^{-1}	-4.26085×10^{-1}	4.98569×10^{-5}

Example 3.2 *Consider the Kawahara Equation (2.1) which has the exact solution*

$$u(x,t) = \frac{-72}{169} + \frac{420\sec h^2[K(x+Ct)]}{169(1+\sec h^2[K(x+Ct)])}. \tag{3.4}$$

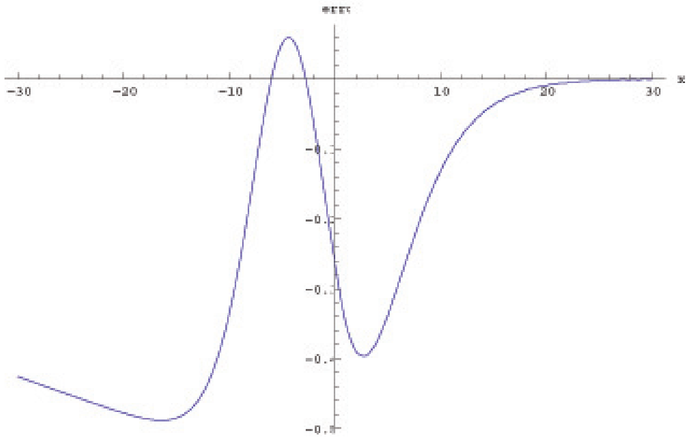

Fig. 1. Figure 1. Error graph at time t=0.5.

The solution (3.4) is to be obtained subject to the initial condition

$$u(x,0) = \frac{-72}{169} + \frac{420\sec h^2(Kx)}{169(1+\sec h^2(Kx))}, \tag{3.5}$$

where the parameters are as the Example 3.1. The numerical results for this example are given in Table 2. In this Table the exact and approximation solutions are calculated in per 20 *place step in the place interval* $0 \leq x \leq 50$ *when the time interval is* $0 \leq t \leq 1$*. The time step size is* $\delta t = 0.001$ *and the number of collocation points is* $N = 100$*. The* L_∞ *error of MQ radial basis function at time* $t = 1$ *is shown in Figure 2. Also solutions at the last time is shown in Figure 3. In this figure distribution of* u_{app} *is shown for* $N = 300$*, in space interval* $-30 \leq x \leq 60$*.*

Table 2: The values of u_{ex}, u_{app} and $|u_{ex} - u_{app}|$ for Example 3.2.

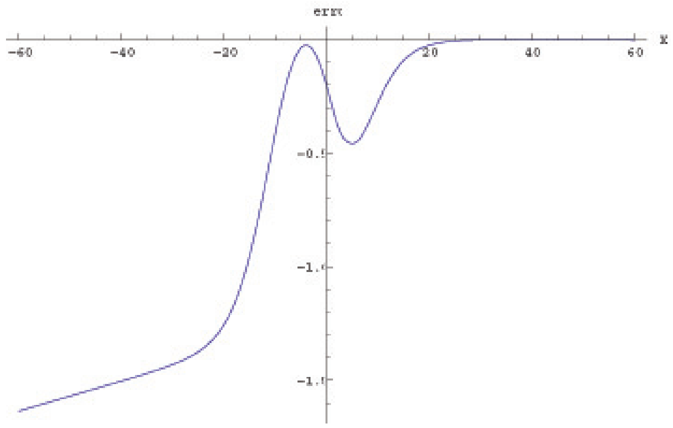

Fig. 2. Figure 2. Error graph at time t=1.

x	u_{ex}	u_{app}	$\lvert u_{ex} - u_{app} \rvert$
0	-4.26036×10^{-1}	8.04777×10^{-1}	3.78741×10^{-1}
10	7.99214×10^{-1}	7.90817×10^{-1}	8.39716×10^{-3}
20	-6.87339×10^{-2}	-5.17561×10^{-2}	1.69778×10^{-2}
30	-3.97238×10^{-1}	-3.96695×10^{-1}	5.42719×10^{-4}
40	-4.24156×10^{-1}	-4.24125×10^{-1}	3.08035×10^{-5}
50	-4.25915×10^{-1}	-4.25718×10^{-1}	1.96359×10^{-4}

4 Conclusion

This paper studied the MQ interpolation method for the Kawahara equation with initial conditions. The efficiency of the method is tested for two problems. The results show that this scheme is accurate and efficient approach for solving the nonlinear partial differential equations.

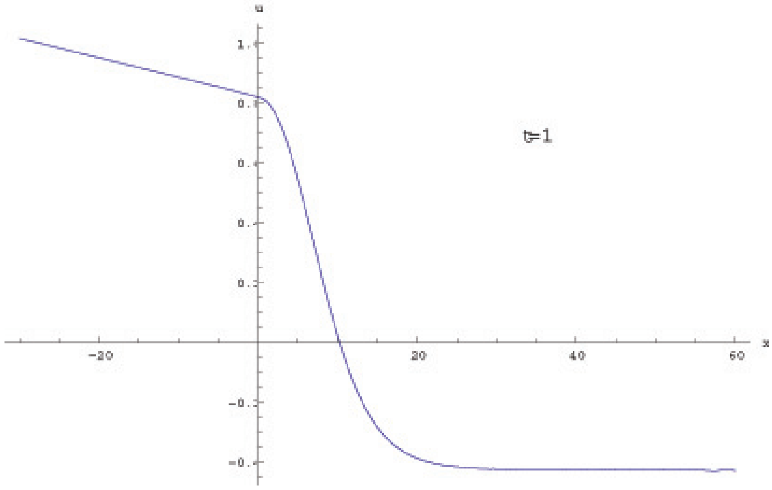

Fig. 3. Figure 3. Solution of Kawahara equation at time $t = 1$.

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