Analytic Solution of Fuzzy Second Order Differential Equations under H-Derivation

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Abstract

In this paper, the solution of linear second order equations with fuzzy initial values are investigated. The analytic general solutions of them using a first solution is founded. The parametric form of fuzzy numbers to solve the second order equations is applied. The solutions are searched in four cases. Finally the example is got to illustrate more and the solutions are shown in figures for four cases.

Key words: Second order equation, Fuzzy initial values, H-derivations, General solution.

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1 Introduction

The differential equations of second order are one of the equations that cannot be solved simply, thus it is necessary to classify them in special equations, for example linear and non-linear equations. Differential equation with constant multipliers is one of the linear equations and the Cauchi differential equation is the nonlinear one. These equations are received in the formulation of applied mathematics problems, but in nature the fuzzy second equations are caught for example in physics problems, mechanical problems and etc.

The H-derivative of fuzzy number-valued function was introduced for solving a fuzzy first order equation in [9]. Under this setting, the existence and uniqueness of a solution of fuzzy differential equations were studied in [3,4]. Under H-derivative. The numerical method for solving differential equations is studied by Pallingkinis through Runge-Kutah method, [21]. The strong general differentiable was introduced in [4]. This concept allows us to solve the problem of H-derivative. The existence of fuzzy differential equations of second order are studied by Allahviranloo et al., in [1] and then by Zhang [27], Under the general H-derivative, [16]. Khastan et al. studied solving second order equations under boundary value problem by the general H-derivative, [14]. Allahviranloo and Hooshangian searched fuzzy second order derivations more, investigated the relationships between fuzzy second order derivations and found the solution of fuzzy constant multipliers and Cauchi second order equations with fuzzy initial values [2]. Some numerical methods for solving fuzzy second order derivations are studied in [10, 17, 19, 25]

In this paper, general derivatives are used to find new solutions for initial value problem of fuzzy linear second order equations with fuzzy initial values. Indeed, with generalized differentiability, the solution for a larger class of them than using H-derivative.

In Section 2, some needed concepts are reviewed. In Section 3, an analytic method in order to find fuzzy second order under H-differential is introduced and our formula for finding solutions of fuzzy Hakuhara differential is obtained and four cases reach the general solutions is searched. Numerical examples are given to show more of our method. In section 4, numerical examples of all cases are given to show more.

2 Basic concepts

The basic definitions of a fuzzy number are given in [9, 11–13] as follows:

Definition 2.1 *u is named a fuzzy number in parametric form that is shown with a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*
2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,*
3. *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

Definition 2.2 *For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$, and scalar k, it is defined addition, subtraction, scalar product by k and multiplication are respectively as follows:*

$$\text{addition}: \quad \underline{u+v}(r) = \underline{u}(r) + \underline{v}(r), \quad \overline{u+v}(r) = \overline{u}(r) + \overline{v}(r)$$
$$\text{subtraction}: \quad \underline{u-v}(r) = \underline{u}(r) - \overline{v}(r), \quad \overline{u-v}(r) = \overline{u}(r) - \underline{v}(r)$$

scalar product:

$$k\tilde{u} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$$

$$\text{multiplication}: \quad \underline{uv}(r) = max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$$
$$\overline{uv}(r) = min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$$

Definition 2.3 *Let $u(r) = [\underline{u}(r), \overline{u}(r)], 0 \leq r \leq 1$ be a fuzzy number, let*

$$u^c = \frac{\underline{u}(r) + \overline{u}(r)}{2}$$

$$u^d = \frac{\overline{u}(r) - \underline{u}(r)}{2}$$

It is clear that $u^d(r) \geq 0$ and $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$

Definition 2.4 *Let $u(r) = [\underline{u}(r), \overline{u}(r)], v(r) = [\underline{v}(r), \overline{v}(r)], 0 \leq r \leq 1$*

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are two fuzzy numbers and also k, s are two arbitrary real numbers. If $w = ku + sv$ then

$$w^c(r) = ku^c(r) + sv^c(r)$$

$$w^d(r) = |k|u^d(r) + |s|v^d(r)$$

Definition 2.5 Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-differential of x, y and it is denoted $x \ominus y$.

Definition 2.6 [3] A function $F : I \longrightarrow \mathbb{R}_F$, $I = (a, b)$, is called $H-differentiable$ on $t \in I$ if for $h > 0$ sufficiently small there exist the $H-differences$ $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and an element $F'(t) \in \mathbb{R}_F$ such that:

$$0 = lim_{h \to 0} D(\frac{F(t_0 + h) \ominus F(t_0)}{h}, F'(t)) = lim_{h \to 0} D(\frac{F(t_0) \ominus F(t_0 - h)}{h}, F'(t))$$

Definition 2.7 [3] Let $F : I \to \mathbb{R}_F$ and $t_0 \in I$. F is differentiable at t_0 if there is $F'(t_0) \in \mathbb{R}_F$ such that either
(i)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0 + h) \ominus F(t_0)$ and $F(t_0) \ominus F(t_0 - h)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t)$$

or
(ii)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0 + h)$ and $F(t_0 - h) \ominus F(t_0)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t)$$

or
(iii)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0 + h) \ominus F(t_0)$ and $F(t_0 - h) \ominus F(t_0)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

or
(iv)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0 + h)$

and $F(t_0) \ominus F(t_0 - h)$ exist and the following limits

$$\lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

Definition 2.8 Let $F : I \longrightarrow \mathbb{R}_F$. For fix $t_0 \in I$ we say F is differentiable of second-order at t_0, if there is $F''(t_0) \in \mathbb{R}_F$ such that either
(i)for $h > 0$ sufficiently close to 0, the H-differences $F'(t_0 + h) \ominus F'(t_0)$ and $F'(t_0) \ominus F'(t_0 - h)$ exist and the following limits

$$\lim_{h \to 0} \frac{F'(t_0 + h) \ominus F'(t_0)}{h} = \lim_{h \to 0} \frac{F'(t_0) \ominus F'(t_0 - h)}{h} = F''(t_0)$$

or
(ii)for $h > 0$ sufficiently close to 0, the H-differences $F'(t_0) \ominus F'(t_0 + h)$ and $F'(t_0 - h) \ominus F'(t_0)$ exist and the following limits

$$\lim_{h \to 0} \frac{F'(t_0) \ominus F'(t_0 + h)}{-h} = \lim_{h \to 0} \frac{F'(t_0 - h) \ominus F'(t_0)}{-h} = F''(t_0)$$

or
(iii)for $h > 0$ sufficiently close to 0, the H-differences $F'(t_0 + h) \ominus F'(t_0)$ and $F'(t_0 - h) \ominus F'(t_0)$ exist and the following limits

$$\lim_{h \to 0} \frac{F'(t_0 + h) \ominus F'(t_0)}{h} = \lim_{h \to 0} \frac{F'(t_0 - h) \ominus F'(t_0)}{-h} = F''(t_0)$$

or
(iv)for $h > 0$ sufficiently close to 0, the H-differences $F'(t_0) \ominus F'(t_0 + h)$ and $F'(t_0) \ominus F'(t_0 - h)$ exist and the following limits

$$\lim_{h \to 0} \frac{F'(t_0) \ominus F'(t_0 + h)}{-h} = \lim_{h \to 0} \frac{F'(t_0) \ominus F'(t_0 - h)}{h} = F''(t_0)$$

Theorem 2.1 [1] If $f : [a, b] \times \mathbb{R}_F \longrightarrow \mathbb{R}_F$ is continuous and let $t_0 \in [a, b]$. A mapping $x : [a, b] \longrightarrow \mathbb{R}_F$ is a solution to the initial value problem

$$x'' = f(t, x(t, r), x'(t, r)), \qquad x(t_0) = k_1, \qquad x'(t_0) = k_2$$

if and only if x and x' are continuous and satisfy one of the following conditions:
(a) $\quad x(t) = k_2(t - t_0) + \int_{t_0}^{t} (\int_{t_0}^{t} f(s, x(s), x'(s)) ds) ds + k_1$

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where x' and x" are (i)-differentials, or

(b) $x(t) = \ominus(-1)(k_2(t-t_0) \ominus (-1) \int_{t_0}^{t}(\int_{t_0}^{t} f(s, x(s), x'(s))ds)ds) + k_1$
where x' and x" are (ii)-differentials, or

(c) $x(t) = \ominus(-1)(k_2(t-t_0) + \int_{t_0}^{t}(\int_{t_0}^{t} f(s, x(s), x'(s))ds)ds) + k_1$
where x' is the (i)-differential and x" is the (ii)-differential, or

(b) $x(t) = k_2(t-t_0) \ominus (-1) \int_{t_0}^{t}(\int_{t_0}^{t} f(s, x(s), x'(s))ds)ds + k_1$
where x' is the (ii)-differential and x" is the (i)-differential.

proof: See theorem (3.1) in Ref. [1].□

Theorem 2.2 [3] *Let $[t_0, T] \times E \times E \longrightarrow E$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that*

$$d(f(t, x_1, x_2), f(t, y_1, y_2)) \leq M_1 d(x_1, y_1) + M_2 d(x_2, y_2)$$

for all $t \in [t_0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}_F$. Then initial value problem mentioned in theorem (2.9) has a unique solution on $[t_0, T]$ for each case (i) or (ii).

3 Analytic Solution of Fuzzy Linear Second Order Equation

In this section an analytic method for solving fuzzy linear second order equation with fuzzy initial value is studied.
Fuzzy linear second order equation with fuzzy initial values is considered by following:

$$u''(t) + p(t)u'(t) = q(t)u(t), \qquad u(a) = u_0, \qquad u'(a) = u'_0$$

that $p(t)$ and $q(t)$ be two crisp functions. If initial values are fuzzy numbers and there is one fuzzy solution $u_1(t)$, it can to be considered four cases.

3.1   Case(1)

If $p(t)$ and $q(t)$ are two crisp functions and positive and u' and u'' are considered (i)-differentiable then:

$$\begin{cases} \overline{u}''(t) + p(t)\overline{u}'(t) = q(t)\overline{u}(t), \\ \underline{u}''(t) + p(t)\underline{u}'(t) = q(t)\underline{u}(t), \\ \underline{u}(a) = \underline{u}_0, \\ \overline{u}(a) = \overline{u}_0, \\ \underline{u}'(a) = \underline{u}_0', \\ \overline{u}'(a) = \overline{u}_0' \end{cases} \tag{3.1}$$

Now the following system can be denoted:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, P(t) = \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix}, Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(t) \end{pmatrix}$$

then it can be written the system (1) in the following equation:

$$U''(t) + P(t)U'(t) = Q(t)U(t) \tag{3.2}$$

Now if $U_1(t) = \begin{pmatrix} \overline{u}_1(t) \\ \underline{u}_1(t) \end{pmatrix}$ Then it is considered that the solution of (2) is given:

$$U_2 = U_1 \int (\frac{1}{U_1^2} e^{\int P(t)dt}) dt$$

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If $p(t)$ and $q(t)$ be two crisp function and are positive and u' and u'' are considered (ii)-differentiable then it can to denote two systems:

$$\begin{cases} \overline{u}''(t) + p(t)\overline{u}'(t) = q(t)\underline{u}(t), \\ \overline{u}(a) = \overline{u}_0, \\ \overline{u}'(a) = \overline{u}_0' \end{cases} \tag{3.3}$$

and

$$\begin{cases} \underline{u}''(t) + p(t)\underline{u}'(t) = q(t)\overline{u}(t), \\ \underline{u}(a) = \underline{u}_0, \\ \underline{u}'(a) = \underline{u}_0', \end{cases} \tag{3.4}$$

Now the following system can be denoted:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, P(t) = \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix}, R(t) = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix}$$

Then it given:
$$U''(t) + P(t)U'(t) = R(t)U(t)$$

Then by using (3) and (4) together it denoted:

$$\begin{cases} u''^c(t) + p(t)u'^c(t) = q(t)u^c(t), \\ u^c(a) = u_0^c, \\ u'^c(a) = u_0'^c, \end{cases} \tag{3.5}$$

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and

$$\begin{cases} u''^d(t) + p(t)u'^d(t) = -q(t)u^d(t), \\ u^d(a) = u_0^d, \\ u'^d(a) = u'^d_0, \end{cases} \qquad (3.6)$$

Now by solving (5) and (6) together and using case (1):

$$u_2^c = u_1^c \int \left(\frac{1}{(u_1^c)^2}e^{\int p(t)dt}\right)dt, \, u_2^d = u_1^d \int \left(\frac{1}{(u_1^d)^2}e^{\int p(t)dt}\right)dt$$

where $u^c = \frac{u+\overline{u}}{2}$, $u'^c = \frac{u'+\overline{u}'}{2}$ and $u''^c = \frac{u''+\overline{u}''}{2}$ also $u^d = \frac{\overline{u}-u}{2}$, $u'^d = \frac{\overline{u}'-u'}{2}$ and $u''^d = \frac{\overline{u}''-u''}{2}$. If we consider $u_1^c = \frac{u_1+\overline{u}_1}{2}$, $u_1^d = \frac{\overline{u}_1-u_1}{2}$. Now by definition (2.3) we can find u_2 and $\overline{u}_2$. Now the general solution is $u(t) = c_1u_1(t) + c_2u_2(t)$ that c_1 and c_2 are two fuzzy numbers.

3.3 Case(3)

If $p(t)$ and $q(t)$ be two crisp positive function and those are considered that u' is (i)-differentiable and u'' is (ii)-differentiable then it can be denoted two systems: Now the following system can be denoted:

$$\begin{cases} \overline{u}''(t) = p(t)\underline{u}'(t) + q(t)\underline{u}(t), \\ \underline{u}(0) = \underline{u}_0, \\ \overline{u}'(0) = \overline{u}'_0 \end{cases} \qquad (3.7)$$

and

$$\begin{cases} \underline{u}''(t) = p(t)\overline{u}'(t) + q(t)\overline{u}(t), \\ \overline{u}(0) = \overline{u}_0, \\ \underline{u}'(0) = \underline{u}'_0, \end{cases} \qquad (3.8)$$

The it given the following:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, S(t) = \begin{pmatrix} 0 & p(t) \\ p(t) & 0 \end{pmatrix}, R(t) = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix}$$

Then the following system is gotten:

$$U''(t) + P(t)U'(t) = R(t)U(t)$$

Now by solving (7) and (8) together we have

$$\begin{cases} u''^c(t) + p(t)u'^c(t) = q(t)u^c(t), \\ u^c(a) = u_0^c, \\ u'^c(a) = u'^c_0, \end{cases} \tag{3.9}$$

and

$$\begin{cases} u''^d(t) - p(t)u'^d(t) = -q(t)u^d(t), \\ u^d(a) = u_0^d, \\ u'^d(a) = u'^d_0, \end{cases} \tag{3.10}$$

Then we have

$$u_2^c = u_1^c \int (\frac{1}{(u_1^c)^2} e^{\int p(t)dt}) dt, u_2^d = u_1^d \int (\frac{1}{(u_1^d)^2} e^{-\int p(t)dt}) dt$$

where $u^c = \frac{u+\overline{u}}{2}$, $u'^c = \frac{u'+\overline{u}'}{2}$, $u''^c = \frac{u''+\overline{u}''}{2}$ and $u^d = \frac{\overline{u}-u}{2}$ also $u'^d = \frac{\overline{u}'-u'}{2}$, $u''^d = \frac{\overline{u}''-u''}{2}$ and $u_1^c = \frac{u_1+\overline{u}_1}{2}$, $u_1^d = \frac{\overline{u}_1-\overline{u}_1}{2}$. Now by definition (2.3) it can be to find $\underline{u}_2$ and $\overline{u}_2$. Thus $u(t) = c_1 u_1(t) + c_2 u_2(t)$ is the general solution.

3.4   Case(4)

If $p(t)$ and $q(t)$ be two crisp positive function and these are considered that u' is (ii)-differentiable and u'' is (i)-differentiable then two following

systems are denoted:

$$\begin{cases} \overline{u}''(t) = p(t)\underline{u}'(t) + q(t)\underline{u}(t), \\ \underline{u}(0) = \underline{u}_0, \\ \overline{u}'(0) = \overline{u}'_0 \end{cases} \tag{3.11}$$

and

$$\begin{cases} \underline{u}''(t) = p(t)\overline{u}'(t) + q(t)\overline{u}(t), \\ \overline{u}(0) = \overline{u}_0, \\ \underline{u}'(0) = \underline{u}'_0, \end{cases} \tag{3.12}$$

Thus the following system is founded:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, S(t) = \begin{pmatrix} 0 & p(t) \\ p(t) & 0 \end{pmatrix}, Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(t) \end{pmatrix}$$

Briefly:

$$U''(t) + S(t)U'(t) = Q(t)U(t)$$

Now by solving (11) and (12) together the following equations are given:

$$\begin{cases} u''^c(t) + p(t)u'^c(t) = q(t)u^c(t), \\ u^c(a) = u^c_0, \\ u'^c(a) = u'^c_0, \end{cases} \tag{3.13}$$

and

$$\begin{cases} -u''^d(t) + p(t)u'^d(t) = -q(t)u^d(t), \\ u^d(a) = u^d_0, \\ u'^d(a) = u'^d_0, \end{cases} \tag{3.14}$$

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Then we have

$$u_2^c = u_1^c \int (\frac{1}{u_1^c} e^{\int p(t)dt}) dt, u_2^d = u_1^d \int (\frac{1}{u_1^d} e^{-\int p(t)dt}) dt$$

where $u^c = \frac{u+\bar{u}}{2}$, $u'^c = \frac{u'+\bar{u}'}{2}$ and $u''^c = \frac{u''+\bar{u}''}{2}$ also $u^d = \frac{\bar{u}-u}{2}$, $u'^d = \frac{\bar{u}'-u'}{2}$, $u''^d = \frac{\bar{u}''-u''}{2}$ and $u_1^c = \frac{u_1+\bar{u}_1}{2}$, $u_1^d = \frac{\bar{u}_1-u_1}{2}$.

Now by definition (2.3) we can find $\underline{u}_2$ and $\bar{u}_2$ and $u(t) = c_1 u_1(t) + c_2 u_2(t)$ is the general solution.

4   Examples

Example 4.1 *Consider the following second order differential equation with initial value:*

$$\begin{cases} u''(t) + \frac{1}{t} u'(t) = \frac{1}{t^2} u(t) \\ u(1) = [2\alpha - 1, 3 - 2\alpha] \\ u'(1) = [6\alpha - 5, 5 - 4\alpha] \end{cases} \qquad (4.1)$$

then by analytic method it can given:
In case (1):
The first solution of this equation is $u_1(t) = t$, then $u_2(t) = \frac{-1}{2t}$. The solution of (15) is:

$$u(t) = [(2\alpha - 3)t + (4\alpha - 2)(-\frac{1}{t}), (4 - 3\alpha)t + (1 - \alpha)(-\frac{1}{t})]$$

In case (2):
The first solution of this equation are $u_1^c(t) = t$ and $u_1^d(t) = \frac{1}{t}$, $u_2^d(t) = \frac{t^3}{4}$, $u^c(1) = 1$, $u^d(1) = 1 - \alpha$, $u'^c(1) = \alpha$, $u'^d(1) = 5 - 5\alpha$. The solution of (15) is given by:

$$u(t) = [(\frac{\alpha+1}{2})t + (\alpha - 1)(\frac{1}{t}) - (6 - 6\alpha)\frac{t^3}{4}, (\frac{\alpha+1}{2})t + (6 - 6\alpha)\frac{t^3}{4}]$$

In case (3):
The first solution of this equation are $u_1^c(t) = t$ and $u_1^d(t) = t$, $u_2^d(t) = tlnt$, $u^c(1) = 1$, $u^d(1) = 1 - \alpha$, $u'^c(1) = \alpha$, $u'^d(1) = 5 - 5\alpha$. The solution of (15) is denoted by following:
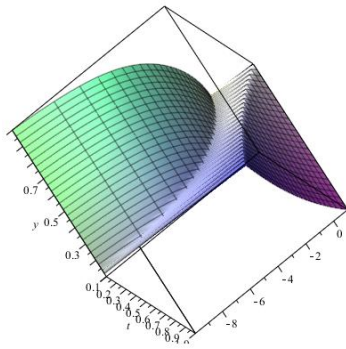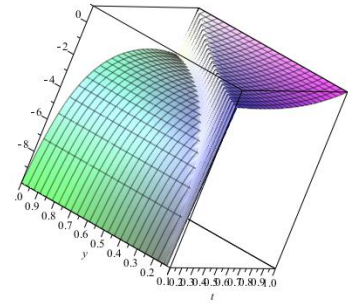
$$u(t) = [(\frac{3\alpha - 1}{2})t + (\alpha-1)(\frac{-1}{2t}) + (4\alpha-4)tlnt, (\frac{3 - \alpha}{2})t + (\alpha-1)(\frac{1}{2t}) + (4-4\alpha)tlnt]$$

In case (4):
The first solution of this equation are $u_1^c(t) = t$ and $u_1^d(t) = t^{1+\sqrt{2}}$, $u_2^d(t) = t^{\sqrt{2}-1}$, $u^c(1) = 1$, $u^d(1) = 1 - \alpha$, $u'^c(1) = \alpha$, $u'^d(1) = 5 - 5\alpha$. The solution of (15) is in the following:
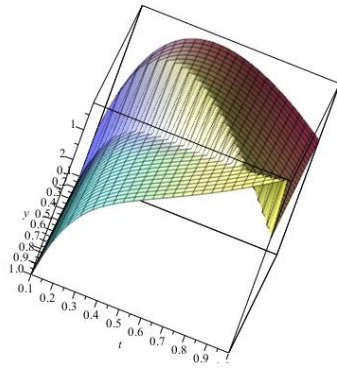$u(t) = [(\frac{\alpha+1}{2})t + (\alpha-1)(\frac{-1}{2t}) + (\frac{4-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}-1} - (\frac{6-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}+1}, (\frac{\alpha+1}{2})t +$
$(\alpha - 1)(\frac{-1}{2t}) - (\frac{4-\sqrt{2}}{2})(\alpha - 1)t^{\sqrt{2}-1} + (\frac{6-\sqrt{2}}{2})(\alpha - 1)t^{\sqrt{2}+1}]$
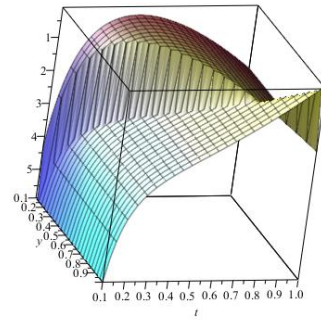
Fig(1): Case (1) in example (4.1)

Fig(2): Case (2) in example (4.1)

Fig(3): Case (3) in example (4.1)

Fig(4): Case (4) in example (4.1)

5 Conclusion

In this work, fuzzy second order differential equations with fuzzy initial values considered. Parametric fuzzy number used to find the general solution with first given solution under Hakuhara derivations in the formula in four cases. Then by using fuzzy initial values four solutions are found.

References

[1] T.Allahviranloo, N.Kiani, N.Barkhordari,*Toward the existence and uniqueness of solution of second order fuzzy differential equations* , Information sciences 179, 2009.

[2] Allahviranloo. T, Hooshangian. L, *Fuzzy Generalized H-Differential and Applications to Fuzzy Differential Equations of Second-Order*, Intelligent and Fuzzy Systems, 26 (2014) 1951-1967.

[3] B.Bede, S.G.Gal, *Generalizations of differentiability of fuzzy number value function with applications to fuzzy differential equations* , Fuzzy sets and systems151, 2005.

[4] B. Bede, I.J. Rudas, A.L. Bencsik*First order linear fuzzy differential equations under genaralizes differentiability* , Information Sciences 177, 2007.

[5] J. Buckley, T. Feuring, M.D. Jimenez-Gamero, *Fuzzy differential equation*, Fuzzy Sets and Systems 110, 2000.

[6] Y. Chalco-Cano, H. Roman-Flores,*Fuzzy differential equations with generalized derivative* , 27th ANFIPS International Conference IEEE, 2008.

[7] Y. Chalco-Cano, H. Roman-Flores, *On new solutions of fuzzy differntial equations* , Chaos solitons and fractals 38, 2008.

[8] Y. Chalco-Cano, H. Roman-Flores, M.D. Jimenez-Gamero, *Generalized derivative and $\pi-$derivative for set-valued functions* , Information Science, 2011.

[9] P. Diamond,*Brief note on the variation of constance formula for fuzzy differential equations* ,fuzzy sets and systems 129 , 2002.

[10] E. Eljaoui, S. Mellani, L. S. Chadli,*Solving second order fuzzy differential equations by fuzzy laplace transfom method*, Advances in Difference Equations , 2015.

[11] O. Kaleva,*Fuzzy differential equations*,fuzzy sets and systems 24, 1987.

[12] O. Kaleva,*A note on fuzzy differential equations* , Nonlinear Analysis 64, 2006.

[13] A. Kauffman, M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Application*, Van Nostrand Reinhold, New York, 1991.

[14] A. Khastan, J.J.Nieto, *Aboundary value problem for second order fuzzy differential equations*, Nonlinear analysis 72, 2010.

[15] V.Lupulescu, *Initial value problem of fuzzy differential equation under dissipative condition* , Information sciences 178, 2008.

[16] J.J. Nieto, R.R. Lopez*Fuzzy differential system under generalized metric space approach* , Dynamic System Application 17, 2008.

[17] N. Parandin,*Numerical solutions of fuzzy second order differential equations of 2nd-order by Runge-Kutta method* , Journal of mathematical extension Vol 7. No 3, 47-62, 2013.

[18] Pallingkinis,*Runge-Kutah methods for fuzzy differential equations* , Applied mathematics and computation 209, 2009.

[19] V. Parimala, P. Rajarajeswari, V. Nirvala*A second order Runge-Kutta method to solve Fuzzy differential equations with fuzzy initial condition* , International Journal of Science and research Vol 3, 428-431, 2014.

[20] M. Puri, D. Ralescu*Differential and fuzzy functions* , Mathematics Analysis and Applications 91, 1983.

[21] Pallingkinis,*Runge-Kutah methods for fuzzy differential equations* , Applied mathematics and computation 209, 2009.

[22] S. Siekalla, *On the fuzzy initial value problem* , Fuzzy sets and systems, 1987.

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[23] L. stefanini, B.Bede, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations* , Nonlinear Analysis 71, 2009.

[24] L. Stefanini ,*A generalization of Hukuhara difference and division for interval and fuzzy arithmetics*, Fuzzy sets and systems 161, 2010.

[25] L. Wang, S. Guo, *Adomian method for second order Fuzzy differential equation*, International Journal of mathematical, computational physical, electrical and computer engineering, Vol 5, 613-616, 2011.

[26] L.A. Zadeh, *Fuzzy Sets*, Information and Control 8, 1965.

[27] D. Zhang, W.Feng, Y.Zhao, J.Qiu, *Global existence of solutions for fuzzy second order differential equations under generalize H-differentiability* , Computers and mathematics with applications 60, 2010.

[28] H.J. Zimmermann, *Fuzzy sets theory and its applications*, Kluwer Academic Press, Dordrecht, 1991.

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