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Analysis of Convergence of Solution of General Fuzzy Integral Equation with Nonlinear Fuzzy Kernels

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Abstract

Fuzzy integral equations have a major role in the mathematics and applications. In this paper, general fuzzy integral equations with nonlinear fuzzy kernels are introduced. The existence and uniqueness of their solutions are approved and an upper bound for them are determined. Finally an algorithm is drawn to show theorems better.

Key words: General Nonlinear fuzzy integral equation; Existence theorem; Uniqueness theorem, Upper bound, Nonlinear fuzzy kernels.

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1 Introduction

In recent years, many basic functions have used to estimate the solutions of integral equations in applied sciences such as physics and mechanics. The topics of fuzzy integral equations which attracted growing interest for some time, in particular in relation to fuzzy control, have been developed. The existence and uniqueness of solutions of fuzzy integral equations are dilemma and it is necessary to approve.

Park and Jeong approved the existence and uniqueness theorem of a solution to the fuzzy Volterra integral equation

$$x(t) = f(t) + \int_{t_0}^{t} k(t, s, x(s))ds$$

where $f : [t_0, t_0 + a] \to E^1$ is level wise continuous and $k : [t_0, t_0 + a] \times [t_0, t_0 + a] \times E^n \to E^n$ satisfies some conditions, [7]. Park and Han studied the problems of existence and uniqueness of the solutions of fuzzy Volterra-Fredholm integral equation of the form

$$x(t) = F(t, x(t), \int_{0}^{t} f(t, s, x(s))ds, \int_{0}^{T} g(t, s, x(s))ds), 0 \leq t \leq T$$

where $x(t)$ is an unknown fuzzy set-valued mapping, [8]. Georgiou and Kougias examined conditions under which all the solutions of the fuzzy integral equation

$$x(t) = \int_{0}^{t} G(t, s)x(s)ds + f(t)$$

and the special case $x(t) = \int_0^t k(t - s)x(s)ds + f(t)$ are bounded that fuzzy integral equations prove useful when studying of fuzzy dynamical control systems, [5]. Balachandran and Prakash proved the existence of solutions of fuzzy Volterra integral equations with deviating arguments which established with the help of the Darbo fixed point theorem. They studied the maximal solution of the fuzzy delay Volterra integral equation, [1]. In [2], Balachandran and Karagarajan proved the existence of solutions of fuzzy integral equations of the form

$$x(t) = \phi(t) + x(t) \int_{0}^{t} k(t, s)f(s, x(s))ds + \int_{0}^{t} g(t, s, x(s))ds$$

where $\phi : [0,T] \to E^n, k : [0,T]\times[0,T] \to R, f : [0,T]\times E^n \to E^n$, and $g : [0,T]\times[0,T]\times E^n \to E^n$ are continuous functions. In all papers mentioned above and in many others the authors considered fuzzy integral equations of the first order. However, integral equations are encountered in various fields of science and applications.

The paper organized as the following: In section 2, some necessary concepts is reviewed briefly. In section 3, general fuzzy integral equation is introduced and conditions for the existence and uniqueness of their solutions are presented. Finally, in section 4 a conclusion is drawn.

2 Basic concepts

The basic definitions of a fuzzy number are given as follows:

Definition 2.1 *[12] A fuzzy number is a fuzzy set like $u : \mathbb{R}\to [0,1]$ which satisfies:*

1. u is an upper semi-continuous function,
2. $u(x) = 0$ outside some interval [a,d],
3. There are real numbers b, c such as $a \le b \le c \le d$ and
3.1 $u(x)$ is a monotonic increasing function on [a, b],
3.2 $u(x)$ is a monotonic decreasing function on [c, d],
3.3 $u(x) = 1$ for all $x \in [b, c]$.

Let $P_K(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = max\{sup_{a\in A} inf_{b\in B}\|a-b\|, sup_{b\in B} inf_{a\in A}\|a-b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n), d)$ becomes a metric space.

Theorem 2.2 *[10] The metric space $(P_K(R^n), d)$ is complete and sepa-*

rable. Let $I = [c, d] \subset R$ *be a compact interval and denote*

$$E^n = \{u : R^n \to [0,1] \mid u \ \ satisfies \ \ (i) - (iv) \ \ below\}$$

where

(i) u *is normal, i.e. there exists an* $x_0 \in R^n$ *such that* $u(x_0) = 1$,
(ii) u *is fuzzy convex*
(iii) u *is upper semicontinuous,*
(iv) $[u]^0 = cl\{x \in R^n | u(x) > 0\}$ *is compact.*

For $0 < \alpha \leq 1$ *denote* $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$, *then from (i)-(iv) it follows that the* α*-level set* $[u]^\alpha \in P_k(R^n)$ *for all* $0 \leq \alpha \leq 1$.

Remark 2.3 E^n *denotes the class of fuzzy subsets of real axis. The metric structure is given by Hausdorff distance satisfying the following properties:*
$D(u(r), v(r)) = Max\{sup|\underline{u} - \underline{v}|, sup|\overline{u} - \overline{v}|\}$
(E^n, D) *is a complete space and the following properties are well known:*
$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in E^n$
$D(ku, kv) = |k| D(u, v), \quad \forall u, v \in E^n, \quad \forall k \in \mathbb{R}$
$D(u + v, w + e) \leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in E^n$

Definition 2.4 *[11] If* $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ *be a fuzzy set on the* $X \in \mathbb{R}$, *the* $\alpha - cut$ *of subsets of* $\tilde{A}$ *is:*

$$\tilde{A}_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$$

that $\mu_{\tilde{A}} : X \to [0,1]$ *is named membership functions of* $\tilde{A}$. *For* $\alpha = 1$, *the 1-cut of* $\tilde{A}$ *is named core of* $\tilde{A}$.

Theorem 2.5 *[7] If* $f : [a, b] \to E^n$ *be integrable and* $c \in [a, b]$, $\lambda \in \mathbb{R}$. *Then:*
(i) $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,
(ii) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
(iii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,
(iv) $D(F, G)$ *is integrable,*
(V) $D(\int_I F(t)dt, \int_I G(t)dt) \leq \int_I D(F, G)$

Definition 2.6 *[6] A mapping* $F : [a, b] \to E^n$ *is strongly measurable if* *for all $\alpha \in [0,1]$ the set-valued mapping $F_\alpha : I \to P_K(R^n)$ defined by*

$$F_\alpha(t) = [F(t)]^\alpha$$

is Lebesque measurable, when $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.7 *[6] Let $F : I \to E^n$. The integral of F over I, denoted by $\int_I F(t)dt$, is defined levelwise by the equation*

$$(\int_I F(t)dt)^\alpha = \int_I F_\alpha(t)dt = \{f(t)dt \mid f : I \to R^n \text{ is a measurable selection for } F_\alpha\}$$

for all $0 < \alpha \leq 1$. A strongly measurable and integrable bounded mapping $F : I \to E^n$ is said to be integrable over I if $\int_I F(t)dt \in E^n$.

3 The existence and uniqueness theorems of general fuzzy integral equations

A general nonlinear fuzzy integral equation with nonlinear fuzzy kernel is defined as follows:

$$x(t) = f(t) + \int_{t_0}^{t} g_1(t,s,x(s))ds + ... + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x(s))ds...ds, \quad t \geq 0 \tag{3.1}$$

Where $x(t)$ is a function of t, $f(t)$ is a set-valued function and $g_1(t,s,x(s))$ and $g_2(t,s,x(s))$,..., $g_n(t,s,x(s))$ are nonlinear fuzzy functions and all continuous and n is natural number.
In the following theorem, the existence and uniqueness of the solution of the general fuzzy integral equation are investigated:

Theorem 3.1 *Let a and L are positive numbers. Assume that Eq.(1) satisfies the following conditions:*

1. *$f : [0,a] \to \mathbb{R}$ is continuous and bonded,*

2. $g_i : [0,a] \to \mathbb{R}$ *for all i=1,2,...n are continuous and satisfy the Lipschitz condition:*
   $D(g_i(t,s,x), g_i(t,s,y)) \leq L_i D(x,y)$,
3. $g_i(t,s,0)$ *are bounded on* $[0,a]$ *for i=1,2,...n.*

then there exists a unique solution $x(t)$ *of Eq.(1) on* $[0,a]$ *and the successive iterations:*

$$\begin{cases} x_0(t) = f(t), \\ x_m(t) = f(t) + \int_{t_0}^{t} g_1(t,s,x_{(m-1)}(s)) + ... + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_{m-1}(s))ds...ds, \\ \\ \text{where } m = 1,2,..., \end{cases} \tag{3.2}$$

are uniformly convergent to $x(t)$ *on* $[0,a]$.

Proof: By mathematical induction, it can be seen that all $x_n(t)$ are level wise continuous mappings on $[t_0, t_0+\varepsilon]$. Indeed, let $t \in [t_0, t_0+\varepsilon]$, for $n = 1$,

$$\begin{aligned} x_1(t) = x_0(t) + \textstyle\int_{t_0}^{t} g_1(t,s,x_0(s))ds... \\ + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_0(s))ds...ds \end{aligned}$$

which proves that $x_1(t)$ is continuous on $[t_0, t_0+a]$, hence on $[t_0, t_0+\varepsilon]$.

For any $\alpha \in [0,1]$, it is written:

$$\begin{aligned} D(x_0(t), x_1(t)) &= D(x_0(t), x_0(t) + \textstyle\int_{t_0}^{t} g_1(t,s,x_0(s))ds + ... \\ &\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_1(s))ds...ds) \\ &\le D(\textstyle\int_{t_0}^{t} g_1(t,s,x_0(s))ds, 0) + ... \\ &\quad + D(\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_0(s))ds...ds], 0) \\ &\le \textstyle\int_{t_0}^{t} D(g_1(t,s,x_0(s)), 0) + ... \\ &\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(g_n(t,s,x_0(s))ds...ds], 0) \\ &\le M(t-t_0) + ... + M\tfrac{(t-t_0)^n}{n!} \\ &\le M(\varepsilon + ... + \tfrac{\varepsilon^n}{n!}) \end{aligned}$$

Then

$$D(x_1(t), x_0(t)) \le M \sum_{i=1}^{n} \frac{\varepsilon^i}{i!} \tag{3.3}$$

According to (3), we have

$$\begin{aligned} D(x_2(t), x_1(t)) &\le D(x_0(t) + \textstyle\int_{t_0}^{t} g_1(t,s,x_1(s))ds + ... \\ &\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_1(s))ds...ds, x_0(t) \\ &\quad + \textstyle\int_{t_0}^{t} g_1(t,s,x_0(s))ds + ... \\ &\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_0(s))ds...ds) \\ &\le M(L_1\tfrac{(t-t_0)^2}{2!} + ... \\ &\quad + L_n\tfrac{(t-t_0)^{n+2}}{n!2!}) \end{aligned}$$

Now assume that $x_{m-1}(t)$ is continuous, then:

$$\begin{aligned}
D(x_m(t), x_{m-1}(t)) \leq\ & D(x_0(t) + \textstyle\int_{t_0}^t g_1(t,s,x_{m-1}(s))ds + ... \\
& + \underbrace{\int_{t_0}^t ... \int_{t_0}^t}_{n} g_n(t,s,x_{m-1}(s))ds...ds \\
& , x_0(t) + \textstyle\int_{t_0}^t g_1(t,s,x_{m-2}(s))ds + ... \\
& + \underbrace{\int_{t_0}^t ... \int_{t_0}^t}_{n} g_n(t,s,x_{m-2}(s))ds...ds) \\
\leq\ & D(\textstyle\int_{t_0}^t g_1(t,s,x_{m-1}(s))ds, \int_{t_0}^t g_1(t,s,x_{m-2}(s))ds + ... \\
& + D(\underbrace{\int_{t_0}^t ... \int_{t_0}^t}_{n} g_n(t,s,x_{m-1}(s))ds...ds, \\
& \underbrace{\int_{t_0}^t ... \int_{t_0}^t}_{n} g_n(t,s,x_{m-2}(s))ds...ds) \\
\leq\ & \textstyle\int_{t_0}^t D(g_1(t,s,x_{m-1}(s))ds, g_1(t,s,x_{m-2}(s))ds) + ... \\
& + \underbrace{\int_{t_0}^t ... \int_{t_0}^t}_{n} D(g_n(t,s,x_{m-1}(s)), [g_n(t,s,x_{m-2}(s)))ds...ds \\
\leq\ & D(x_{m-1}, x_{m-2})(L_1(t-t_0) + ... \\
& + L_n \tfrac{(t-t_0)^n}{n!}) \\
\leq\ & M(L_1^{m-1} \tfrac{(t-t_0)^m}{m!} + ... \\
& + L_n^{m-1} \tfrac{(t-t_0)^{n+m}}{n!m!})
\end{aligned}$$

Therefore it can be written as:

$$D(x_m(t), x_{m-1}(t)) \leq M \sum_{i=1}^{n} L_i^{m-1} \frac{\varepsilon^{(i+m)!}}{i!m!} \tag{3.4}$$

It follows by mathematical induction that Eq. (4) holds for any $n \geq 1$. Consequently, Eq. (3) is uniformly convergent on $[t_0, t_0 + \varepsilon]$, and so is the sequence $\{x_n(t)\}$. Denote by $x(t) = lim_{n\to\infty} x_n(t)$. The function $x(t)$ is level wise continuous on $[t_0, t_0+\varepsilon]$ and it can be seen that it satisfies Eq. (1). Indeed, by condition (3) in theorem and from definition of D we get:

$D(g_i(t,s,x_{m-1}(s)), g_i(t,s,x(s))) \leq L_i D(x_{m-1}(s), x(s)) \to 0 \quad as \quad n \to \infty, \quad t_0 \leq s \leq t \leq t_0+\varepsilon \quad i = 1, 2, ...$
Thus, the existence of a solution is proven.

To prove the uniqueness, let $y(t)$ is a level wise continuous solution of Eq. (1) on $[t_0, t_0+\varepsilon]$. Then

$$y(t) = f(t) + \int_{t_0}^{t} g_1(t,s,x(s))ds + ... + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x(s))ds...ds \quad (3.5)$$

From (2), (5) and the condition (3) of theorem 3.1, for $m \geq 1$ and any $\alpha \in (0,1]$ we obtain:

$$\begin{aligned}
D(y(t), x_m(t)) &\leq D(f(t) + \textstyle\int_{t_0}^{t} g_1(t,s,y(s))ds + ... \\
&\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,y(s))ds...ds, f(t) \\
&\quad + \textstyle\int_{t_0}^{t} g_1(t,s,x_{m-1}(s))ds + ... \\
&\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t,s,x_{m-1}(s))ds...ds) \\
&\leq \textstyle\int_{t_0}^{t} D(g_1(t,s,y(s)), g_1(t,s,x_{m-1}(s)))ds + ... \\
&\quad + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(g_n(t,s,y(s)), g_n(t,s,x_{m-1}(s)))ds...ds \\
&\leq L_1 \textstyle\int_{t_0}^{t} D(y(s), [x_{m-1}(s))ds + ... \\
&\quad + L_n \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(y(s), x_{m-1}(s))ds...ds
\end{aligned}$$

Then we get:

$$D(y(t), x_m(t)) \leq L_1 \int_{t_0}^{t} D(y(s), x_{m-1}(s))ds + ... \\ +L_n \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(y(s), [x_{m-1}(s))ds...ds \tag{3.6}$$

But for $t \in [t_0, t_0 + \varepsilon]$ we know

$$D(y(t), f(t)) \leq M(t - t_0) + ... + M\frac{(t - t_0)^n}{n!}$$

From (6), we have

$$D(y(t), x_1(t)) \leq ML_1\frac{(t - t_0)}{1!} + ... + ML_n\frac{(t - t_0)^{n+1}}{n!}$$

Now assume that:

$$D(y(t), x_{m-1}(t)) \leq ML_1^{m-1}\frac{(t - t_0)^m}{(m-1)!} + ... + ML_n^{m-1}\frac{(t - t_0)^{m+n}}{(m-1)!n!}, t \in [t_0, t_0 + \varepsilon] \tag{3.7}$$

From (6) and (7) we obtain:

$$D(y(t), x_m(t)) \leq ML_1^{m}\frac{(t - t_0)^{m+1}}{m!} + ... + ML_n^{m}\frac{(t - t_0)^{m+n+1}}{m!n!}, t \in [t_0, t_0 + \varepsilon] \tag{3.8}$$

Consequently, (8) holds for any n, which leads to the conclusion $D(y(t), x_n(t)) = D(x(t), x_n(t)) \to 0$ on $[t_0, t_0 + \varepsilon]$ as $n \to \infty$.

This proves the uniqueness of the solution for (1). □

In the following theorem, continuity of solution of the general fuzzy integral equation is searched:

Theorem 3.2 *Assume that $f(t)$ is level wise continuous on $[t_0, t_0 + a]$, $a > 0$, and $g_i(t, s, x)$ for ,$i = 1, 2, ..., n$, are level wise continuous on*

$t_0 \le s \le t \le t_0 + a$. Then there exists at least one level wise continuous solution of Eq. (1), defined in $[t_0, t_0 + a]$.

Proof:

$$
\begin{aligned}
D(x_m(t_1), x_m(t_2)) \le\ & D(f(t_1), f(t_2)) + D(\textstyle\int_{t_0}^{t_1} g_1(t_1, s, x_{m-1}(s))ds \\
& +...+ \underbrace{\int_{t_0}^{t_1} ... \int_{t_0}^{t_1}}_{n} g_n(t_1, s, x_{m-1}(s))ds...ds, \\
& \textstyle\int_{t_0}^{t_2} g_1(t_2, s, x_{m-1}(s))ds + ... \\
& + \underbrace{\int_{t_0}^{t_2} ... \int_{t_0}^{t_2}}_{n} g_n(t_2, s, x_{m-2}(s))ds...ds \\
\le\ & D(f(t_1), f(t_2)) + D(\textstyle\int_{t_0}^{t_1} g_1(t_1, s, x_{m-1}(s))ds, \\
& \textstyle\int_{t_0}^{t_2} g_1(t_2, s, x_{m-2}(s))ds) + ... \\
& +D(\underbrace{\int_{t_0}^{t_1} ... \int_{t_0}^{t_1}}_{n} g_n(t_1, s, x_{m-1}(s))ds...ds \\
& , \underbrace{\int_{t_0}^{t_2} ... \int_{t_0}^{t_2}}_{n} g_n(t_2, s, x_{m-2}(s))ds...ds) \\
\le\ & D(f(t_1), f(t_2)) + \textstyle\int_{t_0}^{t_1} D(g_1(t_1, s, x_{m-1}(s))ds, g_1(t_2, s, x_{m-1}(s))ds) \\
& + \textstyle\int_{t_1}^{t_2} D(g_1(t_2, s, x_{m-1}(s))ds, 0) + ... \\
& + \underbrace{\int_{t_0}^{t_1} ... \int_{t_0}^{t_1}}_{n} D(g_n(t_1, \\
& s, x_{m-1}(s)), g_n(t_2, s, x_{m-1}(s)))ds...ds \\
& + \underbrace{\int_{t_1}^{t_2} ... \int_{t_1}^{t_2}}_{n} D(g_n(t_2, s, x_{m-1}(s)), 0) \\
\le\ & D(f(t_1), f(t_2)) + (t_1 - t_0)D(g_1(t_1, s, x_{m-1}(s))ds, g_1(t_2, s, x_{m-1}(s))ds) \\
& +M(t_2 - t_1) + ... + \tfrac{(t_1-t_0)^n}{n!} D(g_1(t_1, s, x_{m-1}(s))ds, g_1(t_2, s \\
& , x_{m-1}(s))ds) + M\tfrac{(t_2-t_1)^n}{n!}
\end{aligned}
$$

Then it is gotten:

$$D(x_m(t_1), x_m(t_2)) \to \tilde{0} \quad as \quad t_2 \to t_1$$

Thus the sequence $x_m(t)$ is continuous on $[t_0, t_0 + a]$. $\Box$

Algorithm:

step 1: For $i = 1$ set $x_i(t) = f(t)$;
step 2: Obtain that $D(x_i, 0)$ is bounded;
step 3: Prove that x_i is continuous;
step 4: Obtain that $D(x_{i+1}(t), x_i(t))$ is bounded;
step 5: Obtain that $D(y(t), x_i(t))$ is bounded for all continuous y(t);
step 6: Let $i + 1 = i$ and set $x_i^{(n)}(t) = f(t, s, x_{i-1})x_i(t) = f(t) + \int_{t_0}^{t} g_1(t, s, x_{i-1})ds + ... + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} g_n(t, s, x_{i-1})ds...ds$ then go to step 2.

4 Conclusion

In this work, the general fuzzy integral equations with nonlinear fuzzy kernels were studied. The existence and uniqueness of solutions of these fuzzy integral equations were proved by theorems. An algorithm was presented to show the conditions of existence and uniqueness of solutions better. These results will be useful in future research for obtaining the solutions of general nonlinear fuzzy integral equations.

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