Homological dimensions of complexes of R-modules

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Abstract

Let R be an associative ring with identity, $\mathbf{C}(R)$ be the category of complexes of R-modules and $\mathrm{Flat}(\mathbf{C}(R))$ be the class of all flat complexes of R-modules. We show that the flat cotorsion theory $(\mathrm{Flat}(\mathbf{C}(R)), \mathrm{Flat}(\mathbf{C}(R))^{\perp})$ have enough injectives in $\mathbf{C}(R)$. As an application, we prove that for each flat complex $\mathbf{F}$ and each complex $\mathbf{Y}$ of R-modules, $\mathrm{Ext}^i(\mathbf{F}, \mathbf{Y}) = 0$, whenever R is n-perfect and $i > n$.

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1 Introduction

In 1966 Spencer E. Dickson [2] introduced *torsion* theories for abelian categories by exploiting the Hom-functor. By replacing formally the Hom-functor with the Ext-functor one get the basic tools of a *cotorsion theory* in an abelian category, which naturally extends the *classical* cotorsion theory. The classical cotorsion theory, where it is developed in the 60s by Harrison and many other algebraists, is the pair (Torsion-free, Cotorsion). Based on this idea, in 1978 Luigi Salce [6], introduced the notion of cotorsion theories in the category of abelian groups.

The main task of Salce is a detailed description of the cotorsion theory $(^{\perp}(S^{\perp}), S^{\perp})$ cogenerated by S, where $S \subseteq \mathbb{Q}$ is a rank-1 group such that $1 \in S$, see [6]. These cotorsion theories (for any $S \subseteq \mathbb{Q}$) are called rational cotorsion theories. Salce in [6, Problem 2, p. 31] raised the question of whether rational cotorsion theories have enough projectives (injectives).

In 1998, Gobel and Shelah answered this question, see [5, Theorem 6.1]. They proved that any cotorsion theory of abelian groups, which is cogenerated by a set H of rank-1 groups, has enough injectives and projectives. Therefore rational cotorsion theories have enough injectives and projectives.

In 1981 , Enochs raised the question of whether every module has flat cover. Also he proved that if a module has a flat precover, then it has a flat cover. Let $\mathcal{F}$ stands for the class of all flat R-modules. It is easy to see that the problem of the existence of $\mathcal{F}$-precovers is equivalence to the problem of completeness of the cotorsion theory $(\mathcal{F}, \mathcal{F}^{\perp})$.

The existence of covers and envelopes are essential tools of *relative homological algebra*. Let $\mathcal{X}$ be a class of R-modules, which is closed under isomorphism, the completeness of the cotorsion theories $(^{\perp}\mathcal{X}, (^{\perp}\mathcal{X})^{\perp})$ and $(^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$, induces a relative homology with respect to $\mathcal{X}$. Hence complete cotorsion theories, provide us

to have homological algebra in Grothendieck categories.

In 2000, Eklof and Trlifaj proved that any cotorsion theory of R-modules which is cogenerated by a set, is a complete cotorsion theory, see [4]. Therefore by [7] the flat cotorsion theory $(\mathcal{F}, \mathcal{F}^{\perp})$ is cogenerates by a set and hence it is complete. Thus the category of R-modules admits $\mathcal{F}$-covers and $\mathcal{F}^{\perp}$-envelopes.

2 Complete cotorsion theories in the category of complexes of R-modules

Throughout this section, let $\mathcal{G}$ be a Grothendieck category with projective generator and $\mathcal{A} = \mathbf{C}(R)$ be the category of all complexes of R-modules. Let $\mathcal{X}$ be a class of objects of $\mathcal{G}$ such that it is closed under isomorphisms, finite direct sums and direct summands. In this section, we will give a general definition of relative homological algebra.

Definition 2.1 *The right(left) orthogonal of $\mathcal{X}$ in $\mathcal{G}$ is defined as*

$$\mathcal{X}^{\perp} = \{\mathbf{Y}|\ \mathrm{Ext}^1_{\mathcal{A}}(\mathbf{X}, \mathbf{Y}) = 0, \forall \mathbf{X} \in \mathcal{X}\}(^{\perp}\mathcal{X} = \{\mathbf{Y}|\ \mathrm{Ext}^1_{\mathcal{A}}(\mathbf{Y}, \mathbf{X}) = 0,$$
$$\forall \mathbf{X} \in \mathcal{X}\}).$$

The pair $(\mathcal{X}, \mathcal{Y})$ is said to be a cotorsion theory in $\mathcal{G}$ if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. If there exists a class $\mathcal{S}$ of objects in $\mathcal{X}$ such that $\mathcal{S}^{\perp} = \mathcal{Y}$, we say that $(\mathcal{X}, \mathcal{Y})$ is cogenerated by $\mathcal{S}$.

Definition 2.2 *A cotorsion theory $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{G}$ is said to have enough injectives(projectives), if for any object $\mathbf{M}$ of $\mathcal{A}$, there exists a short exact sequence*

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathbf{X} \longrightarrow \mathbf{Y} \longrightarrow 0\ (\ 0 \longrightarrow \mathbf{Y} \longrightarrow \mathbf{X} \longrightarrow \mathbf{M} \longrightarrow 0\ )$$

for some $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$. It also called complete if it has enough injectives and projectives.

Example 2.1 *The ordinary homological algebra, induced by the complete cotorsion theories (Proj R,R-Mod) and (R-Mod,Inj R), where Proj R (Inj R) is the class of all projective (injective) R-modules. Those cotorsion theories have enough injectives and projectives.*

Example 2.2 *The cotorsion theory (R-Mod, Inj R) is cogenerated by the set of modules R/I where I is a left ideal. Therefore it has enough injectives.*

Example 2.3 *In the classical cotorsion theory (Torsion-free, Cotorsion), every torsion-free abelian group J can be embedded in an exact sequence already guarantees that G is cotorsion and hence, the definition of cotorsion groups may also be given as groups G satisfying $Ext(\mathbb{Q}, G) = 0$. Therefore the classical cotorsion theory is cogenerated by the rationals $\mathbb{Q}$, i.e. (Torsion-free, Cotorsion) = $(^{\perp}(\mathbb{Q}^{\perp}), \mathbb{Q}^{\perp})$. Therefore it has enough injectives.*

Proposition 2.1 *If a cotorsion theory $(\mathcal{X}, \mathcal{Y})$ having enough injectives in $\mathcal{G}$, then it also have enough projectives.*

Proof. Let $(\mathcal{X}, \mathcal{Y})$ has enough injectives and $\mathbf{M}$ be an object of $\mathcal{G}$. The category $\mathcal{G}$ is a Grothendieck category with projective generators. Then there exists an exact sequence $0 \longrightarrow \mathbf{T} \longrightarrow \mathbf{P} \longrightarrow \mathbf{M} \longrightarrow 0$ with $\mathbf{P}$ projective. By assumption, there exists an exact sequence $0 \longrightarrow \mathbf{T} \longrightarrow \mathbf{Y} \longrightarrow \mathbf{X} \longrightarrow 0$ with $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$. Using the pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbf{T} & \longrightarrow & \mathbf{P} & \longrightarrow & \mathbf{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{Y} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{X} & = & \mathbf{X} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns. Since $\mathcal{X}$ is closed under extensions, then $\mathbf{Z} \in \mathcal{X}$. Hence the middle row is the desired exact sequence.

Proposition 2.2 *Let* $\mathbf{S}$ *be a nonempty subset of objects of* $\mathcal{A}$*. Then the cotorsion theory* $(^{\perp}(\mathbf{S}^{\perp}), \mathbf{S}^{\perp})$ *has enough injectives.*

Proof. Let $\mathbf{B}$ be the direct sum of the objects in $\mathbf{S}$, $\mathbf{X}$ be an object of $\mathcal{A}$ and κ be an infinite cardinal number such that $\kappa > |\mathbf{B}| + |\mathbf{X}| + |R|$. Let $\beta = 2^{\kappa}$. By [4, Theorem 2] there exists an exact sequence $0 \longrightarrow \mathbf{X} \longrightarrow \mathbf{Y} \longrightarrow \mathbf{K} \longrightarrow 0$ of objects in $\mathcal{A}$ such that $\mathrm{Ext}^1(\mathbf{B}, \mathbf{Y}) = 0$. To prove that $\mathbf{K} \in {}^{\perp}(\mathbf{S}^{\perp})$, it suffices to show that $\mathrm{Ext}^1(\mathbf{K}, \mathbf{T}) = 0$ whenever $\mathrm{Ext}^1(\mathbf{B}, \mathbf{T}) = 0$. However, $\mathbf{K} = \cup_{\alpha < \beta} \mathbf{K}_{\alpha}$ where $\mathbf{K}_{\alpha} = \mathbf{K}_{\alpha}/\mathbf{X}$, so $\mathbf{K}_0 = 0$ and for each $\alpha < \beta$, $\mathbf{K}_{\alpha} + 1/\mathbf{K}_{\alpha} \cong \mathbf{Y}_{\alpha+1}/\mathbf{Y}_{\alpha} \cong \mathbf{B}$. Hence, by [4, Lemma 1], $\mathrm{Ext}^1(\mathbf{K}, X) = 0$ when $\mathrm{Ext}^1(\mathbf{B}, \mathbf{X}) = 0$. Then $(^{\perp}(\mathbf{S}^{\perp}), \mathbf{S}^{\perp}) = (^{\perp}(\mathbf{B}^{\perp}), \mathbf{B}^{\perp})$ has enough injectives.

3   The projective dimension of complexes of R-modules

Recall that, an acyclic complex $(\mathbf{F}, (\delta^n)_{n \in \mathbb{Z}})$ of flat R-modules is called *flat* if, for any $n \in \mathbb{Z}$, $\ker \delta^n$ is also flat R-module. We denote by $\mathrm{Flat}(\mathbf{C}(R))$ the class of all flat complexes in $\mathcal{A} = \mathbf{C}(R)$. We will show that the cotorsion theory $(\mathrm{Flat}(\mathbf{C}(R)), \mathrm{Flat}(\mathbf{C}(R))^{\perp})$ is a complete in $\mathcal{A}$. Let κ be a cardinal number such that $\kappa \geq \max\{|R|, \aleph_0\}$. Let $\mathbf{X} = (X^i, \delta^i_{\mathbf{X}})$ and $\mathbf{Y} = (Y^i, \delta^i_{\mathbf{Y}})$, the complex $\mathbf{Hom}^{\bullet}(\mathbf{X}, \mathbf{Y})$ is defined as follows:

$$\mathbf{Hom}^{\bullet}(\mathbf{X}, \mathbf{Y})^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X^i, Y^{i+n})$$

and its chain map is given by

$$\delta_{\mathbf{Hom}^{\bullet}(\mathbf{X}, \mathbf{Y})} = \delta_{\mathbf{Y}} \circ f - (-1)^n f \circ \delta_{\mathbf{X}} (f \in \mathbf{Hom}^{\bullet}(\mathbf{X}, \mathbf{Y})^n).$$

Theorem 3.1 *The cotorsion theory* $(\mathrm{Flat}(\mathbf{C}(R)), \mathrm{Flat}(\mathbf{C}(R))^{\perp})$ *has enough injectives.*

Proof.

Let $\mathbf{F} = (F^i, \delta^i)$ be a flat complex and $n \in \mathbb{Z}$. Let T be a subset of F^n with $|T| < \kappa$. We find a flat subcomplex $\mathbf{F}_0 = (F_0^i, \delta_0^i)$ of F such that, $T \subseteq F_0^n$, $|\mathbf{F}_0| \leq \kappa$ and $\frac{\mathbf{F}}{\mathbf{F}_0}$ is a flat complex.

Without loss of generality, let $n = 0$. There exists the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0^0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\delta^0) & \longrightarrow & F^0 & \longrightarrow & \mathrm{Ker}(\delta^1) & \longrightarrow & 0,
\end{array}
$$

of flat modules which is pure exact in rows and columns such that, $T \subseteq F_0^0$ and $\max\{|F_0^0|, |K_0^0|, |K_0^1|\} \leq \kappa$.
We use an inductive procedure to obtain, for every $i \leq 0$, a pure exact sequences $0 \rightarrow K_0^{i-1} \rightarrow F_0^{i-1} \rightarrow K_0^i \rightarrow 0$, such that $\max\{|F_0^{i-1}|, |K_0^{i-1}|, |K_0^i|\} \leq \kappa$. Set $F_0^1 := K_0^1$, $F_0^i := 0$, for all $i > 1$, and $\delta_0^i := \delta^i|_{F_0^i}$, for all i. The complex $\mathbf{F}_0 = (F_0^i, \delta_0^i)$ provides the required complex.

Hence, for a flat complex $\mathbf{F}$, we can construct a continuous chain $\{\mathbf{F}_\alpha \mid \alpha \leq \gamma\}$ of flat subcomplexes of $\mathbf{F}$ with $\mathbf{F} = \cup_{\alpha \leq \gamma} \mathbf{F}_\alpha$ such that $|\mathbf{F}_0| \leq \kappa$, for all $\alpha \leq \gamma$, $|\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}| \leq \kappa$ and $\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}$ is a flat complex. Let Y be representative set of flat complexes $\mathbf{F}$ with $|\mathbf{F}| \leq \kappa$. Then $(\mathrm{Flat}(\mathbf{C}(R)), \mathrm{Flat}(\mathbf{C}(R))^\perp)$ cogenerated by Y and so by Proposition 2.2, it has enough injectives.

Corollary 3.1 *The cotorsion theory* $(\mathrm{Flat}(\mathbf{C}(R)), \mathrm{Flat}(\mathbf{C}(R))^\perp)$ *is complete.*

Proof. The result follows from Theorem 3.1 and Proposition 2.1.

Recall that a ring R is called n-perfect if $n = \sup\{\mathrm{cd}F | \mathrm{F}$ is a flat R-module$\} = \sup\{\mathrm{pd}F | \mathrm{F}$ is a flat R-module$\}$. In the remainder of this section we

let R be an n-perfect ring.

Proposition 3.1 *Let $\mathbf{C}$ be a complex of R-modules. Then $\mathbf{C} \in$ $\mathrm{Flat}(\mathbf{C}(R))^{\perp}$ if and only if it is a complex of cotorsion R-modules.*

Proof. Let $\mathbf{C} \in \mathrm{Flat}(\mathbf{C}(R))^{\perp}$. By [3], it is a complex of cotorsion R-modules and $\mathbf{Hom}^{\bullet}(\mathbf{F}, \mathbf{C})$ is an acyclic complex of R-modules for each flat complex $\mathbf{F}$.

Conversely, let $\mathbf{C}$ be a complex of cotorsion R-modules. Then the cotorsion envelope $0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{C}' \longrightarrow \mathbf{F} \longrightarrow 0$ is degree-wise split. So, $\mathbf{F}$ be a pure acyclic complex of cotorsion flat R-modules and hence it is contractible by n-perfectness of R. Therefore $\mathbf{Hom}^{\bullet}(\mathbf{F}, \mathbf{C})$ is an acyclic complex for each $\mathbf{F} \in \mathrm{Flat}(\mathbf{C}(R))$. Then by [3], $\mathbf{C} \in \mathrm{Flat}(\mathbf{C}(R))^{\perp}$.

Theorem 3.2 *Let $\mathbf{F}$ be a flat complex and $\mathbf{Y}$ be a complex of R-modules. Then $\mathrm{Ext}^{i}(\mathbf{F}, \mathbf{Y}) = 0$, for each $i > n$.*

Proof. Let $\mathbf{Y}$ be a complex of R-modules and

$$0 \longrightarrow \mathbf{Y} \longrightarrow \mathbf{C}^{0} \longrightarrow \mathbf{C}^{1} \longrightarrow \cdots \longrightarrow \mathbf{C}^{n-1} \xrightarrow{\delta^{n-1}} \mathbf{C}^{n} \longrightarrow \cdots$$

be it's minimal cotorsion resolution by Proposition 3.1. Since $\mathrm{Ext}^{i}(\mathbf{F}, \mathbf{C}^{j}) = 0$ for every flat complex $\mathbf{F}$ and $i > 0$, $j \geq 0$, Then $\mathrm{Ext}^{n}(\mathbf{F}, \mathbf{Y}) \cong \mathrm{Ext}^{1}(\mathbf{F}, \mathrm{Im}\delta^{n-1})$. Since R is n-perfect then $\mathrm{Im}\delta^{n-1}$ is a complex of cotorsion R-modules and hence it belongs to $\mathrm{Flat}(\mathbf{C}(R))^{\perp}$ by Proposition 3.1. Then $\mathrm{Ext}^{i}(\mathbf{F}, \mathbf{Y}) = 0$, for each $i > n$.

Proposition 3.2 *The ring R is n-perfect if and only if every complex of R-modules has finite cotorsion dimension.*

Proof.

Let R be n-perfect, $\mathbf{Y}$ be an R-module and $0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{F}' \longrightarrow \mathbf{Y} \longrightarrow 0$ be the flat cover of $\mathbf{Y}$. Then for any flat complex $\mathbf{F}$ we have the following exact sequence

$$0 = \mathrm{Ext}^{n+1}(\mathbf{F}, \mathbf{C}) \longrightarrow \mathrm{Ext}^{n+1}(\mathbf{F}, \mathbf{F}') \longrightarrow \mathrm{Ext}^{n+1}(\mathbf{F}, \mathbf{Y}) \longrightarrow \mathrm{Ext}^{n+2}(\mathbf{F}, \mathbf{C}) = 0.$$

Then $\mathrm{Ext}^{n+1}(\mathbf{F}, \mathbf{Y}) = 0$ and hence cd $\mathbf{Y} \leq n$.

The converse is trivial.

Aknowledgements

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